Lecture 6

Transition to global stochasticity

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June 4.th, 2025

Advanced Plasma Physics III: Nonlinear Plasma Physics (2)
– Part II. Non-linear charged particle dynamics –
May 19 – June 4, 2025, IFTS – ZJU, Hangzhou

General concepts

- For near-integrable systems with two degrees of freedom, chaotic regions always exist near the separatrices and persist for finite amplitude perturbations even for $\epsilon \to 0$ (see Lecture 5)

- Therefore, there is not abrupt *transition to stochasticity* above some critical value of ϵ and one should carefully devise the meaning of this concept.

- In this Lecture, we examine a criterion based on the sharp transition one has in systems with two degrees of freedom
  - for sufficiently small ϵ, regions of stochastic motions are limited to the order of the separatrix width $\Delta J/J = \mathcal{O}(\epsilon^{1/2})$: *local/isolated/weak stochastic regions*
  - for stronger ϵ the excursion of the action in the stochastic region can be $\Delta J/J = \mathcal{O}(1)$: *global/connected/strong stochastic regions*

- This transition provides important information concerning the system and can be defined as *barrier transition* or *transition to global stochasticity*.

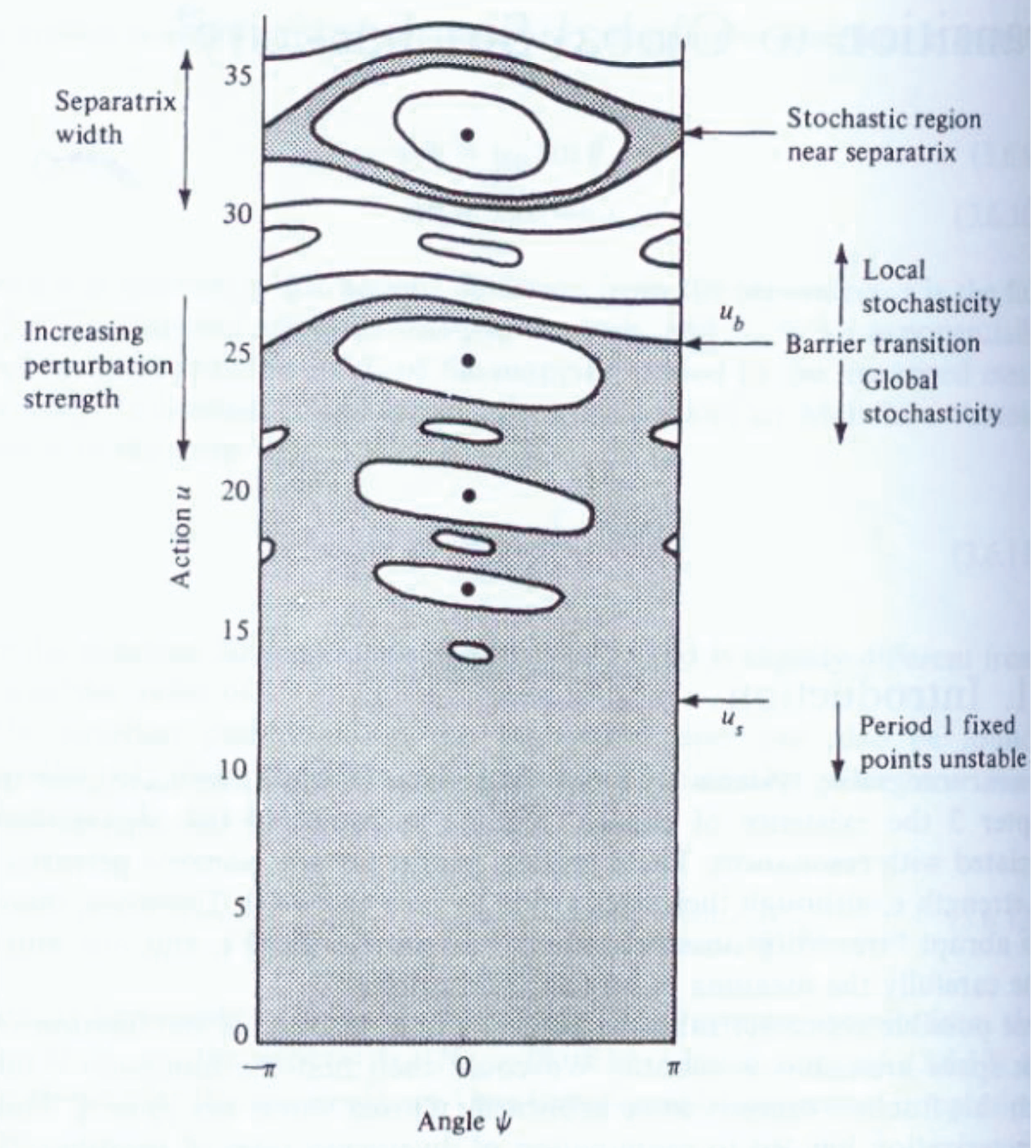


Illustration of the transition from local to global stochasticity as the perturbation strength is increased (*Lieberman and Lichtenberg*, 2010).

- There are other possible characterization of the transition to stochasticity, applicable to systems with $N > 2$ degrees of freedom. These are based on the fraction of the phase space area that is stochastic, so on finding the value of ϵ for which the stochastic region is above a given fraction of the phase space.

- This characterization leads to the examination of divergence rates for neighboring trajectories

  - calculation of the Lyapunov exponents (*Benettin* et al, 1979)
  - calculation of the related Kolmogorov-Sinai entropy (*Chirikov*, 1979)
  - both methods distinguish between regular and stochastic motions and the strength of stochasticity within the stochastic regions

Qualitative criteria

- Overlap criterion. Advanced by (*Chirikov*, 1960) and refined later (*Chirikov*, 1979). It states that the last KAM surface between two lowest-order resonances is destroyed when the sum of the half width of the two resonances, calculated independently, equates the distance between the resonances.
- Rigorously, this criterion is neither necessary nor sufficient, since the KAM surfaces can be strongly perturbed by secondary islands and the separatrix width can be modified for the self-consistent problem.
- The overlap criterion can be refined and made sharper (see below). But it remains qualitatively correct even in its simplest form.
- A similar refinement of the Chirikov criterion is based on the growth of the second order islands (*Jaeger and Lichtenberg*, 1972), whose relative size ratio to resonance spacing at resonance overlap is comparable to that of primary island size to spacing and so on, by induction (see Lecture 4, p.29,33).

- In practice, at resonance overlap, the winding number of the primary resonance is $\alpha = 1/4$, bringing four second order islands, that overlap when their respective winding number is also $\alpha = 1/4$ and so on by induction. In numerical simulations, transition to global stochasticity is measured for a separatrix width smaller by a factor 2/3 (perturbation amplitude smaller by 4/9), corresponding to a $k = 6$ island chain.

- Linear stability can also be used for determining the transition. The loss of linear stability of lowest order islands $k = 1$ is a too strong criterion. However the stability of high k number islands close to a KAM surface was found numerically to give a sharp transition criterion (*Greene*, 1968).

□ The qualitative criterion is the existence of a KAM surface at a given α between two primary resonances $k = 1$. The irrational number that is furthest away from neighboring rationals is the golden mean $(\sqrt{5}-1)/2$, which is expected to be the last KAM surface to disappear, as confirmed numerically by Greene. This is found by looking at the stability of the rational iterates of the golden mean

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} = [a_1, a_2, a_3, \ldots] \quad ; \quad a_n \in \mathbb{N}^+$$

$$\alpha_I = [1, 1, \ldots,] = \frac{\sqrt{5}-1}{2} \quad ; \quad |\alpha - \frac{r_n}{s_n}| < \frac{1}{s_n s_{n-1}}$$

This procedures is similar to a *renormalization.*

- Renormalization method (*Escande and Doveil*, 1981) for determining the amplitudes of two nearest neighbor island chains that are required to destroy a KAM surface. The general solution is known for two primary resonances with arbitrary separation (see below) and does not give the golden mean criterion.
- Determination of a KAM torus form a variational approach (*Percival*, 1974).

The standard mapping

- Transition to global stochasticity can be studied with the standard mapping, introduced in Lecture 5 p.12.

$$I_{n+1} = I_n + K \sin\theta_n \ \ ; \ \ \theta_{n+1} = \theta_n + I_{n+1}$$

- The standard mapping is locally equivalent to cases of physical interest and the *stochasticity parameter* K is related to the action (energy) of the system in different part of the phase space for systems with two degrees of freedom.

- Example 1. Consider the *Fermi mapping*

$$u_{n+1} = |u_n + \sin\psi_n| \ \ ; \ \ \psi_{n+1} = \psi_n + 2\pi M/u_{n+1}$$

- This gives back the standard mapping by linearization near a period 1 fixed point $2\pi M/u_1 = 2\pi m, m \in \mathbb{Z}$, by letting $u_n = u_1 + \Delta u_n$, $I_n = (-2\pi M \Delta u_n)/u_1^2$, $\theta_n = \psi_n - \pi, \theta_n \in (-\pi, \pi]$ and $K = 2\pi M/u_1^2$

E: Show step by step the local reduction of the Fermi mapping to the standard mapping.

- **Example 2.** Consider the *separatrix mapping* (Lecture 5 p.32), for $r = 1$ with $Q_0 = \Omega/(r\omega_0) = \Omega/\omega_0$ and $w(J) = -(\epsilon F + \Omega J)/(\epsilon F)$ representing the deviation of the energy from the separatrix

$$w_{n+1} = w_n - w_0 \sin\theta_n \quad ; \quad \theta_{n+1} = \theta_n + Q_0 r \ln\left|\frac{32}{w_{n+1}}\right|$$

$$w_0 = \Omega\frac{f}{F} = 8\pi\frac{\Lambda}{F}Q_0^2 \exp(-\pi Q_0/2)$$

- The standard mapping is obtained by linearization around period 1 fixed points $\theta_1 = 0, \pi$ and

$$Q_0 r \ln \left| \frac{32}{w_1} \right| = 2\pi m \;\;;\;\; w_1 = \pm 32 \exp(\frac{-2\pi m}{Q_0 r}) \;\;;\;\; m \in \mathbb{Z}$$

- Posing $w_n = w_1 + \Delta w_n$, the standard mapping is obtained for

$$I_n = -\frac{Q_0 \Delta w_n}{w_1} \;\;;\;\; K = \frac{Q_0 w_0}{w_1}$$

E: Show step by step the local reduction of the separatrix mapping to the standard mapping.

- Given the standard mapping, the period 1 fixed points are obtained requiring that phase (mod 2π) and action are stationary, i.e. $I_1 = 2\pi m, m \in \mathbb{Z}$ and $\theta_1 = 0, \pi$.

E: Show that this is the case.

- There are two fixed points for each integer m. Expanding about them, the linearized transformation matrix (see Lecture 5 p.18,33,34) is given by (upper sign corresponds to $\theta_1 = 0$)

$$\boldsymbol{A} = \begin{pmatrix} 1 & \pm K \\ 1 & 1 \pm K \end{pmatrix}$$

E: Show that the matrix $\boldsymbol{A}$ is area preserving.

- For stability (see Lecture 5 p.33) $|\mathrm{Tr}\boldsymbol{A}| < 2$

$$|2 \pm K| < 2$$

- The fixed point at $\theta_1 = 0$ is always unstable while that at $\theta_1 = \pi$ is unstable for $K > 4$. For $K > 4$ there is no stable motion about period 1 fixed points.

- Period 2 fixed points are obtained iterating the map twice

$$I_2 = I_1 + K \sin \theta_1 \;\; ; \;\; \theta_2 = \theta_1 + I_2 - 2\pi m_1 \;\; ; \;\; m_1 \in \mathbb{Z}$$

$$I_1 = I_2 + K \sin \theta_2 \;\; ; \;\; \theta_1 = \theta_2 + I_1 - 2\pi m_2 \;\; ; \;\; m_2 \in \mathbb{Z}$$

- Stability follows from the trace of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & K \cos \theta_2 \\ 1 & 1 + K \cos \theta_2 \end{pmatrix} \begin{pmatrix} 1 & K \cos \theta_1 \\ 1 & 1 + K \cos \theta_1 \end{pmatrix}$$

E: Show that this is indeed the form for the linear transformation matrix. Is the transformation area preserving? Why?

- Stability condition is generally given by

$$-4 < 2K(\cos\theta_1 + \cos\theta_2) + K^2 \cos\theta_1 \cos\theta_2 < 0$$

E: Show that this is the case from the condition $|\mathrm{Tr}\boldsymbol{A}| < 2$.

E: Show that $\sin\theta_1 + \sin\theta_2 = 0$, i.e., that there are two cases to be considered: $\theta_2 = -\theta_1$ and $\theta_2 = \theta_1 - \pi$, with $0 < \theta_1 \leq \pi$.

- For the case $\theta_2 = -\theta_1$, the mapping equations give

$$2\pi p - 4\theta_1 = K\sin\theta_1 \quad ; \quad p = m_1 - m_2$$

- Fixed points are classified in two families:
    - primary families: exist for $K \to 0$
    - bifurcation families: exist for K above a given threshold

- For the case $\theta_2 = -\theta_1$, the primary family is given by $p = 1$ ($p > 1$ gives bifurcation families) and the stability condition is

$$-4 < K \cos\theta_1 < 0$$

- The primary family is always unstable, since it is found at $I_{1,2} = \pi(2m_2 + 1)$, $\theta_{1,2} = \pm\pi/2 \mp K/4$ for $K \ll 1$.

- The bifurcation family $p = 2$ appears at $K = 4$ and remains stable for $4 < K < 2\pi$. There are stability windows for arbitrarily high K as p increases.

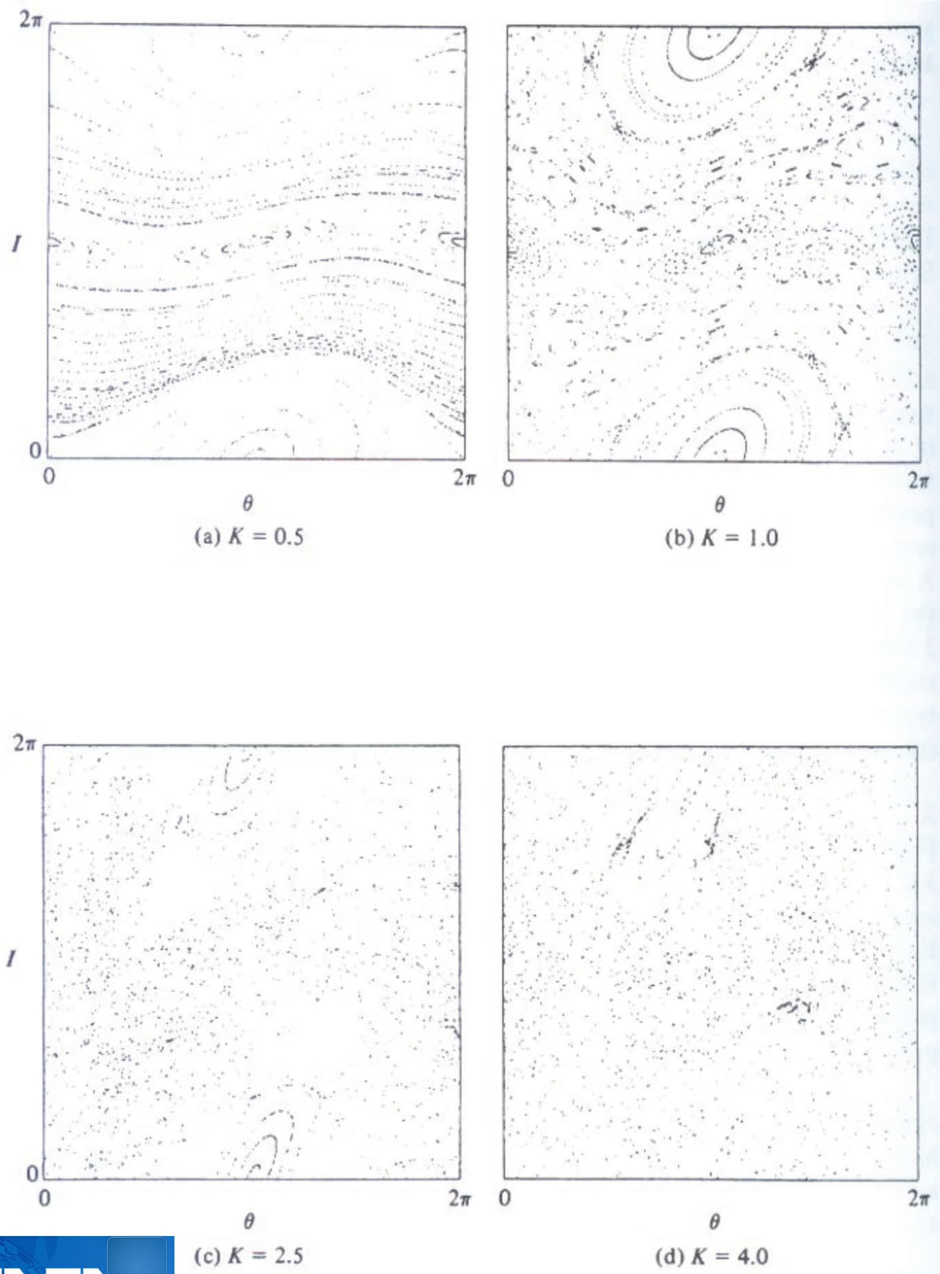
- For the case $\theta_2 = \theta_1 - \pi$, the mapping equations give

$$2\pi(p-1) = K \sin\theta_1 \quad ; \quad p = m_1 - m_2$$

with the stability condition

$$K^2 \cos^2\theta_1 < 4$$

- The primary family is stable for $K < 2$, since it is found at $I_{1,2} = \pi(2m_2+1)$, $\theta_1 = \pi$, $\theta_2 = 0$.
- The bifurcation family $p = 2$ appears at $K = 2\pi$ and remains stable for $6.28 < K < 6.59$. As in the other case, there are stability windows for arbitrarily high K as p increases.



Phase plane for the standard mapping for various values of K (*Lieberman and Lichtenberg*, 2010).

(a) period 1 and period 2 isolated resonances

(b) KAM surface between period 1 and period 2 islands is destroyed, leading to global stochasticity

(c) period 2 fixed points are unstable

(d) period 1 fixed point is unstable

Resonance Overlap

- Following Lecture 5 and introducing the "time-variable" n, the Hamiltonian for the standard mapping is

$$H = \frac{I^2}{2} + K \cos\theta \sum_{m=-\infty}^{\infty} \exp(i2\pi mn)$$

- The criterion of slow variation $|d\theta/dn| \ll 2\pi$ helps identifying the perturbation part that averages to zero. The Hamiltonian reduces then to

$$H = \frac{I^2}{2} + K \cos\theta$$

E: Determine the period 1 fixed points of this Hamiltonian and show that the stable fixed point is located at $I = 0$, $\theta = \pi$, while the frequency of the respective small librations is $K^{1/2}$. Show that the maximum excursion is $\Delta I_{\max} = 2K^{1/2}$ and that the distance between primary resonances is $\delta I = 2\pi$ (why?).

- One time iteration $n \to n+1$ corresponds to $\omega_0/2\pi = K^{1/2}/2\pi$ dimensionless time step in the small libration. Thus, the rotation number is

$$\frac{2\Delta I_{\max}}{\delta I} = 4\alpha_0 = \frac{4}{Q_0}$$

$Q_0 = 1/\alpha_0$ giving the number of iterations per libration period.

E: Comment this point in the light of the remarks given at p.5 and of the results of Lecture 4, p.29,33. One criterion for *barrier transition* is $\alpha_0 = 1/6$, i.e. $2\Delta I_{\max}/\delta I \simeq 2/3$.

- We saw already that the primary resonance overlap criterion is qualitatively correct but quantitative too severe:

$$2\Delta I_{\rm max}/\delta I = 1 \;\rightarrow\; 4K^{1/2} = 2\pi \quad ; \quad K = (\pi/2)^2$$

- Period 2 fixed points are found (see p.16) for $\theta = \pi$, $I = (2p+1)\pi$, $p \in \mathbb{Z}$.

- The second order island width is computed using the method of Lie generating function (see Lecture 4) and the Hamiltonian

$$H = \frac{I^2}{2} + \epsilon K \sum_{m=-\infty}^{\infty} \cos(\theta - 2\pi n m)$$

- The first order equation is (see Lecture 4 p.42)

$$\left(\partial/\partial n + I\partial/\partial\theta\right) w_1 = \bar{H}_1 - H_1$$

- This can be solved for $\bar{H}_1 = 0$ and yields the non singular expression

$$w_1 = K \sum_{m=-\infty}^{\infty} \frac{\sin(\theta - 2\pi m n)}{2\pi m - I}$$

E: Explain why, at second order resonances, the expression for w_1 is non singular.

- At second order (see Lecture 4 p.42)

$$\left(\frac{\partial}{\partial n} + I\frac{\partial}{\partial\theta}\right) w_2 = 2(\bar{H}_2 - H_2) - [w_1, \bar{H}_1 + H_1] \;\; ; \;\; \bar{H}_1 = H_2 = 0$$

- One should choose $\bar{H}_2$ such that the average on the unperturbed orbit $\theta = In$ of the RHS vanishes

$$\bar{H}_2 = \frac{1}{2}\langle [w_1, H_1] \rangle = -\frac{1}{2}\left\langle \frac{\partial w_1}{\partial I}\frac{\partial H_1}{\partial \theta} \right\rangle$$

- Doing some algebra explicitly:

$$\begin{aligned}\bar{H}_2 &= \frac{K^2}{2}\left\langle \sum_{m,m'} \frac{\sin(\theta - 2\pi m n)}{(2\pi m - I)^2} \sin(\theta - 2\pi m' n) \right\rangle \\ &= \frac{K^2}{4}\left\langle \sum_{m,m'} \frac{\cos[2\pi(m'-m)n] - \cos[2\theta - 2\pi(m+m')n]}{(2\pi m - I)^2} \right\rangle\end{aligned}$$

- The phases are rapidly oscillating. So the first term vanishes unless $m' = m$ yielding a constant and ignorable term. The second term vanishes unless, using the resonance condition (remember $\theta = In$ for the unperturbed motion)

$$2I - 2\pi(m' + m) = 2\pi(2p + 1) - 2\pi(m + m') = 0 \;\Rightarrow m + m' = 2p + 1$$

- The final result is

$$\bar{H}_2 = -\frac{K^2}{4} \sum_m \frac{\cos[2\theta - 2\pi(2p+1)n]}{(2\pi)^2 (m - p - 1/2)^2} = -\frac{K^2}{16} \cos 2\hat{\theta}$$

having noted

$$\sum_m \frac{1}{(m - p - 1/2)^2} = \pi^2 \quad ; \quad \hat{\theta} = \theta - \pi(2p+1)n$$

- Up to second order the Hamiltonian is

$$\bar{H} = \frac{1}{2} I^2 - \frac{K^2}{16} \cos 2\hat{\theta} \quad ; \quad \omega_s = \frac{K}{4} \quad ; \quad \Delta I_{2\max} = \frac{K}{2}$$

- The overlap criterion including primary and secondary resonances becomes (*Chirikov*, 1979)

$$2K^{1/2} + \frac{1}{2} K = \pi \quad ; K = 1.46$$

- Thickness of the separatrix layer. The criterion can be made even sharper including the thickness of the separatrix layer

- The separatrix mapping has been put in connection with the standard mapping on p.10,11 (considering $\Lambda/F = 1$)

$$Q_0 = \frac{2\pi}{K^{1/2}} \;\; ; \;\; w_0 = 8\pi Q_0^2 \exp(-\pi Q_0/2) = 4(2\pi)^3 K^{-1} \exp(-\pi^2/K)$$

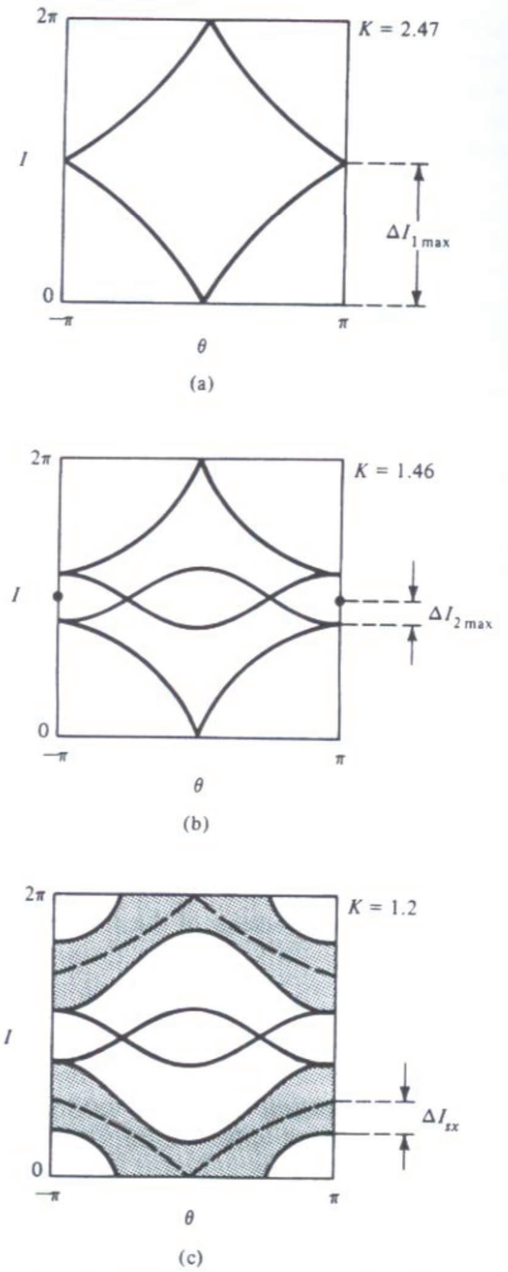
$$K = \frac{Q_0 w_0}{w_1} \;\; ; \;\; w_1 = 4(2\pi)^4 K^{-5/2} \exp(-\pi^2/K)$$

- The fractional deviation from the separatrix energy K is w_1 (p.10). In terms of actions

$$I_{\rm sx}\Delta I_{\rm sx} = \Delta H = K w_1 \quad \Rightarrow \Delta I_{\rm sx} = w_1 \frac{K^{1/2}}{2}$$

- The barrier transition criterion including the separatrix width is

$$(1 + w_1/4) 2K^{1/2} + K/2 = \pi \;\; ; \;\; K = 1.2$$



Overlap criteria (*Lieberman and Lichtenberg*, 2010).

(a) primary resonance overlap

(b) overlap of primary and secondary resonances

(c) improved overlap criterion including separatrix width of the primary resonance

Growth of second order islands

- The method has been proposed by (*Jaeger and Lichtenberg*, 1972) (see p. 5)
- Lecture 4 derived the results we are interested in for the resonant wave-particle interaction problem (see p.29,33): the ratio of island size to resonance spacing is essentially the same for first and second order islands, and so forth by induction.
- Therefore, at island overlap as for the Chirikov criterion, the barrier transition has already occurred. This happens when the winding number is $\alpha = 1/4$ corresponding also to 4 second order islands.
- In practice, at the barrier transition, the winding number of the primary resonance is $1/6 \lesssim \alpha < 1/5$, and all higher order islands have about the same rotation number and the same ratio of the separatrix width to resonance spacing as the primary resonance.

- At the barrier transition to global stochasticity, we expect 5 second order islands near a primary island. For the standard mapping, this criterion (*Jaeger and Lichtenberg*, 1972) gives

$$K^{1/2} = \frac{2\pi}{6} \;\; ; K \approx 1.1$$

not far away from the estimate of the improved Chirikov criterion.

Renormalization of two resonances

- A method for determining the destruction of a KAM surface (torus) between two primary resonances was developed by (*Escande and Doveil*, 1981).
- This is a renormalization procedure for looking at structures near the KAM torus on finer and finer scales.
- Consider the Hamiltonian of a particle with two waves (*Escande and Doveil*, 1981)

$$H_w = \frac{p^2}{2m} - V_1 \cos(k_1 x - \omega_1 t) - V_2 \cos(k_2 x - \omega_2 t)$$

  where the subscript 1 is for the main resonance M and 2 for the perturbing resonance P.

- Introduce the notations $v_1 = \omega_1/k_1$, $v_2 = \omega_2/k_2$

$$\psi = k_1 x - \omega_1 t \ ; \ I = \frac{p/m}{v_2 - v_1} \ ; \ \Omega = k_1(v_2 - v_1)$$

$$H_I = \frac{1}{2}I^2 - M\cos\psi - P\cos k(\psi - \Omega t)$$

$$k = k_2/k_1 \ ; \ M = \frac{V_1}{m(v_2 - v_1)^2} \ ; \ P = \frac{V_2}{m(v_2 - v_1)^2}$$

E: Show that you can recover this change of coordinates with the generating function $F_2 = J(k_1 x - \omega_1 t)$ and considering that the correct time-like parameter for preserving the Hamiltonian structure in the equations of motion is $\tau = (v_2 - v_1)t$. So that

$$\psi = \frac{\partial F_2}{\partial J} = k_1 x - \omega_1 t \ ; \ p = \frac{\partial F_2}{\partial x} = k_1 J \ ; \ K = \frac{k_1^2 J^2}{2m} - J\omega_1 - V_1\cos\psi - V_2\cos k(\psi - \Omega t)$$

- The actual form of the Hamiltonian is then

$$H_I = \frac{1}{2}I^2 - M\cos\psi - P\cos k(\psi - \Omega t) = \frac{1}{2}I^2 - M\cos\psi - P\cos k(\psi - k_1\tau)$$

- As for the analyses of the second order resonances (see Lecture 4), we move to action angle coordinates of the main resonance, which implies using expressions involving elliptic integrals (see Lecture 2, p.8–10)

- We study the system in the region between two primary resonances, i.e. in the region of rotation for the main resonance. Thus

$$\psi = \theta + \chi(J, \theta)$$

with $\chi(J, \theta)$ a periodic function of θ expressed in term of elliptic integrals.

- The new Hamiltonian can be expressed as Fourier series

$$H = H_0(J) - P\cos k[\psi(J,\theta) - \Omega t] = H_0(J) - P\sum_{n=-\infty}^{\infty} U_n(J)\cos[(k+n)\theta - k\Omega t]$$

- Following Lecture 4, one seeks secondary resonances between the rotation frequency $\omega(J_n) = \dot{\theta}(J_n)$ and the driving frequency $k\Omega$ that occur when

$$(k+n)\omega(J_n) - k\Omega = 0$$

- In terms of the inverse rotation number $Q_n = 1/\alpha_n$, the secondary resonance condition is

$$Q_n = (k+n) + \frac{k\Omega}{\omega(J_n)}$$

- We want to determine the condition under which a regular trajectory (KAM torus) at $J = J_0$ between main and perturbing resonances is broken. The trajectory has (if not, exchange main and perturbing resonance definition)

$$Q(J_0) = \frac{k\Omega}{\omega(J_0)} > k+1$$

- The KAM torus, thus, lies between the secondary resonances n and $n+1$ with $n > 1$ and

$$z \equiv Q - k = n + \delta k \quad \Rightarrow \quad n = \mathrm{INT}\,(Q-k)$$

- As usual, one can expand for small action deviations $\Delta J = J - J_0$ introducing the nonlinearity parameter $G = \partial^2 H_0/\partial J_0^2$

$$H \simeq H_0(J_0) + \omega(J_0)\Delta J_0 + \frac{1}{2}G(J_0)\Delta J^2 - PU_n(J_0)\cos[(k+n)\theta - k\Omega t]$$
$$-PU_{n+1}(J_0)\cos[(k+n+1)\theta - k\Omega t]$$

- Note the similarity of this problem with the original problem. We have a regular trajectory between 2 resonances, which now are secondary resonances. We can call main resonance the closest to Q and perturbing resonance the other one, depending on whether δk is larger or smaller than 1/2.
- Introducing $\lambda = 0, \delta k < 1/2$, $\lambda = 1, \delta k \geq 1/2$ and

$$\beta = \frac{(k+n)(k+n+1)}{k}$$

we can once more move to action angle coordinates of the main resonance and write the new Hamiltonian as

$$\bar{H} = \frac{1}{2}\bar{I}^2 - \bar{M}\cos\bar{\psi} - \bar{P}\cos k(\bar{\psi} - \bar{\Omega}t)$$

E: Show this result and derive the explicit expressions of the constants appearing in the new Hamiltonian.

- Here, the definition of the parameters is

$$\bar{\psi} = (k + n + \lambda)\theta - k\Omega t \;\; ; \;\; \bar{\Omega} = \frac{(2\lambda - 1)k\Omega}{k + n + 1 - \lambda}$$

$$\bar{z} \equiv \bar{Q} - \bar{k} \equiv \bar{n} + \delta\bar{k} = \frac{1 - \lambda - \delta k}{\delta k - \lambda} \;\; ; \;\; \bar{k} = \frac{k + n + 1 - \lambda}{k + n + \lambda}$$

$$\bar{M} = PU_{n+\lambda}\beta^2 G \;\; ; \;\; \bar{P} = PU_{n+1-\lambda}\beta^2 G$$

- This is a renormalization transform

$$\mathcal{I} : (\delta k, n, k, M, P) \mapsto (\delta\bar{k}, \bar{n}, \bar{k}, \bar{M}, \bar{P})$$

given as non-Hamiltonian five-dimensional mapping.

- Analytic treatment of this mapping is difficult (see *Escande and Doveil*, 1981). The simplified approach is based on the assumption that, at each step of the renormalization procedure, if $\bar{M}$ and $\bar{P}$ converge to zero, the KAM torus is preserved, otherwise is destroyed. The result gives (see *Escande and Doveil*, 1981)

$$Q = 2 + \frac{\sqrt{5} - 1}{2}$$

consistent with the result of (*Greene*, 1968).

- This result is approximate, since only two resonances are computed rather than all. The corresponding value of $K \approx 1.2$ for the standard map is obtained with the 2/3 rule.

E: Show that this expression for Q gives the value

$$\alpha = \frac{1}{Q} = \frac{K^{1/2}}{2\pi} = \frac{1}{2}\left(3 - \sqrt{5}\right)$$

which is not the golden mean defined at p.7

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