Lecture 3

Classical perturbation theory

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General description of perturbation methods

- The basic method for solving nonlinear dynamical problems is by perturbations of known integrable solutions.
- The known solutions are those of an integrable Hamiltonian and perturbations are computed as series expansions in a small parameter ϵ by which the system of interest differs from the known integrable Hamiltonian.
- The fundamental underlying assumptions of this procedure is that the solution we are looking for actually exists. However, we know that only systems with one degree of freedom are always integrable. Most multidimensional nonlinear systems are not integrable.
- When systems are not integrable, chaotic trajectories associated with resonances among different degrees of freedom are densely distributed among regular trajectories and have finite measure in the phase space.

- Perturbation theory fails to describe the complexity of chaotic motion.

- Classical perturbation theory (this lecture) also fails to describe the change in topology of regular solutions when islands in phase space are form (librations), e.g. when particular resonances among the degrees of freedom occur. Treating these situation is the topic of secular perturbation theory (see Lecture 4).

E: Discuss the concept of resonances among degrees of freedom. Can you provide an argument for showing that wave-particle resonances fall in this category?

- Perturbation theory provides series expansions of the system trajectories that should approximate the actual solutions. It is important to be aware of the accuracy of the approximate solutions. Uniformly convergent series can be generally obtained of one degree of freedom.

- In general series are not convergent or at best asymptotically convergent. Typical example of the failure of a series expansion to converge is the occurrence of *small denominators* (*resonances*) or the corresponding *secular terms* (unbounded in time).

E: Can you explain why secular terms in systems with one degree of freedom can be avoided? Can you draw analogies between the present general discussion about the occurrence of secular terms with the similar discussion made on p.19 in Lecture 2 of Spring 2010 lecture notes?

- Among most efficient methods to cure divergent series there is the method of averaging, which leads directly to computing *adiabatic invariants*, i.e. approximate integrals of motion obtained by averaging over a fast variable.

- Connected with the convergence of the formal series expansions that are the basis of perturbation theory, there is the fact that the approximate solution can approximate the motion in a coarse-grained sense, if chaotic motion is confined to a thin separatrix layer bounded by regular trajectories. However, perturbation theory certainly fails to describe, even qualitatively, the chaotic motion in regions where main resonances overlap.

- In systems with more than two degrees of freedom, this is even more true, due to the presence of *Arnold diffusion*. In this case, adiabatic invariant may not be even approximately conserved on time scales longer than the characteristic *slow* time scale of the system.

Removal of secular terms in systems with one degree of freedom

- Consider the pendulum Hamiltonian (see Lecture 2) and expand the motion for small $\phi = x$ up to cubic terms in x

$$H = \frac{1}{2}Gp^2 - F\cos\phi = E$$

- Posing $\omega_0 = (FG)^{1/2}$ and adding formally an ϵ in front of cubic terms in x as a dimensionless parameter of smallness, the equation of motion is

$$\ddot{x} + \omega_0^2 x = \frac{\epsilon}{6}\omega_0^2 x^3$$

- The approximate solution of this problem can be obtained with the formal expansion

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

E: Show that the solution one obtains with this method is

$$x_0 = A\cos\omega_0 t \quad ; \quad x_1 = \frac{A^3}{192}\left(3\omega_0 t\sin\omega_0 t + 6\cos\omega_0 t - \cos 3\omega_0 t\right)$$

- This solution is incorrect, since it contains secular terms while we know that the system is integrable. The correct solution is obtained assuming a formal series expansion not only for x but for ω as well

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

E: Show that correct solution one obtains with this method is

$$x_0 = A\cos\omega_0 t \quad ; \quad x_1 = \frac{A^3}{192}\left(\cos\omega_0 t - \cos 3\omega_0 t\right) \quad ; \quad \omega_1 = -\frac{A^2}{16}\omega_0$$

E: Consider the driven oscillator $\ddot{x} + \omega_0^2 x = g(t) = (a_0/2) + \sum_{n=1}^{\infty}(a_n \cos n\Omega t + b_n \sin n\Omega t)$. Show that the solution can be written as

$$x = A\cos\omega_0 t + B\sin\omega_0 t + \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty}\frac{a_n\cos n\Omega t + b_n \sin n\Omega t}{\omega_0^2 - n^2\Omega^2}$$

- The resonant denominators are evidence of the failure of the series to converge. The system in this case has two degrees of freedom, since it is time dependent.
- When the oscillator is nonlinear, all harmonics of ω_0 characterize the motion. So resonances are possible when ω_0/Ω is a rational number.

- The oscillation frequency is also function of the oscillation amplitude. So we can say that there is a dense set of resonances at the rationals when the oscillation amplitude is varied. This is sign that the character of phase space trajectories is changed.

Systems with one degree of freedom

- The general canonical perturbation theory for systems with one degree of freedom is given by Poincaré (*Poincaré*, 1892) and Von Zeipel (*Von Zeipel*, 1916).

- Consider the Hamiltonian

$$H = H_0(J) + \epsilon H_1(J, \theta) + \epsilon^2 H_2(J, \theta) + \ldots$$

- The integrable part $H_0(J)$ is given in action-angle form, so that the solution is

$$J = J_0 \;\; ; \;\; \theta = \omega t + \beta \;\; ; \;\; \omega = \frac{\partial H_0}{\partial J}$$

- The perturbative solution is found seeking for a canonical transform to new variables $\bar{J}, \bar{\theta}$ via the generating function $F_2 = S(\bar{J}, \theta)$, such that the new Hamiltonian $\bar{H} = \bar{H}(\bar{J})$.

- By construction, $F_2 = S(\bar{J}, \theta)$ should reduce to the identity transform for $\epsilon = 0$, so

$$S = \bar{J}\theta + \epsilon S_1 + \ldots \;\; ; \;\; \bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 + \ldots \;\; ; \;\; J = \bar{J} + \epsilon \frac{\partial S_1(\bar{J}, \theta)}{\partial \theta} + \ldots \;\; ; \;\; \bar{\theta} = \theta + \epsilon \frac{\partial S_1(\bar{J}, \theta)}{\partial \bar{J}} + \ldots$$

- By definition

$$\bar{H}(\bar{J}, \bar{\theta}) = H\left(J(\bar{J}, \bar{\theta}), \theta(\bar{J}, \bar{\theta})\right)$$

This means that one must invert the implicit expressions of the old coordinates as a function of the new ones.

- This procedure is straightforward but tedious and not practical, for the canonical transformations are always expressed as a mixture of old and new coordinates. This is the limitation of the application of canonical methods to perturbation theories beyond first order, which favors the introduction of non-canonical methods based, e.g., on the Lie generating function (see Lecture 1).

- Up to first order one gets

$$J = \bar{J} + \epsilon \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{\theta}} + \ldots \; ; \quad \theta = \bar{\theta} - \epsilon \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{J}} + \ldots$$

$$H_0\left(J(\bar{J},\bar{\theta})\right) = H_0\left(\bar{J}\right) + \epsilon \frac{\partial H_0(\bar{J})}{\partial \bar{J}} \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{\theta}} + \ldots \; ; \quad H_1\left(J(\bar{J},\bar{\theta}), \theta(\bar{J},\bar{\theta})\right) = H_1(\bar{J},\bar{\theta}) + \ldots$$

$$\bar{H}_0 = H_0(\bar{J}) \; ; \quad \bar{H}_1 = \omega(\bar{J}) \frac{\partial S_1(\bar{J},\bar{\theta})}{\partial \bar{\theta}} + H_1(\bar{J},\bar{\theta})$$

- The condition that $\bar{H} = \bar{H}(\bar{J})$ up to first order requires

$$\bar{H} = H_0(\bar{J}) + \epsilon \langle H_1 \rangle + \ldots \;\; ; \;\; \langle H_1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\bar{\theta} H_1(\bar{J}, \bar{\theta})$$

$$\omega \frac{\partial S_1(\bar{J}, \bar{\theta})}{\partial \bar{\theta}} = - \left[H_1 - \langle H_1 \rangle \right] = - \{ H_1 \}$$

- The solution can be written given the Fourier expansions

$$\{H_1\} = \sum_{n \neq 0} H_{1n} e^{in\bar{\theta}} \;\; ; \;\; S_1 = \sum_n S_{1n} e^{in\bar{\theta}} \;\; ; \;\; S_{1n} = i \frac{H_{1n}}{n\omega} \quad n \neq 0$$

- One can directly verify that the Fourier series converges for $\omega(\bar{J}) \neq 0$.

E: Apply this general procedure to the pendulum Hamiltonian. Show that

$$S_1 = -\frac{GJ^2}{192\omega_0}(8\sin 2\theta - \sin 4\theta)$$

$$H = \omega_0 J - \epsilon\frac{GJ^2}{48}(3 - 4\cos 2\theta + \cos 4\theta)$$

$$\bar{H} = \omega_0 \bar{J} - \frac{\epsilon}{16} G\bar{J}^2 \ \ ; \ \ \bar{\omega} = \frac{\partial \bar{H}}{\partial \bar{J}} = \omega_0 - \frac{\epsilon}{8} G\bar{J}$$

Systems with two or more degrees of freedom

- The method of the previous section is readily extended more than one degree of freedom. In this case

$$H(\boldsymbol{J}, \boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta}) + \epsilon^2 H_2(\boldsymbol{J}, \boldsymbol{\theta}) + \ldots$$

- In action-angle form, we can assume H_1 as a multiply periodic function of the angles ($m_i \in \mathbb{Z}$)

$$H_1 = \sum_m H_{1m}(\boldsymbol{J}) e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}} \quad ; \quad \boldsymbol{m}\cdot\boldsymbol{\theta} = m_1\theta_1 + \ldots m_N\theta_N$$

- To obtain the new Hamiltonian in the form $\bar{H} = \bar{H}(\bar{\boldsymbol{J}})$ one seeks a canonical transformation to new variables $\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}$ via the generating function $F_2 = S(\bar{\boldsymbol{J}}, \boldsymbol{\theta})$, such that

$$S = \bar{\boldsymbol{J}} \cdot \boldsymbol{\theta} + \epsilon \sum_m S_{1m}(\bar{\boldsymbol{J}}) e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}} + \ldots$$

- Proceeding as for one degree of freedom, up to first order one gets

$$\bar{H} = H_0\left(\bar{\boldsymbol{J}}\right) + \epsilon\left(\boldsymbol{\omega}(\bar{\boldsymbol{J}}) \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \bar{\theta}} + H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})\right) + \ldots \;\; ; \;\; \boldsymbol{\omega}(\bar{\boldsymbol{J}}) = \frac{\partial H_0(\bar{\boldsymbol{J}})}{\partial \bar{\boldsymbol{J}}}$$

- The condition that $\bar{H} = \bar{H}(\bar{\boldsymbol{J}})$ up to first order requires

$$\bar{H} = H_0(\bar{\boldsymbol{J}}) + \epsilon \left\langle H_1 \right\rangle + \ldots \;\; ; \;\; \boldsymbol{\omega} \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \bar{\boldsymbol{\theta}}} = -\left\{H_1\right\}$$

E: Derive these results step by step and show that

$$\frac{dS_1}{dt} = \frac{d\bar{\boldsymbol{\theta}}}{dt} \cdot \frac{\partial S_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}})}{\partial \bar{\boldsymbol{\theta}}}$$

What information do you need to use for deriving this result? Is that useful for the solution of the perturbation expansion?

- Having solved the exercise above, one can write

$$S_1 = -\int^t \left\{ H_1(\bar{\boldsymbol{J}}, \bar{\boldsymbol{\theta}}(t')) \right\} dt'$$

- Another equivalent form of S_1 can be obtained integrating the Fourier series for H_1, i.e.,

$$S(\bar{\boldsymbol{J}}, \boldsymbol{\theta}) = \bar{\boldsymbol{J}} \cdot \boldsymbol{\theta} + i\epsilon \sum_{m \neq 0} \frac{H_{1m}(\bar{\boldsymbol{J}})}{\boldsymbol{m} \cdot \boldsymbol{\omega}(\bar{\boldsymbol{J}})} e^{i\boldsymbol{m}\cdot\boldsymbol{\theta}} + \dots$$

- When $\boldsymbol{m}\cdot\boldsymbol{\omega}(\bar{\boldsymbol{J}}) = 0$ there is a resonance and the series fails to converge. This is connected with the change of topology of trajectories near resonances, for which one must use secular perturbation theory (see Lecture 4).

- The perturbative solution written above is still a good approximation sufficiently far away from resonances or in a coarse-grain sense.

E: Show that the resonant denominators correspond to secular terms, i.e. terms that are unbounded in time.

- A simple application of the above general result is the problem of a Hamiltonian with one degree of freedom and explicit time dependence. This system is equivalent to an autonomous system with two degree of freedom in the extended phase space, which we take as $J_1, J_2, \theta_1, \theta_2$, with $J_1 = J$, $\theta_1 = \theta$, $J_2 = -H$ and $\theta_2 = \Omega t$, with Hamiltonian H and extended phase Hamiltonian K given by

$$H = H_0(J) + \epsilon \sum_{l,m} H_{1lm}(J) e^{il\theta_1 + m\theta_2} + \ldots = H_0(J) + \epsilon \sum_{l,m} H_{1lm}(J) e^{il\theta + m\Omega t} + \ldots$$

$$K = H_0(J_1) + \epsilon \sum_{l,m} H_{1lm}(J_1) e^{il\theta_1 + m\theta_2} + J_2 + \ldots$$

E: Apply the general formalism and show that S_1 and the new action invariant are

$$S_1 = i \sum_{l,m\neq 0} \frac{H_{1lm}(\bar{J})}{l\omega(\bar{J}) + m\Omega} e^{i(l\theta + m\Omega t)} \quad ; \quad \bar{J} = J - \epsilon \frac{\partial}{\partial\theta} S_1(J,\theta,t)$$

- Wave particle interaction of a charge moving in a constant $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$ field, i.e. $\boldsymbol{A}(\boldsymbol{x}) = -B_0 y \hat{\boldsymbol{x}}$

$$H = H_0 + \epsilon H_1 \quad ; \quad H_0 = \frac{1}{2M} \left| \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} \right|^2 \quad ; \quad H_1 = e\Phi_0 \sin(k_z z + k_\perp y - \omega t)$$

- Transformation to guiding center variables is obtained with the generating function, given $\Omega = eB_0/(Mc)$ and $\tan\phi = -v_x/v_y$,

$$F_1 = M\Omega \left[\frac{1}{2}(y - Y)^2 \cot\phi - xY \right]$$

- One readily has

$$p_x = Mv_x - M\Omega y = \frac{\partial F_1}{\partial x} = -M\Omega Y \;\; ; \;\; p_y = Mv_y = \frac{\partial F_1}{\partial y} = M\Omega(Y - y)\cot\phi$$

from which

$$Y = y - v_x/\Omega \;\; ; \;\; v_y = -v_x \cot\phi$$

$$P_\phi = -\frac{\partial F_1}{\partial \phi} = \frac{M\Omega}{2}(y-Y)^2(1+\cot^2\phi) = \frac{Mv_\perp^2}{2\Omega} = \;\; ; \;\; v_\perp^2 = v_x^2 + v_y^2$$

$$P_Y = -\frac{\partial F_1}{\partial Y} = M\Omega\left(\frac{v_x}{\Omega}\cot\phi + x\right) = M\Omega\left(-\frac{v_y}{\Omega} + x\right) = M\Omega X \;\; ; \;\; X = x - \frac{v_y}{\Omega}$$

E: Discuss the fact that X, Y and ϕ are not independent variables. In the guiding center generating function, it is assumed that Y, ϕ play the role of new canonical coordinates. On the basis of which argument can we introduce X?

E: Show that the new coordinates are ϕ, Y, z, with new momenta $P_\phi, P_Y =$

$M\Omega X, P_z$

- At the lowest order, the new Hamiltonian K is

$$K_0 = \frac{P_z^2}{2M} + \Omega P_\phi$$

- For the unperturbed motion, P_ϕ, ϕ represent action-angle coordinates, with P_ϕ the conserved action. Other constants of motion are P_z and P_Y (or X); so the motion is integrable, as expected.

- At the next order, with $Y = y + \rho \sin\phi$ and $\rho = v_\perp/\Omega = \rho(P_\phi) = (2P_\phi/M\Omega)^{1/2}$

$$K_1 = e\Phi_0 \sin\left(k_z z + k_\perp Y - k_\perp \rho \sin\phi - \omega t\right)$$

- The new Hamiltonian does not depend on P_Y, therefore $Y = \text{const.}$ and the $k_\perp Y$ can be eliminated by a shift in z or t.

- With another canonical transformation it is possible to eliminate explicit time dependences, since it appears only in combinations $k_z z - \omega t$:

$$F_2 = (k_z z - \omega t) P_\psi + P_\phi \phi \quad ; \quad P_z = \frac{\partial F_2}{\partial z} = k_z P_\psi \quad ; \quad \psi = \frac{\partial F_2}{\partial P_\psi} = (k_z z - \omega t)$$

$$\bar{H} = K + \frac{\partial F_2}{\partial t} = \frac{k_z^2 P_\psi^2}{2M} + \Omega P_\phi - \omega P_\psi + \epsilon e \Phi_0 \sin\left(\psi - k_\perp \rho \sin\phi\right)$$

- The first order term can be expanded in a series of Bessel functions as

$$\bar{H} = \frac{k_z^2 P_\psi^2}{2M} + \Omega P_\phi - \omega P_\psi + \epsilon e \Phi_0 \sum_m J_m(k_\perp \rho) \sin\left(\psi - m\phi\right)$$

- This shows that the original system has been reduced to a two degrees of freedom system in action angle coordinates, whose unperturbed motion has characteristic frequencies

$$\omega_\phi = \frac{\partial \bar{H}}{\partial P_\phi} = \Omega \;\; ; \;\; \omega_\psi = \frac{\partial \bar{H}}{\partial P_\psi} = k_z^2 \frac{P_\psi}{M} - \omega = k_z v_z - \omega$$

- Resonance between the two degrees of freedom occurs when $\omega_\psi - m\omega_\phi = 0$, i.e.

$$P_\psi = \frac{M}{k_z^2}(\omega + m\Omega) \;\; , \;\; \text{for } k_z \neq 0 \;\; ; \;\; \omega + m\Omega = 0 \;\; , \;\; \text{for } k_z = 0$$

- For $k_z = 0$ the resonance condition is not met and it is possible to apply the classical perturbation theory. Thus (see p.17), with $\rho(P_\phi) = (2P_\phi/M\Omega)^{1/2}$,

$$S_1 = -e\Phi_0 \sum_m \frac{J_m(k_\perp \rho)}{\omega + m\Omega} \cos(\psi - m\phi)$$

$$P_\psi = \bar{P}_\psi + \frac{\partial S_1}{\partial \psi} \quad ; \quad \bar{P}_\psi = P_\psi - e\Phi_0 \sum_m \frac{J_m(k_\perp \rho)}{\omega + m\Omega} \sin(\psi - m\phi) = \text{const.}$$

$$P_\phi = \bar{P}_\phi + \frac{\partial S_1}{\partial \phi} \quad ; \quad \bar{P}_\phi = P_\phi + e\Phi_0 \sum_m \frac{mJ_m(k_\perp \rho)}{\omega + m\Omega} \sin(\psi - m\phi) = \text{const.}$$

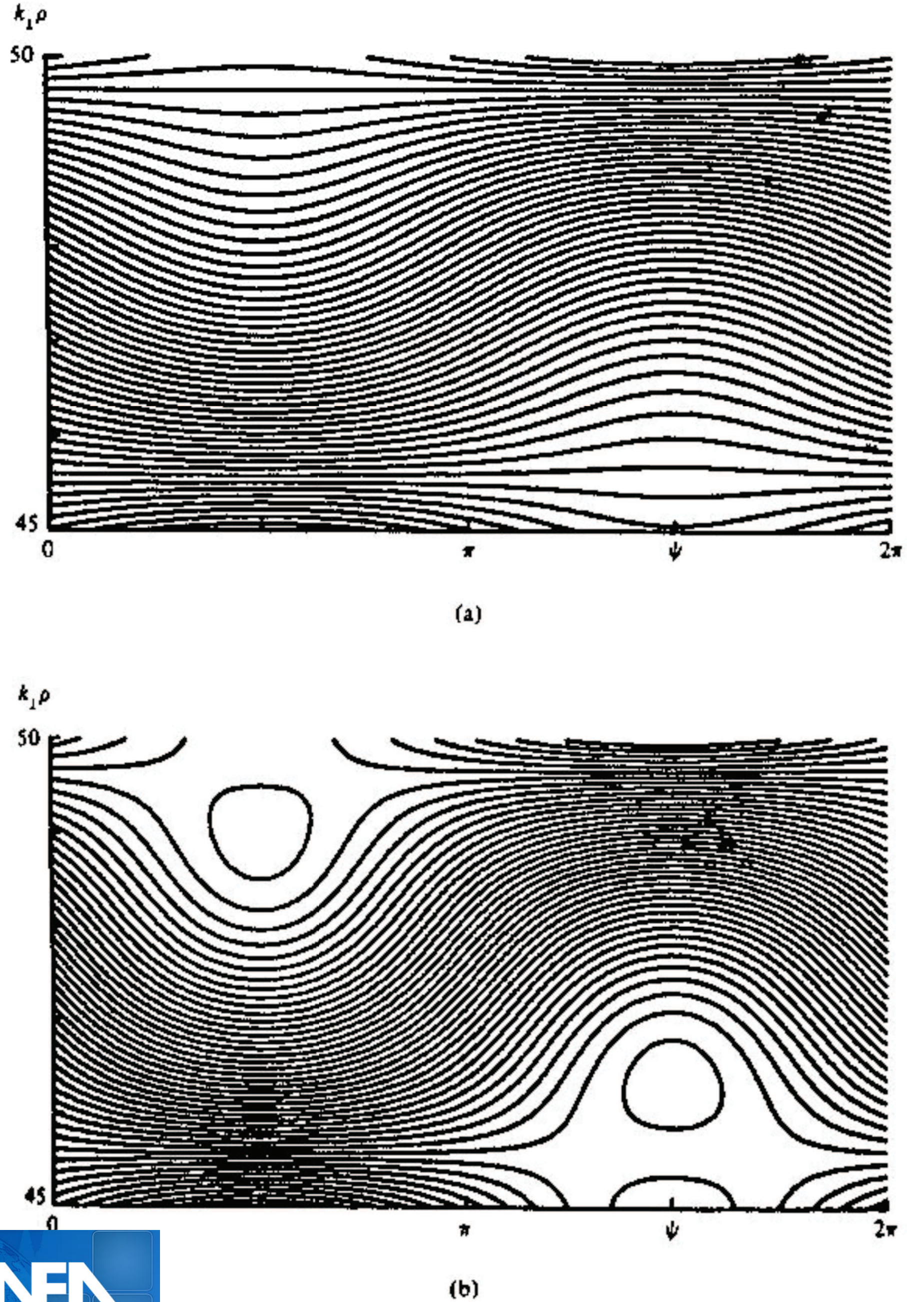

Fig 2.4 p91 Lieberman. Non-resonant invariant curves $k_\perp\rho$ vs. ψ in a surface of section $\phi = \pi$ (Note that this is not the actual surface of section plot P_ϕ vs ψ. Here $\omega/\Omega = 30.11$, while (a) and (b) are, respectively, low and high amplitude cases (*Karney*, 1977).

E: Case (b) shows evidence of trapping. Do you have trapping in case (a)?

- Canonical adiabatic theory is another straightforward application of the perturbation theory above, elucidating its applicability limit.

- An adiabatic invariant, at the lowest order, is connected with the action J associated with the fast degree of freedom. This system can be generally written in the form

$$H = H_0(J, \epsilon \boldsymbol{p}, \epsilon \boldsymbol{q}, \epsilon t) + \epsilon H_1(J, \theta, \epsilon \boldsymbol{p}, \epsilon \boldsymbol{q}, \epsilon t) + \ldots$$

  where ϵ in front of $\boldsymbol{p}, \boldsymbol{q}, t$ indicates the *slow* dependences that are remaining in the Hamiltonian system.

- Following the perturbation theory approach, one looks for a near-identity transformation

$$F_2 = \bar{J}\theta + \bar{\boldsymbol{p}} \cdot \boldsymbol{q} + \epsilon S_1(\bar{J}, \theta, \epsilon \bar{\boldsymbol{p}}, \epsilon \boldsymbol{q}, \epsilon t) + \ldots$$

  such that the new Hamiltonian $\bar{H}$ is independent of $\bar{\theta}$.

- Up to first order, proceeding as usual,

$$J = \bar{J} + \epsilon \frac{\partial S_1}{\partial \bar{\theta}} \quad ; \quad \theta = \bar{\theta} - \epsilon \frac{\partial S_1}{\partial \bar{J}}$$

$$\boldsymbol{p} = \bar{\boldsymbol{p}} + \epsilon \frac{\partial S_1}{\partial \bar{\boldsymbol{q}}} \quad ; \quad \boldsymbol{q} = \bar{\boldsymbol{q}} - \epsilon \frac{\partial S_1}{\partial \bar{\boldsymbol{p}}}$$

$$\bar{H} = H_0(\bar{J}, \epsilon\bar{\boldsymbol{p}}, \epsilon\bar{\boldsymbol{q}}, \epsilon t) + \epsilon \left[\omega(\bar{J}) \frac{\partial S_1}{\partial \bar{\theta}} - \frac{\partial H_0}{\partial \bar{\boldsymbol{q}}} \cdot \frac{\partial S_1}{\partial \bar{\boldsymbol{p}}} + \frac{\partial H_0}{\partial \bar{\boldsymbol{p}}} \cdot \frac{\partial S_1}{\partial \bar{\boldsymbol{q}}} + \frac{\partial S_1}{\partial t} + H_1(\bar{J}, \bar{\theta}, \epsilon\bar{\boldsymbol{p}}, \epsilon\bar{\boldsymbol{q}}, \epsilon t) \right]$$

- Considering that terms containing $\partial S_1/\partial \bar{\boldsymbol{p}}, \partial S_1/\partial \bar{\boldsymbol{q}}, \partial S_1/\partial t$ appear formally at higher order, one readily has

$$\bar{H}(\bar{J}, \epsilon\bar{\boldsymbol{p}}, \epsilon\bar{\boldsymbol{q}}, \epsilon t)) = H_0 + \epsilon \langle H_1 \rangle_{\bar{\theta}} \quad ; \quad \omega \left(\partial S_1/\partial \bar{\theta} \right) = - \{ H_1 \}_{\bar{\theta}}$$

$$\bar{J} = J - \epsilon \left(\partial S_1/\partial \theta \right) \quad ; \quad \bar{J} = J + \epsilon \left(\{ H_1 \}_{\bar{\theta}} / \omega \right)$$

- The formal disappearance of resonances seems to have resolved the issue of non-convergence of the perturbation series, which is *not the case*. It is the adiabatic-invariant $\bar{J}$ formalism that hides the problem.

- Keeping terms that are formally of higher order and assuming the slow dependences are in action-angle form $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}} = \bar{\boldsymbol{J}}_q, \bar{\boldsymbol{\theta}}_q$ $(\partial H_0/\partial \bar{\boldsymbol{\theta}}_q = -\dot{\bar{\boldsymbol{J}}}_q = 0)$, then

$$\omega \frac{\partial S_1}{\partial \bar{\theta}} + \epsilon \boldsymbol{\omega}_q \cdot \frac{\partial S_1}{\partial \bar{\boldsymbol{\theta}}_q} + \epsilon \frac{\partial S_1}{\partial t} = - \{H_1\}_{\bar{\theta}}$$

$$S_1 = i \sum_{k \neq 0, m, l} \frac{H_{1k,l,m}(\bar{J}, \bar{\boldsymbol{J}}_q)}{k\omega + \epsilon \boldsymbol{m} \cdot \boldsymbol{\omega}_q + \epsilon l \Omega} e^{i\left(k\bar{\theta} + \epsilon \boldsymbol{m} \cdot \bar{\boldsymbol{\theta}}_q + \epsilon l \Omega t\right)}$$

- Resonant denominators appear back at higher order resonances, where resonance between fast and slow degrees of freedom is possible.

- This is consequence of the fact that adiabatic series is asymptotic and is applicable on time scales less than the slow time scales.

E: Review the notion of asymptotic series expansion

$$X_n(t,\epsilon) = \sum_{i=0}^{n} \epsilon^i x_i(t) \quad ; \quad \lim_{\epsilon\to 0} \epsilon^{-n} \left[x(t,\epsilon) - X_n(t,\epsilon) \right] = 0$$

E: Show that two different functions may have the same asymptotic series expansions. Hint: take two functions that differ for an exponentially small term.

E: Review the notion of optimal truncation of an asymptotic series expansion. This is connected with the formal divergence of such expansions

$$\lim_{n\to\infty} \left[x(t,\epsilon) - X_n(t,\epsilon) \right] \to \infty$$

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