Lecture 1

Overview of basic concepts in Hamiltonian mechanics

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Advanced Plasma Physics III: Nonlinear Plasma Physics (2)
– Part II. Non-linear charged particle dynamics –
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Lecture Plan and Teaching Material

Abstract: This course on *Advanced Plasma Physics III: Nonlinear Plasma Physics (2)* is devoted to *Non-linear charged particle dynamics* and will be lectured by Prof. Fulvio Zonca in Spring 2025. Prerequisite of the course is a preliminary knowledge of general plasma physics and classical mechanics. Part II of this course is meant to serve complementing the first part of the course, given by Prof. Yong Xiao, and to introduce fundamental concepts in non-linear charged particle dynamics.

Main areas that will be explored are:

i) Overview of basic concepts in Hamiltonian mechanics (Lecture 1)
ii) Notions of integrable, near integrable and dissipative systems (Lecture 2)
iii) Classical perturbation theory (Lecture 3)
iv) Secular perturbation theory (Lecture 4)
v) Mappings and linear stability (Lecture 5)
vi) Transition to global stochasticity (Lecture 6)

Sources: Large portions of these lecture notes are based on the book by A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Second Edition, Springer (2010).

Lecture Notes: Available in electronic form. At the end of each lecture and during the Q&A time, a list of specific reading material is given explicitly.

Exercises: In the lecture notes, Exercises (E) of various difficulty levels are suggested. These are meant to be important part of the lectures themselves.

Time and Location: Lectures (2:15pm-3:55pm) will be held at IFTS on May 19-21-26-28 and June 2-4, with corresponding Q&A Sessions (6pm-7pm) of the same days.

Lagrangian and Hamiltonian formulation of classical mechanics

- Equations of motion must be equivalent in all coordinate systems and can be obtained from one another by coordinate transformations.
- Introduce vector position and velocity $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ and the Lagrangian function

$$L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) = T(\dot{\boldsymbol{q}}, \boldsymbol{q}) - U(\boldsymbol{q}, t)$$

where $T(\dot{\boldsymbol{q}}, \boldsymbol{q})$ is the kinetic energy and $U(\boldsymbol{q}, t)$ is the potential energy.

- Equations of motion in the Lagrangian form are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

E: Give a physical reason why $T(\dot{\boldsymbol{q}}, \boldsymbol{q})$ does not depend explicitly on time. Can you give an example in which the potential energy can be function of $\dot{\boldsymbol{q}}$, i.e. it can be expressed in the form $U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$?

E: Show that the Euler-Lagrange form of the equations of motions coincide with Newton's law. What is the general form of the force field?

E: Show that the Euler-Lagrange form of the equations of motions can be derived from the Least Action Principle, i.e. the variational form $\delta \int L dt = 0$.

- The Hamiltonian function is defined by

$$H(\boldsymbol{p}, \boldsymbol{q}, t) \equiv \sum_i \dot{q}_i p_i - L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$$

where $\dot{\boldsymbol{q}}$ is considered as a function of $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{p}, \boldsymbol{q})$.

- By definition one has

$$
\begin{aligned}
dH &= \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \\
&= \sum_i \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\
&= \sum_i \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dq_i - \frac{\partial L}{\partial t} dt
\end{aligned}
$$

- This is consistent if and only if equations of motion in the Hamiltonian form are satisfied

$$
\dot{q}_i = \frac{\partial H}{\partial p_i} \ ; \ \dot{p}_i = - \frac{\partial H}{\partial q_i} \quad \left(p_i = \frac{\partial L}{\partial \dot{q}_i} \right) \ ; \ \frac{\partial L}{\partial t} = - \frac{\partial H}{\partial t} = - \frac{dH}{dt}
$$

- Any set of variables $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying equations of motion in the Hamiltonian form are called canonical and p_i and q_i are called canonical conjugate variables.

E: Show that, given the Hamiltonian form of equations of motion, $H = T + U$. What properties of T do you need to use? Show that, if H does not depend on time explicitly, then H is conserved and the energy is conserved.

E: Show that, given the Hamiltonian H is the Legendre transform of the Lagrangian L. Note that the construction of the Hamiltonian is always possible because T is a quadratic form of $\dot{q}_i$, i.e. $T = (1/2) \sum_{i,j} a_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$.

E: Show that, for particles with charge e and mass m, the canonical momentum in an e.m. field is $\boldsymbol{p} = m\boldsymbol{v} + (e/c)\boldsymbol{A}$. Hint: recall that a general force can be given by the expression $\boldsymbol{F} = -\partial_{\boldsymbol{q}} U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) + (d/dt)\partial_{\dot{\boldsymbol{q}}} U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$

Poisson brackets

- An important dynamical quantity is the Poisson bracket for a pair of functions u, v of the generalized coordinates $\boldsymbol{p}$ and $\boldsymbol{q}$, defined by

$$[u, v] = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i} \frac{\partial u}{\partial p_i} \right)$$

- A more general definition of the Poisson brackets is by their properties

$$[u, v] = -[v, u]$$

$$[[u, v], w] + [[w, u], v] + [[v, w], u] = 0$$

$$[u + av, w] = [u, w] + a[v, w]$$

$$[uv, w] = u[v, w] + [u, w] v$$

E: Show the last statement.

- By definition of the Poisson brackets, introducing $\boldsymbol{z} = (q_1, ..., q_N, p_1, ..., p_N)$, equations of motion in the Hamiltonian form can be written as

$$\dot{\boldsymbol{z}} = [\boldsymbol{z}, H]$$

- More generally, given a function $f(\boldsymbol{p}, \boldsymbol{q}, t)$

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}$$

- This shows that if f does not depend explicitly on time and $[f, H] = 0$ (f commutes with H), then f is a constant of motion. A particular case is $f = H$, showing that if H does not depend explicitly on time (autonomous system), H is constant and is the energy of the system.

- This also shows that if f is one of the momenta p_i and H does not depend explicitly on the conjugate q_i, then p_i itself is a constant of motion.

- With p_i constant of motion, the equation for q_i is

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \omega_i$$

- If a canonical transformation (see next section) can be found such that all p_i are constants, then the system is easily integrated and

$$q_i = \omega_i t + \beta_i$$

- The solution of the Hamilton-Jacobi equation gives the generating function for such coordinate transformation, i.e. the Hamilton's characteristic function (see section after next).

Generating functions and canonical transformations

- Canonical transformations preserve the Hamiltonian structure of the equations of motions. Thus, they must satisfy the same Least Action Principle

$$\delta \left[\int_{t_1}^{t_2} \left(\boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{p}, \boldsymbol{q}, t) \right) dt \right] = 0$$

- The integrands, expressed in old $\boldsymbol{p}, \boldsymbol{q}$ and new $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ coordinates, should differ at most of a complete differential, i.e.

$$\boldsymbol{p} d\boldsymbol{q} - H(\boldsymbol{p}, \boldsymbol{q}, t) dt = \bar{\boldsymbol{p}} d\bar{\boldsymbol{q}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) dt + dF_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$$

with $F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$ being the generating function of the canonical transformation from $\boldsymbol{p}, \boldsymbol{q}$ and new $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ coordinates.

- From the differential forms above, one readily derives

$$\boldsymbol{p} = \frac{\partial F_1}{\partial \boldsymbol{q}} \quad ; \quad \bar{\boldsymbol{p}} = -\frac{\partial F_1}{\partial \bar{\boldsymbol{q}}} \quad ; \quad \bar{H} = H + \frac{\partial F_1}{\partial t}$$

- Introduce the generating function $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$ via the Legendre transform $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t) = F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) + \bar{\boldsymbol{p}}\bar{\boldsymbol{q}}$ and substitute dF_2 into the differential forms

$$\boldsymbol{p}d\boldsymbol{q} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = -\bar{\boldsymbol{q}}d\bar{\boldsymbol{p}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$$

One derives readily

$$\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{q}} \quad ; \quad \bar{\boldsymbol{q}} = \frac{\partial F_2}{\partial \bar{\boldsymbol{p}}} \quad ; \quad \bar{H} = H + \frac{\partial F_2}{\partial t}$$

- Introduce the generating function $F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t)$ via the Legendre transform $F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t) = F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) - \boldsymbol{p}\boldsymbol{q}$ and substitute dF_3 into the differential forms

$$-\boldsymbol{q}d\boldsymbol{p} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = \bar{\boldsymbol{p}}d\bar{\boldsymbol{q}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t)$$

One derives readily

$$\boldsymbol{q} = -\frac{\partial F_3}{\partial \boldsymbol{p}} \;\; ; \;\; \bar{\boldsymbol{p}} = -\frac{\partial F_3}{\partial \bar{\boldsymbol{q}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_3}{\partial t}$$

- Introduce the generating function $F_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t)$ via the Legendre transform $F_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t) = F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t) + \bar{\boldsymbol{p}}\bar{\boldsymbol{q}}$ and substitute dF_4 into the differential forms

$$-\boldsymbol{q}d\boldsymbol{p} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = -\bar{\boldsymbol{q}}d\bar{\boldsymbol{p}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t)$$

One derives readily

$$\boldsymbol{q} = -\frac{\partial F_4}{\partial \boldsymbol{p}} \;\; ; \;\; \bar{\boldsymbol{q}} = \frac{\partial F_4}{\partial \bar{\boldsymbol{p}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_4}{\partial t}$$

Hamilton - Jacobi equation

- With canonical transformation theory, one can formally solve the equations of motion. For example, choosing $\bar{H} = 0$, the new canonical coordinates will be constants of motion; i.e., they can be considered as initial values of the untransformed coordinates. Thus, the equations for coordinate transformation give positions and momenta at any time of the considered motion.

- In this case, $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$ is called Hamilton's principal function and can be determined from (note that $\bar{\boldsymbol{p}}$ plays the role of a parameter)

$$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}, t\right) + \frac{\partial F_2}{\partial t} = 0$$

- For an autonomous Hamiltonian, $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}})$ is called Hamilton's characteristic function and, setting $\bar{H} = E$, can be determined from the Hamilton - Jacobi equation, i.e.

$$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}\right) = E$$

- As noted before, the solution of the Hamilton-Jacobi equation gives the Hamilton's characteristic function for generating the canonical transformation such that all p_i are constants and $q_i = \omega_i t + \beta_i$.

- An alternative method is based on the Lie generating function $w(\boldsymbol{p}, \boldsymbol{q}, s)$ (see Lecture 4), which generates a canonical transformation from old $\boldsymbol{p}, \boldsymbol{q}$ to new $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ coordinates by means of the Hamilton's equations

$$\frac{dp_i}{ds} = -\frac{\partial w}{\partial q_i} \quad ; \quad \frac{dq_i}{ds} = \frac{\partial w}{\partial p_i}$$

where s is a parameter and $\bar{\boldsymbol{p}} = \boldsymbol{p}(s)$, $\bar{\boldsymbol{q}} = \boldsymbol{q}(s)$.

- Use of the Lie generating function simplifies calculation of higher order perturbation theories, for old and new variables remain unmixed.
- Computing Hamilton's principal or characteristic functions is generally difficult unless the corresponding equations are separable. Integrability and separability of the Hamilton-Jacobi equation are linked in a subtle way.
- The Hamilton-Jacobi formalism is generally very useful for obtaining approximate (series) solutions of systems that are nearly separable or nearly-integrable (see Lecture 2).

E: Discuss examples of integrable (separable) Hamiltonian systems on the basis of your knowledge.

Phase space motion

- The equations of motion in the Hamiltonian form for a system with N-degrees of freedom have $2N$ constants of motion, i.e. the initial values of canonical coordinates and momenta.
- The motion of the system can be visualized as trajectories in the $2N$-dimensional space $\boldsymbol{p}, \boldsymbol{q}$, called the phase space.
- The motion of the system can be interpreted as the result of a phase flow, i.e. the one-parameter group of transformation in the phase space

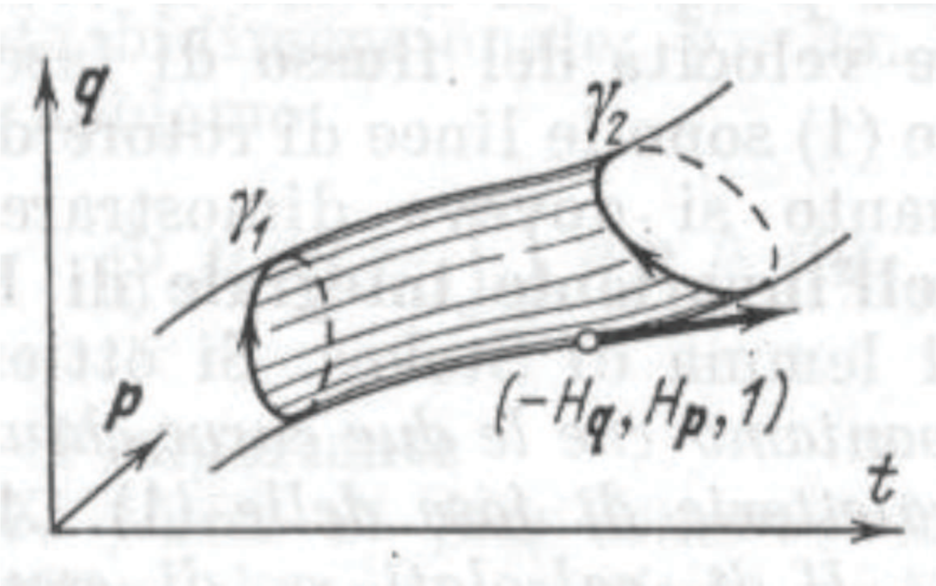
$$g^t : (\boldsymbol{p}(0), \boldsymbol{q}(0)) \mapsto (\boldsymbol{p}(t), \boldsymbol{q}(t))$$

with $\boldsymbol{p}(t), \boldsymbol{q}(t)$ solutions of the equations of motion in the Hamiltonian form.

E: Show that g^t is a group. Show that this implies that phase space trajectories corresponding to the solution of the equations of motion cannot intersect.

E: Discuss the relationship of this property with the invariance under time reversal of the equations of motion.

E: Discuss the concept of integrability in the light of the statements above.



Hamiltonian field and characteristic lines of the 1-form $\omega^1 = \boldsymbol{p}d\boldsymbol{q} - Hdt$ in the $2N+1$-dimensional phase space $\boldsymbol{p}, \boldsymbol{q}, t$ (*Arnold*, 1989)

For $\gamma_1 - \gamma_2 = \partial\sigma$, Stokes theorem implies

$$\oint_{\gamma_1} \boldsymbol{p}d\boldsymbol{q} - Hdt = \oint_{\gamma_2} \boldsymbol{p}d\boldsymbol{q} - Hdt$$

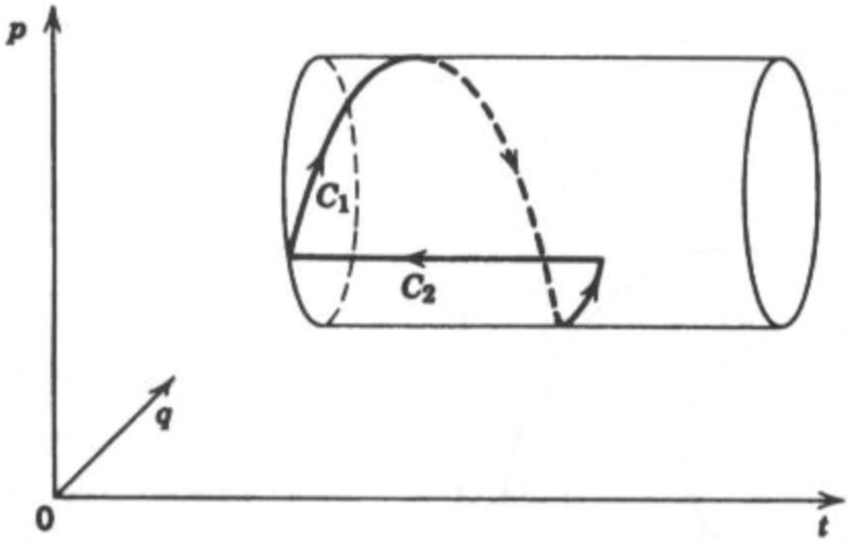
This is the Poincaré – Cartan invariant.

- The Poincaré – Cartan invariant can be written for an arbitrary closed contour in the in the $2N+1$-dimensional phase space $\boldsymbol{p}, \boldsymbol{q}, t$. For a one dimensional periodic system, the integral can be taken along the contour of the characteristic lines; i.e.

$$\oint p dq = \text{const} = 2\pi J$$

where J is the so called action integral defined as

$$J = \frac{1}{2\pi} \oint p dq$$



Path for action integral calculation. (*Lichtenberg and Lieberman*, 2010)

The action integral is the canonical momentum in action angle coordinates and has important applications as adiabatic constant of motion.

- Consider the ensemble of initial conditions and the density distribution if points in the $2N+1$-dimensional phase space $\boldsymbol{p}, \boldsymbol{q}, t$

$$\tau = \tau(\boldsymbol{p}, \boldsymbol{q}, t)$$

- The joint probability that at time t the system is described by a point in the volume between $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{p} + d\boldsymbol{p}, \boldsymbol{q} + d\boldsymbol{q}$ is given by

$$d\mathcal{N} = \tau(\boldsymbol{p}, \boldsymbol{q}, t)\Pi_i dp_i dq_i$$

- The time evolution of $d\mathcal{N}$ is obtained from the continuity equation

$$\frac{\partial d\mathcal{N}}{\partial t} + \sum_i \left(\frac{\partial}{\partial p_i}(d\mathcal{N}\dot{p}_i) + \frac{\partial}{\partial q_i}(d\mathcal{N}\dot{q}_i) \right) = 0$$

- Dividing the above equation by the volume element in the phase space, there remains the following equation for the time evolution of $\tau(\boldsymbol{p}, \boldsymbol{q}, t)$, i.e.

$$\frac{\partial \tau}{\partial t} + \sum_i \left(\dot{p}_i \frac{\partial \tau}{\partial p_i} + \dot{q}_i \frac{\partial \tau}{\partial q_i} \right) = 0$$

E: Fill in the missing details for the demonstration of the incompressibility of the flow in the phase space. This is known as the Liouville theorem.

E: Discuss examples you are familiar with, for which the Liouville theorem plays an important role.

The extended phase space

- The Least Action Principle, underlying the derivation of the equations of motion, can be rewritten introducing a parameter ζ, with respect to which all variations are independent; i.e.

$$\delta\left[\int\left(\boldsymbol{p}\frac{d\boldsymbol{q}}{d\zeta}-H(\boldsymbol{p},\boldsymbol{q},t)\frac{dt}{d\zeta}\right)d\zeta\right]=0$$

- From the form of the variational principle, it is intuitive that we can define an extended phase space setting in $2N+2$ dimensions by letting $\bar{p}_i = p_i$, $\bar{q}_i = q_i$ for $i = 1, N$ and $\bar{p}_{N+1} = -H$, $\bar{q}_{N+1} = t$.

- This corresponds to introducing the generating function

$$F_2 = \sum_{i=1}^{N} \bar{p}_i q_i - Ht \;\; ; \;\; \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = H(\boldsymbol{p}, \boldsymbol{q}, t) - H$$

- Equations of motion in the Hamiltonian form are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} \;\; ; \;\; \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i}$$

- $\bar{H}$ is independent of the parameter (time-like) ζ, so that $\bar{H} = \text{const}$, while $t(\zeta) = \zeta + \zeta_0 = \zeta$, for ζ_0 can be arbitrarily set to zero.

The reduced phase space

- Given the concept of the extended phase space, every system can be described as autonomous system, i.e. with an Hamiltonian that does not explicitly depend on time; i.e.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0$$

- Conversely, one could use this equation to solve one of the momenta, say p_N, as a function of $\bar{p}_i = p_i$ and $\bar{q}_i = q_i$ for $i = 1, N-1$ and $\zeta = q_N$ considered as a parameter; $p_N = p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$. Letting $\bar{H} = -p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$, the equations of motion in the $2N-2$ dimensional reduced phase space are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} = -\frac{1}{\dot{q}_N}\frac{\partial H}{\partial q_i} \quad ; \quad \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i} = \frac{1}{\dot{q}_N}\frac{\partial H}{\partial p_i}$$

- If the momentum p_N is a constant, since the original $H(\boldsymbol{p}, \boldsymbol{q})$ did not explicitly depend on q_N, then $\bar{H}$ is constant and the reduced phase space can be further reduced.

E: Comment this last statement in the light of the remark given in Lecture 3 of the Spring 2010 intensive course on *Resonant and non-resonant interactions in beam-plasma systems*; i.e. that the general problem of fluctuations induced transport by gyrokinetic turbulence in toroidal systems is equivalent to an autonomous Hamiltonian with 3 degrees of freedom.

- A very useful application of the notion of reduced phase space is the definition of a Poincaré surface of section.
- For a 2D system, the motion is bound to occur within the energy surface $H(p_1, p_2, q_1, q_2) = H_0$, which can be written as

$$p_2 = p_2(p_1, q_1, q_2)$$

- If the motion is bounded in the phase space, the motion can repeatedly cross the plane $q_2 = \text{const}$, which is a convenient choice of the surface of section, coinciding with the reduced phase space of the original Hamiltonian system.

- In general, the subsequent crossings of the motion in the (p_1, q_1) surface of section can occur everywhere. However, if a constant of motion exist, in addition to H_0, then $I(p_1, p_2, q_1, q_2) = \text{const}$ in addition to $p_2 = p_2(p_1, q_1, q_2)$. Therefore

$$p_1 = p_1(q_1, q_2)$$

  and the subsequent crossings of the motion in the (p_1, q_1) surface of section must lie on one curve.

- Vice-versa, the fact that the motion in the (p_1, q_1) surface of section lies on a curve can be used as evidence of the existence of a constant of motion in addition to H_0.

- An important property is that the area bounded by a close curve in the surface of section is conserved on subsequent crossings of the surface. One can see this as follows. Given the initial position and momentum in the surface of section (λ, μ), then

$$d\lambda = \frac{\partial \lambda}{\partial q} dq + \frac{\partial \lambda}{\partial \mu} d\mu \;\; ; \;\; dp = \frac{\partial p}{\partial q} dq + \frac{\partial p}{\partial \mu} d\mu$$

- Partial derivatives can be computed in terms of the generating function $F_2(\lambda, p)$ and the relations above can be inverted as

$$dq = \left(F_{2\lambda p} - \frac{F_{2pp} F_{2\lambda\lambda}}{F_{2\lambda p}} \right) d\lambda + \frac{F_{2pp}}{F_{2\lambda p}} d\mu \;\; ; \;\; dp = -\frac{F_{2\lambda\lambda}}{F_{2\lambda p}} d\lambda + \frac{1}{F_{2\lambda p}} d\mu$$

E: Fill in the missing details and show that the Jacobian of the transformation from (λ, μ) to (q, p) is one, i.e. the transformation is area preserving.

- More generally, for $N > 2$ degrees of freedom, we can construct the reduced phase space $p_N = p_N(\boldsymbol{p}, \boldsymbol{q}, q_N)$ and define the surface of section $q_N =$ const. In the reduced phase space, the motion is volume preserving. If one or more constants of motion exists, other than H_0, the system motion will lie on surfaces of dimensionality less than $2N - 2$, where trajectories will be volume preserving.

- If the motion is exactly separable in the (p_i, q_i) coordinates, the motion in each (p_i, q_i) plane is area preserving, a constant of motion (action integral) exists in each degree of freedom and the motion in each (p_i, q_i) plane lies on a smooth curve.

- In general, for a system with N degrees of freedom, the system trajectories projected in the (p_i, q_i) surface of section do not lie on a smooth curve but lie on an annulus of finite area, whose size is related with the nearness to the exact separability of the motion in the (p_i, q_i) coordinates (see Lecture 2).

Action-angle variables

- In this Lecture, it was already pointed out that separability and integrability are concepts related in a subtle way. Assume we can write the generating function F_2 as
$$F_2 = \sum_i S_i(q_i, \alpha_1 ... \alpha_N)$$
where the constants α_i are the new momenta associated with the N constants of motion.
- Note that, for a system with N-degrees of freedom, with $N > 1$, N constants of motion that decouple the N degrees of freedom can be found if the Hamilton-Jacobi equation is completely separable in some coordinate system (for $N = 1$ this is obvious for the constants coincides with the energy).

- If the Hamiltonian can be written in the separated form (not the most general one)

$$H = \sum_i H_i \left(\frac{\partial S_i}{\partial q_i}, q_i \right)$$

then the Hamilton-Jacobi equation can be separated into N conditions

$$H_i \left(\frac{\partial S_i}{\partial q_i}, q_i \right) = \alpha_i \;\; ; \;\; \sum_i \alpha_i = H_0$$

- In general, the new momenta can be taken as generic independent functions of the α_i, i.e. $J_i = J_i(\boldsymbol{\alpha})$ and $\alpha_i = \alpha_i(\boldsymbol{J})$. Thus

$$F_2 = \sum_i S_i(q_i, \boldsymbol{\alpha}(\boldsymbol{J})) = \sum_i \bar{S}_i(q_i, \boldsymbol{J}) \;\; ; \;\; \bar{H}(\boldsymbol{J}) = \sum_i \alpha_i(\boldsymbol{J})$$

- A particular choice of $\boldsymbol{J}$ is useful for completely separable periodic systems, i.e. system for which in every degree of freedom

  - p_i and q_i are periodic functions of t with the same period: the motion is called libration

  - p_i is a periodic function of q_i: the motion is called rotation

- The periods in each degree of freedom need not be the same. If the periods are not in the ratio of rational numbers the motion is called conditionally periodic.

- The convenient choice of J_i are via the action integrals (constants of motion)

$$J_i = \frac{1}{2\pi} \oint p_i dq_i$$

- The conjugate coordinates θ_i are given by

$$\dot{\theta}_i = \frac{\partial \bar{H}}{\partial J_i} = \omega_i \quad ; \quad \theta_i = \omega_i t + \beta_i$$

- The conjugate coordinates θ_i are angles in the sense that, after one period

$$\Delta\theta_i = \oint d\theta_i = \oint \frac{\partial}{\partial q_i}\frac{\partial \bar{S}}{\partial J_i} dq_i = \frac{\partial}{\partial J_i}\oint \frac{\partial \bar{S}}{\partial q_i} dq_i = \frac{\partial}{\partial J_i}\oint p_i dq_i = 2\pi$$

- From this condition, the period of the motion is $T = 2\pi/\omega_i$ and ω_i is the angular frequency of the motion.
- When studying near-integrable systems, the integrable part of the system is usually represented in action-angle variables, while the further response is investigated using perturbation theory or other methods.

References and reading material

A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion*, Springer-Verlag (1983).

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V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag (1989).