Lecture 2

Notions of integrable, near integrable and dissipative systems

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Integrable systems

- From Lecture 1, we know that – given a system with N degrees of freedom – the system is integrable (also completely integrable or completely separable) if the Hamilton-Jacobi equation is separable into N independent equations.

- The constants α_i

$$H_i\left(\frac{\partial S_i}{\partial q_i}, q_i\right) = \alpha_i \;\; ; \;\; \bar{H} = \sum_i \alpha_i = H_0$$

are called isolating integrals or global invariants of the motion, for they isolate each degree of freedom by the property

$$\frac{\partial H}{\partial p_i} = f(q_i)$$

in some canonical coordinate system.

- A Hamiltonian with N degrees of freedom is integrable if and only if N independent isolating integrals exist.
- A sufficient condition for integrability is that the N independent integrals are in involution, i.e. their Poisson brackets with each other vanish

$$[\alpha_i, \alpha_j] = 0$$

- This condition ensures that $\boldsymbol{\alpha}$ is a complete set of new momenta in some transformed coordinate system. Any complete set of N functions of $\boldsymbol{\alpha}$, such that the action variables $\boldsymbol{J}$, are a set of isolating integrals, for their Poisson brackets automatically vanish.

Systems with one degree of freedom

- For an autonomous system with one degree of freedom, the Hamiltonian is conserved and is the energy of the system (see Lecture 1)

$$H(p,q) = E$$

- All such systems are integrable. In fact

$$p = p(q,E) \;\; ; \;\; t = \int_{q_0}^{q} \frac{dq}{\dot{q}} = \int_{q_0}^{q} \frac{dq}{\partial H/\partial p}$$

- As notable example of systems with one degree of freedom, one can take the pendulum Hamiltonian.

The pendulum Hamiltonian

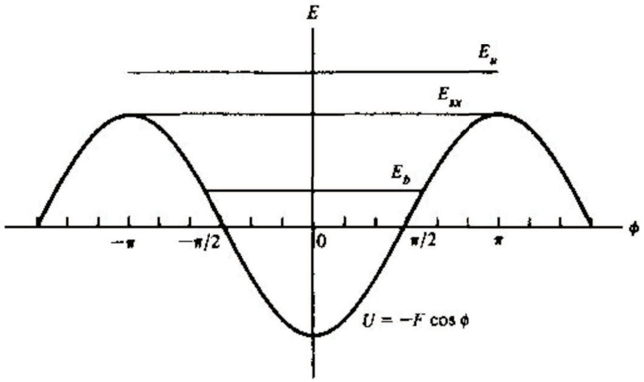
- Given ϕ, the angle from the vertical, and p the angular momentum conjugate to ϕ, the equations of motion of a pendulum of mass m and length h, with $F = mgh$ and $G = (mh^2)^{-1}$, are

$$\dot{p} = -F \sin\phi \quad ; \quad \dot{\phi} = Gp$$

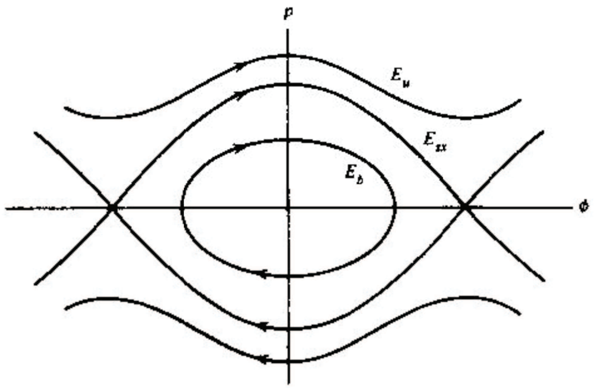
- The Hamiltonian is given by

$$H = \frac{1}{2}Gp^2 - F\cos\phi = E$$

- It is useful to analyze the system behavior, comparing energy diagram and phase space diagram



(a)



(b)

Correspondence between energy diagram and phase space diagram for the pendulum Hamiltonian. (*Lichtenberg and Lieberman*, 2010)

For $E > E_{sx} = F$ the motion is unbounded (rotation). For $-F < E < E_{sx} = F$, the motion is bounded (libration). These motion are divided by a separatrix, where the oscillation period is infinite.

- The motion has a stable singular point at $p = 0, \phi = 0$ (trajectories near the singular point remain in its neighborhood) and an unstable singular point at $p = 0, \phi = \pm\pi$ (trajectories near the singular point diverge from it).

- The period is a function of the energy and can be computed in terms of elliptic integrals as

$$T = \oint \frac{d\phi}{\dot{\phi}} = \frac{1}{(2G)^{1/2}} \oint \frac{d\phi}{(E + F\cos\phi)^{1/2}}$$

- The period at the separatrix becomes infinite since both velocity and restoring force vanish at $\phi = \pm\pi$.

- Transformation to action-angle variables (J, θ) is obtained as

$$J(E) = \frac{\alpha}{\pi} \int_0^{\phi_{\max}} \left[\frac{2}{G} (E + F\cos\phi') \right]^{1/2} d\phi'$$

$$\theta(\phi, E) = \left(G \frac{dJ}{dE} \right)^{-1} \int_0^{\phi} \frac{d\phi'}{[(2/G)(E + F\cos\phi')]^{1/2}}$$

- Here, $\alpha = 1, \phi_{\max} = \pi$ for rotations and $\alpha = 2, \cos\phi_{\max} = -E/F$ for librations.

E: Fill in the missing details for deriving action-angle variables. Hint: derive the generating function $F_2 = F_2(\phi, J)$ and note that $J = J(E)$ with the new Hamiltonian given by $\bar{H} = E$.

- Introduce the variables $\kappa^2 = (1 + E/F)/2$ and $\sin\eta = \kappa^{-1}\sin(\phi/2)$ (for $\kappa < 1$), so that $\kappa < 1$ for librations and $\kappa > 1$ for rotations. Then, action-angle coordinates become (*Smith*, 1977, *Rechester*, 1979)

$$J = \frac{8}{\pi}\left(\frac{F}{G}\right)^{1/2} \begin{cases} \boldsymbol{E}(\kappa) - (1-\kappa^2)\boldsymbol{K}(\kappa) & ; \quad \kappa < 1 \\ (\kappa/2)\boldsymbol{E}(\kappa^{-1}) & ; \quad \kappa > 1 \end{cases}$$

with $\boldsymbol{K}(\kappa)$ and $\boldsymbol{E}(\kappa)$ the complete elliptic integrals of first and second kind

$$\boldsymbol{K}(\kappa) = \int_0^{\pi/2} \frac{d\eta}{\sqrt{1-\kappa^2\sin^2\eta}} \quad ; \quad \boldsymbol{E}(\kappa) = \int_0^{\pi/2} \sqrt{1-\kappa^2\sin^2\eta}\, d\eta$$

- The frequency of the motion can be obtained from $\omega^{-1} = (dJ/dE) = (4F\kappa)^{-1}(dJ/d\kappa)$: i.e., with $\omega_0 = (FG)^{1/2}$ the frequency of deeply trapped librations and noting $\kappa(d\boldsymbol{K}/d\kappa) = \boldsymbol{E}/(1-\kappa^2) - \boldsymbol{K}$, $\kappa(d\boldsymbol{E}/d\kappa) = \boldsymbol{E} - \boldsymbol{K}$

$$\frac{\omega}{\omega_0} = \frac{\pi}{2}\begin{cases} 1/\boldsymbol{K}(\kappa) \ ; & \kappa < 1 \\ 2\kappa/\boldsymbol{K}(\kappa^{-1}) \ ; & \kappa > 1 \end{cases}$$

- For $\kappa \to 0$, $\omega/\omega_0 \to 1$. Meanwhile, using the limiting expression of $\boldsymbol{K}(\kappa)$ as $\kappa \to 1$, it is clear that

$$\lim_{k\to 1}\frac{\omega}{\omega_0} = \begin{cases} (\pi/2)/\ln\left[\dfrac{4}{(1-\kappa^2)^{1/2}}\right] \ ; & \kappa < 1 \\ \pi/\ln\left[\dfrac{4}{(\kappa^2-1)^{1/2}}\right] \ ; & \kappa > 1 \end{cases}$$

E: Can you explain why the limiting expression for $\kappa > 1$ is twice that for $\kappa < 1$?

- From previous page expressions, for $\sin\eta = \kappa^{-1}\sin(\phi/2)$ (for $\kappa < 1$) and with $F(\phi,\kappa)$ the incomplete elliptic integral of the first kind, one can also obtain the angle conjugate to J as

$$\theta = \frac{\pi}{2}\begin{cases} F(\eta,\kappa)/\boldsymbol{K}(\kappa) & ; \quad \kappa < 1 \\ 2F(\phi/2,\kappa^{-1})/\boldsymbol{K}(\kappa^{-1}) & ; \quad \kappa > 1 \end{cases}$$

$$F(\phi,\kappa) = \int_0^{\phi} \frac{d\eta}{\sqrt{1-\kappa^2\sin^2\eta}}$$

E: Fill in the missing details and derive expressions above step by step.

- For the separatrix

$$p_{sx} = \pm(2\omega_0/G)\cos(\phi_{sx}/2) \quad ; \quad \dot{\phi}_{sx} = Gp_{sx}$$

- Integrating in time, we have

$$\omega_0 t = \int_0^{\phi_{sx}/2} \frac{d\xi}{\cos\xi} = \ln\tan\left(\frac{\phi_{sx}}{4}+\frac{\pi}{4}\right) \quad ; \quad \phi_{sx} = 4\tan^{-1}\left[\exp(\omega_0 t)\right] - \pi$$

- The importance of the former study of the pendulum Hamiltonian comes from its role in the analysis of near-integrable systems with many degrees of freedom where a resonance between degrees of freedom exists.

- Since this Hamiltonian always emerges from Fourier analysis of the perturbation, in combination of secular perturbation theory and the method of averaging (see Lecture 4) it has been said to provide the "universal description of a nonlinear resonance" (*Chirikov*, 1979) and is also called the "standard Hamiltonian" (*Lichtenberg*, 2010).

Systems with more than one degree of freedom

- In the case of systems with more than one degree of freedom, one can generally write

$$dt = \frac{dq_1}{\dot{q}_1} = \frac{dq_1}{\partial H/\partial p_1} = ... = \frac{dq_N}{\dot{q}_N} = \frac{dq_N}{\partial H/\partial p_N}$$

- Unlike the case with one degree of freedom, the motion cannot be reduced to quadratures.
- The solution can partly reduced to quadratures if $\partial H/\partial p_1 = f(q_1)$ (see Lecture 1): in that case the first equation can be solved (by quadratures) as would be the case of other degrees of freedom satisfying the same property.
- In general, one has to solve the whole set of coupled differential equations simultaneously to obtain a complete solution.

- Reducing the number of simultaneous equations to be solved coincides with the identification of an isolating integral, i.e. of a constant of motion in addition to the Hamiltonian, which makes it possible to write $\partial H/\partial p_i = f(q_i)$ in some transformed coordinate system.

- Transformation to action-angle coordinates satisfies this requirement, since it ensure the even more restrictive condition $\partial H/\partial p_i =$ const.

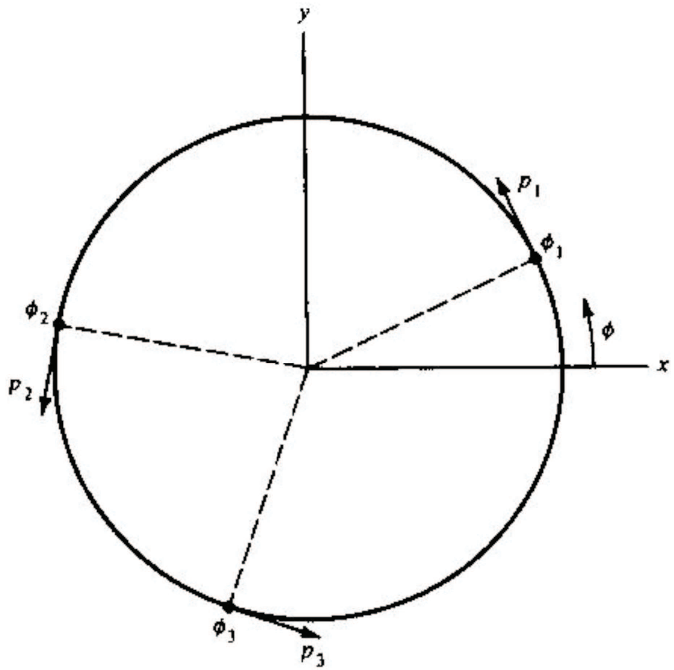
- Generally, the existence of isolating integrals can be associated with symmetries of the system. The symmetries are sometimes obvious and the coordinate transformation required to reduce the solution of the motion to quadratures is straightforward. This is the case of the central force field.

- There is no procedure (presently known) to determine all the isolating integrals of a Hamiltonian system or even to find their number. However it is possible to make numerical experiments that suggest the presence of a hidden isolating integral and may lead to its uncovering.

The Toda lattice

- The three-particle Toda Lattice is described by the following Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \exp\left[-(\phi_1 - \phi_3)\right] + \exp\left[-(\phi_2 - \phi_1)\right] + \exp\left[-(\phi_3 - \phi_2)\right] - 3$$



The three-particle Toda Lattice. (*Lichtenberg and Lieberman*, 2010)

The three particles move on a ring with exponentially decreasing forces between them.

- The obvious symmetry of the Hamiltonian is its invariance under rigid rotation $\phi_i \to \phi_i + \phi_0$. So the obvious isolating integral is the total momentum

$$P_3 = p_1 + p_2 + p_3$$

- The canonical transformation is easily obtained with the generating function

$$F_2 = P_1\phi_1 + P_2\phi_2 + (P_3 - P_2 - P_1)\phi_3$$

$$\Phi_1 = \frac{\partial F_2}{\partial P_1} = \phi_1 - \phi_3 \;\; ; \;\; \Phi_2 = \frac{\partial F_2}{\partial P_2} = \phi_2 - \phi_3 \;\; ; \;\; \Phi_3 = \frac{\partial F_2}{\partial P_3} = \phi_3$$

$$K = \frac{1}{2}\left[P_1^2 + P_2^2 + (P_3 - P_2 - P_1)^2\right] + \exp(-\Phi_1) + \exp\left[-(\Phi_2 - \Phi_1)\right] + \exp(\Phi_2) - 3$$

- Since K is independent of Φ_3, we have $P_3 = \text{const}$ that can be assumed to be $P_3 = 0$.

- Given this system, there seems to be no other obvious isolating integral. Which is instead the case, as it has been shown numerically, looking at the surface of section plots of the corresponding motion.

- The Toda lattice has interesting limiting cases. To see this one has to make two subsequent further coordinate transformations of the Hamiltonian K with two degrees of freedom. The first one is canonical and has generating function

$$F_2' = (4\sqrt{3})^{-1}\left[(\pi_x - \sqrt{3}\pi_y)\Phi_1 + (\pi_x + \sqrt{3}\pi_y)\Phi_2\right]$$

$$q_x = \frac{\partial F_2'}{\partial \pi_x} = (4\sqrt{3})^{-1}(\Phi_1 + \Phi_2) \;\; ; \;\; q_y = \frac{\partial F_2'}{\partial \pi_y} = (\Phi_2 - \Phi_1)/4$$

The second one is non-canonical (see next section) and is given by

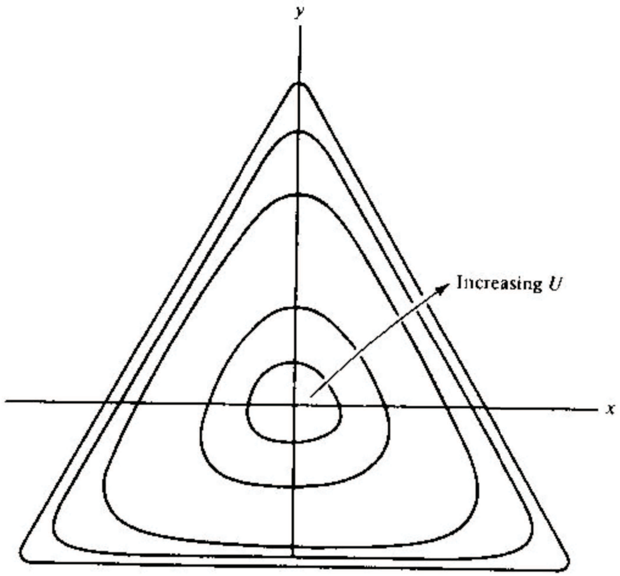
$$\pi_x = 8\sqrt{3}p_x \;\; ; \;\; q_x = x \;\; ; \;\; \pi_y = 8\sqrt{3}p_y \;\; ; \;\; q_y = y \;\; ; \;\; \bar{H} = K'/\sqrt{3}$$

E: Work out the detailed derivations connected with the coordinate transformations above. Why is the second transformation non-canonical? (see next section).

E: Derive the final form of the Toda Hamiltonian as follows.

- The final form of the Toda Hamiltonian is

$$\bar{H} = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{24}\left[\exp\left(2y + 2\sqrt{3}x\right) + \exp\left(2y - 2\sqrt{3}x\right) + \exp\left(-4y\right)\right] - \frac{1}{8}$$

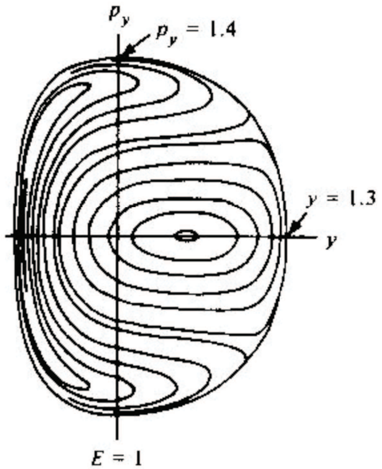


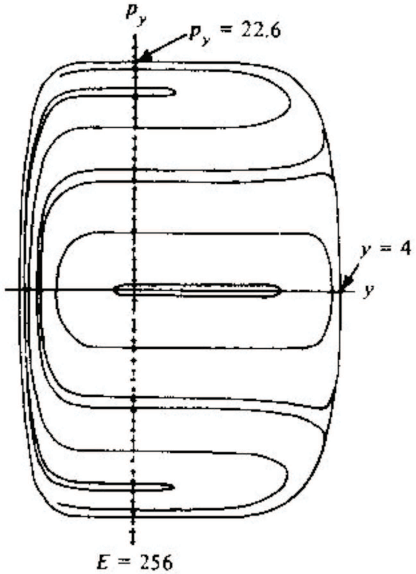
The equipotential lines for the Toda Hamiltonian. (*Lichtenberg and Lieberman*, 2010)

- When expanded for small x, y up to cubic terms, the Toda Hamiltonian reduces to the Hénon and Heiles Hamiltonian (*Hénon and Heiles*, 1964)

$$\bar{H}' = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

- This Hamiltonian is known to be non-integrable, since it was found numerically to yield a transition from ordered to stochastic motion for increasing $\bar{H}' = E$.





The Toda Hamiltonian, instead, was numerically found not to produce the same transition for increasing $\bar{H}' = E$ (*Ford*, 1973).

- Thanks to this hint, Hénon found that the explicit analytic form of the "hidden" isolating integral is (*Hénon*, 1974)

$$I = 8p_x\left(p_x^2 - 3p_y^2\right) + \left(p_x + \sqrt{3}p_y\right)\exp\left(2y - 2\sqrt{3}x\right) - 2p_x\exp\left(-4y\right)$$
$$+ \left(p_x - \sqrt{3}p_y\right)\exp\left(2y + 2\sqrt{3}x\right) = \text{const.}$$

- This procedure is an example of how numerical methods can be use to uncover and eventually find "hidden" isolating integrals.
- In general, however, there are no general methods to test the integrability of a given Hamiltonian.

Non-canonical variables

- Using the definition of Poisson brackets, equations of motion for $z = (q_1, ..., q_{N/2}, p_1, ..., p_{N/2})$ in the Hamiltonian form can be written as (see Lecture 1)

$$\dot{z} = [z, H]$$

- Defining $\mathbf{0}$ and $\mathbf{1}$ as null and unit $(N/2) \times (N/2)$ matrices and introducing the canonical Poisson tensor $\boldsymbol{J}$

$$\boldsymbol{J} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}$$

Poisson brackets are written as

$$[u, v] = J_{ij} \frac{\partial u}{\partial z_i} \frac{\partial v}{\partial z_j}$$

- Hamilton's equations of motion can be written as

$$\dot{z}_i = J_{ij} \frac{\partial H}{\partial z_j}$$

- Given the coordinate transformation $(z, H) \to (z', H')$ one has

$$\dot{z}_i = \frac{\partial z_i}{\partial z'_j} \dot{z}'_j = J_{ij} \frac{\partial H'}{\partial z'_k} \frac{\partial z'_k}{\partial z_j}$$

- Introducing the local transformation matrix $M_{ij} = \partial z'_i / \partial z_j$ from z to z' coordinates, with non-vanishing Jacobian, then

$$\boldsymbol{M}^{-1} \dot{\boldsymbol{z}}' = \boldsymbol{J} \boldsymbol{M}^T \frac{\partial H'}{\partial \boldsymbol{z}'}$$

- The Hamilton's equations of motion in the new coordinates are

$$\dot{z}'_i = J'_{ij}\frac{\partial H'}{\partial z'_j} \quad ; \quad J'_{ij} = M_{ik}M_{jl}J_{kl} \quad ; \quad \boldsymbol{J}' = \boldsymbol{M}\boldsymbol{J}\boldsymbol{M}^T$$

- One can introduce the new brackets, with the same formal properties of the Poisson brackets

$$\{u, v\} = J'_{ij}\frac{\partial u}{\partial z'_i}\frac{\partial v}{\partial z'_j}$$

- Only if $\boldsymbol{J}' = \boldsymbol{J}$, the new brackets coincide with the Poisson brackets and the transformation is canonical.

E: Show that the last transformation used for the Toda Hamiltonian is non-canonical.

- A typical example, where use of non-canonical variables gives advantages, is gyrokinetic theory. The starting point of guiding-center perturbation theory can be obtained for the Hamiltonian describing a particle moving in a static magnetic field

$$H = \frac{1}{2m}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{q})\right|^2$$

- The non-canonical transformation in this case is (*Littlejohn*, 1979, 1983)

$$\boldsymbol{r}' = \boldsymbol{q} \;\; ; \;\; \boldsymbol{v}' = \frac{1}{m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(\boldsymbol{q})\right) \;\; ; \;\; H' = \frac{m}{2}v'^2$$

$$\{u, v\} = \frac{1}{m}\left(\frac{\partial u}{\partial \boldsymbol{r}'}\cdot\frac{\partial v}{\partial \boldsymbol{v}'} - \frac{\partial u}{\partial \boldsymbol{v}'}\cdot\frac{\partial v}{\partial \boldsymbol{r}'}\right) + \frac{e}{m^2 c}\boldsymbol{B}\cdot\left(\frac{\partial u}{\partial \boldsymbol{v}'}\times\frac{\partial v}{\partial \boldsymbol{v}'}\right)$$

- In this case, introducing the $(N/2) \times (N/2)$ matrix $\Omega_{ij} = (e/mc)\epsilon_{ijk} B_k$, the $\boldsymbol{J}'$ tensor can be written as

$$\boldsymbol{J}' = \frac{1}{m} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \boldsymbol{\Omega} \end{pmatrix}$$

E: Derive these results step by step.

- When $\boldsymbol{J}'$ is non-singular and corresponds to a non-canonical transform, it is always possible (Darboux's theorem) to construct new canonical coordinates from any non-singular bracket (*Littlejohn*, 1979).

- When $\boldsymbol{J}'$ is singular, other invariant quantities, called *Casimirs*, exist due to the degeneracy of the brackets.

- If $K < N$ is the rank of the $N \times N$ tensor $\boldsymbol{J}'$, $\boldsymbol{J}'$ has a null space of dimension $M = N - K$, where one can construct M linearly independent vectors $\boldsymbol{V}$ satisfying $\boldsymbol{J}'\boldsymbol{V} = \boldsymbol{0}$.

- It is also a general result that M functions $C_i(\boldsymbol{z}')$ exist, which are degenerate with the brackets for any $u(\boldsymbol{v}')$

$$\{u, C_i\} = 0$$

These are the *Casimirs*.

Near-integrable systems

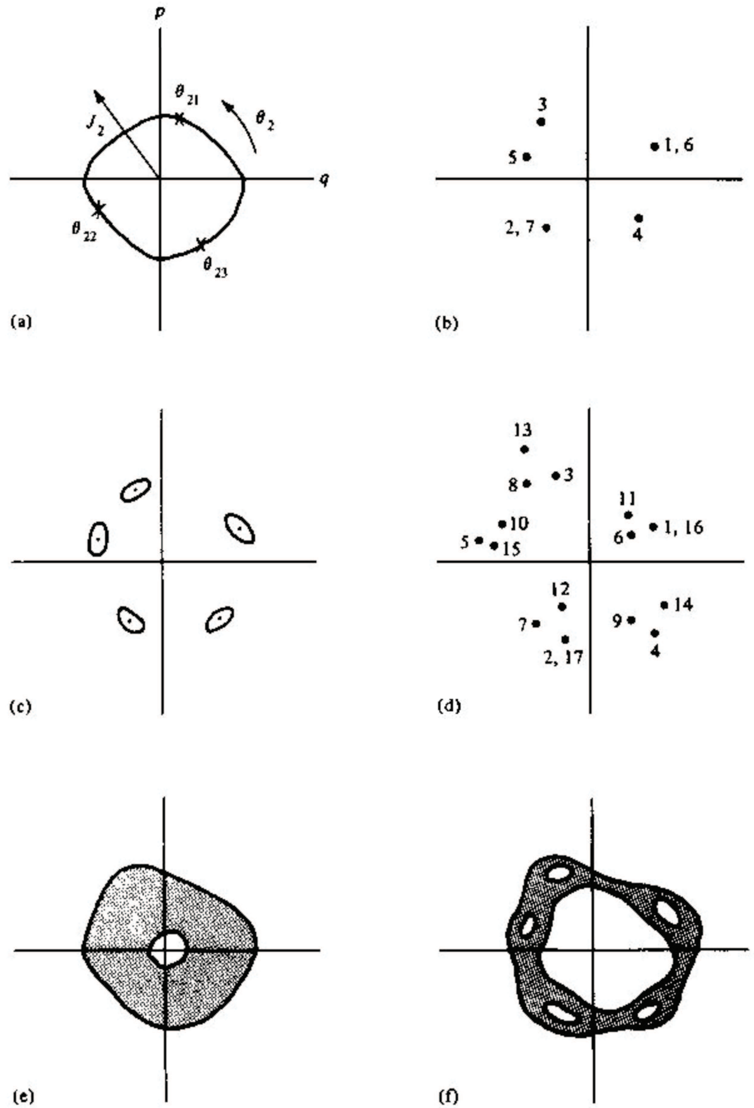
- From the definition of integrability given in Lecture 1 and Lecture 2 so far, we can define a system as near-integrable if it can be described as perturbation of an integrable system.
- The peculiar nature of near-integrable systems is to be characterized by both regular trajectories and regions of stochasticity, which are all consequence of the deterministic solutions of Hamilton's equations.
- For systems with two degrees of freedom, stochastic regions are separated from regular trajectories by regular curves.
- In systems with more than two degrees of freedom, regular trajectories no longer separate stochastic regions and the systems are characterized by the phenomenon of *Arnold diffusion*.

Systems with two degrees of freedom

- The Hamiltonian of a near integrable periodic system with two degrees of freedom, in the action-angle variables of the unperturbed motion $\boldsymbol{J}, \boldsymbol{\theta}$, can be written as

$$H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \epsilon H_1(J_1, J_2, \theta_1, \theta_2)$$

- The trajectories lie in the 3-dimensional surface $H = \text{const.}$ of the four dimensional phase space.
- Regular trajectories depend discontinuously on initial conditions: their presence does not imply the existence of an isolating integral (global invariant) of the motion (or a corresponding symmetry).
- Still, where regular trajectories exist, they represent exact invariants of the motion.



Surface of section plots showing type of trajectories:
(a) regular trajectory of a conditionally periodic system
(b) fixed (or periodic) points of a primary resonance for a periodic system
(c) trajectories form primary islands near fixed points
(d) secondary resonance due to the coupling of periodic motion around a primary island and the unperturbed motion
(e) annular layer of stochasticity between two invariant curves
(f) stochastic layer bounded by primary and secondary invariant curves
(*Lieberman and Lichtenberg*, 2010).

- Regular trajectories are either
  - conditionally periodic: they densely cover a toroidal surface of constant action, where the angle coordinates run around the two directions of the surface with frequencies, whose ratio is not a rational number
  - closed periodic curves, winding around the torus an integral number of turns. This is the case of a resonance, with $k\omega_1(\boldsymbol{J}) + l\omega_2(\boldsymbol{J}) = 0$
- Primary resonances give rise to primary islands, which give rise to secondary resonances and their islands and so on.
- Regions of stochastic trajectories fill a finite portion of the energy surface in phase space. Periodic trajectories exist in these regions, but trajectories near them do not move on stable islands near them (see Lecture 4).

- Stochastic motion always occurs near separatrices between invariant curves and their islands. This occurs because:
  - the island oscillation frequency ω goes to zero with respect to the frequency of the unperturbed motion ω_0 (see p.9)
  - the separation between actions of neighboring resonances (k and $k \pm 1$), $k\omega - \omega_0$, tends to zero as the separatrix is approached
  - the stochasticity region formed near a separatrix is called the *resonance layer*
- For small ϵ and for systems with two degrees of freedom, resonance layers are thin and separated by invariant curves.
- For increasing ϵ, invariant curves separating different island chains are strongly perturbed and eventually destroyed.
- Merging of primary resonance layers leads to the appearance of global or strong stochasticity (see Lecture 5 and Lecture 6).

Systems with more than two degrees of freedom

- In systems with two degrees of freedom, resonance layers are divided and isolated from each other by two dimensional KAM surfaces.

E: Show that two dimensional KAM surfaces divide the three dimensional space $H(\boldsymbol{J}, \boldsymbol{\theta}) = E$ into a set of closed volumes.

- For systems with three degrees of freedom or more, the three dimensional (or more) KAM surfaces cannot divide the five-dimensional (or more) $H(\boldsymbol{J}, \boldsymbol{\theta}) = E$ into closed volumes.
- For $N > 2$ degrees of freedom, the N-dimensional KAM surfaces do not divide the $2N - 1$-dimensional constant-energy volume into distinct regions and all stochastic layers of the energy surface in the phase space are connected into a single complex network, the *Arnold web*.
- The Arnold web permeates the entire phase space intersecting or lying arbitrarily close to every point.

- For initial conditions within the web, the subsequent motion will eventually intersect every finite region of the constant-energy volume even for $\epsilon \to 0$.
- The merging of stochastic trajectories into a single web was demonstrated by Arnold for a specific nonlinear Hamiltonian (*Arnold*, 1964). A general proof for a general Hamiltonian does not exist yet, but many numerical examples are available.
- The diffusion rate along resonant layers has been given by Chirikov (*Chirikov*, 1979) for the case of three resonances.
- For the coupling among many resonances, a rigorous upper bound of the diffusion rate has been obtained by Nekhoroshev (*Nekhoroshev*, 1977), which however generally overestimates the actual rate by orders of magnitude.

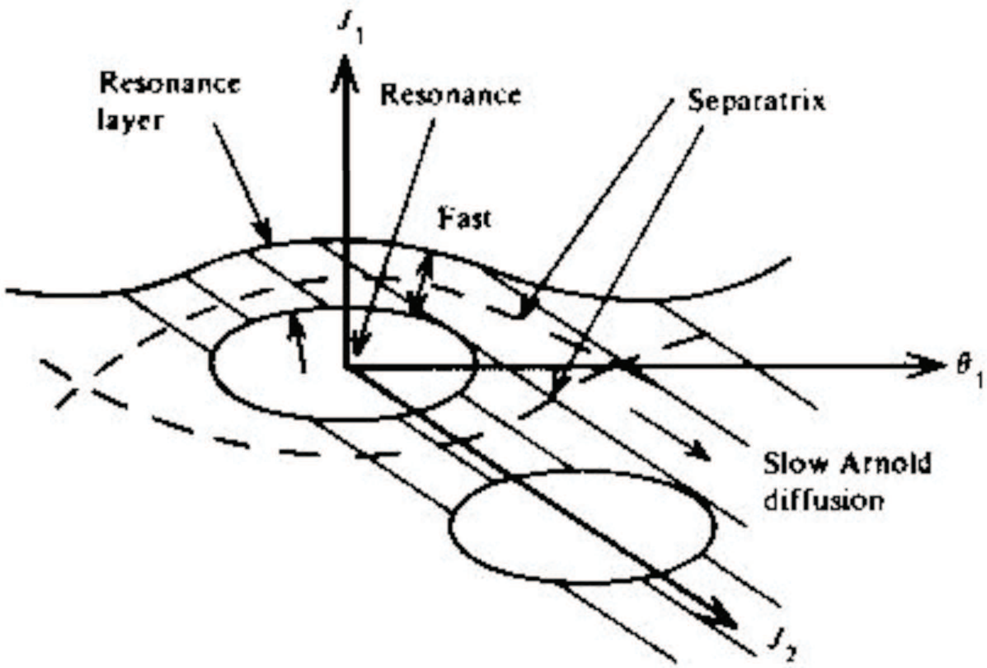


Illustration of processes of fast diffusion across a resonance layer and slow diffusion along the resonance layer (*Lichtenberg and Lieberman*, 2010).

Dissipative systems

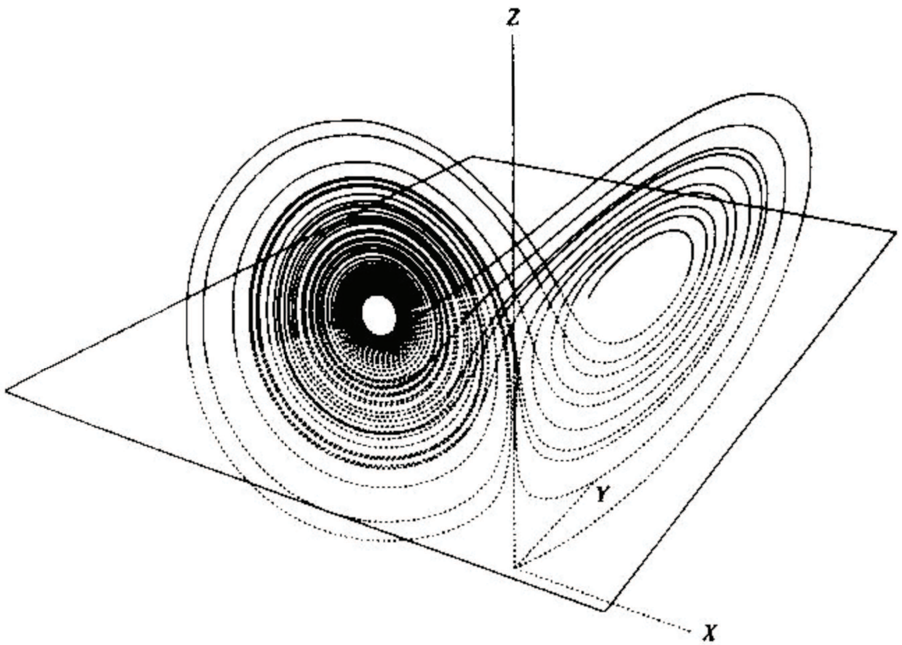
- In contrast to Hamiltonian systems, characterized by the conservation of the phase space volume as a constraint of the motion, dissipative systems are characterized by continued contraction of the phase space volume with increasing time.
- The contraction of the motion onto a surface of lower dimensionality than the original phase space cannot be described by canonical equations, but is still generally described in the form of a set of first order differential equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{V}(\boldsymbol{x})$$

- The trajectory of $\boldsymbol{x}$ describes a N-dimensional flow.

E: Discuss the difference between this set of first order differential equations and the equations of motion in Hamiltonian form.

- For regular motion, the *attractor* of the flow represents a simple motion such as a fixed point (sink) or a singly periodic orbit (limit cycles). For flows in two dimensions these are the only possibilities.
- For three dimensional flows, doubly periodic orbits are also possible.
- For three or more dimensions, attractors exist with very complicated geometric structures, which can be characterized by fractional dimension in the case of *strange attractors*.
- The motion on strange attractors is chaotic, i.e. is a random motion in these dissipative system. Note the difference with the notion of stochastic motion, used for Hamiltonian systems. Often the two words are used as synonymous.
- Dissipative systems are not the subject of these lecture notes.



A chaotic trajectory for the Lorenz attractor at $r = 28$; the horizontal plane is at $Z = 27$ (*Lanford*, 1977).

The Lorenz system is the first example where a strange attractor was studied (*Lorenz*, 1963).

$$
\begin{aligned}
\dot{X} &= -\sigma X + \sigma Y \\
\dot{Y} &= -XZ + rX - Y \\
\dot{Z} &= XY - bZ
\end{aligned}
$$

References and reading material

A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, Springer-Verlag (1983).

A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Second Edition, Springer-Verlag (2010).

V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag (1989).

G. R. Smith, *Stochastic Acceleration by a single wave in a magnetized plasma*, Ph. D. Thesis UC Berkeley, CA (1977) (Lawrence Berkeley Laboratory Report, LBL-6824).

A. B. Rechester and T. H. Stix, *Phys. Rev. A* **19**, 1656 (1979).

B. V. Chirikov, *Phys. Rep.* **52**, 265 (1979).

M. Toda, *Progr. Theoret. Phys. Suppl.* **45**, 174 (1970).

M. Hénon and C. Heiles, *Astron. J.* **69**, 73 (1964).

J. Ford, S. D. Stoddard and J. S. Turner, *Progr. Theoret. Phys.* **50**, 1547 (1973).

M. Hénon, *Phys. Rev. B* **9**, 1925 (1974).

R. Littlejohn, *J. Math. Phys.* **20**, 2445 (1979).

R. G. Littlejohn, *J. Plasma Phys.* **29**, 111 (1983).

E. N. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963).

O. Lanford, in *Turbulence Seminar*, P. Benard and T. Ratiu (eds.), Springer Lecture Notes in Mathematics 615, p.114, Springer-Verlag (1977).

V. I. Arnold, *Russian Math. Surveys* **18**, 85 (1964).

N. N. Nekhoroshev, *Usp. Mat. Nauk. USSR* **32**, 6 (1977).