

Lecture 4

Wave propagation in slowly varying, weakly non-uniform media:
Eikonal formulation of wave equations.
Wave-packet amplitude evolution equation.

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The integral form of the dispersion relation...

- Summary from [Lecture 3](#): Assuming an isolated singular layer at $r = r_0$, the variational principle obtained for the SAW energy functional becomes an integral form of the dispersion relation, which reads

$$i|s|\Lambda = \delta\hat{W}_f .$$

- Expressions of the [general inertia \$\Lambda\$](#) and of the [normalized potential energy \$\delta\hat{W}_f\$](#) (in the fluid MHD limit) are given in [Lecture 3](#).
- The [integral \(global\) form of the dispersion relation](#) derived above is the simplest form of the general fishbone like dispersion relation (GFLDR) [Z & C 2014], which can be further reduced to a [local WKB dispersion relation](#) for short wavelength modes.

- Reconsider the general inertia expression obtained in [Lecture 3](#)

$$\delta I = 2\pi^2 \sum_{r_0} \left[\frac{k_\vartheta^2 c^2 (d\psi/dr)}{\mathcal{J} B_0^2 |sk_\vartheta|} \right]_{r=r_0, \vartheta=0} \left| \hat{\phi}_{sn0+}(r_0) \right|^2 i\Lambda(r_0) .$$

and specialize it to short wavelength modes.

- Here, summation $\sum_{r_0}$ is extended to all resonances (non only a single isolated one) and we have used the definition of Λ as

$$i\Lambda(r_0) = \left(2 \left| \hat{\phi}_{sn0+}(r_0) \right|^2 \right)^{-1} \left[\hat{\phi}_{sn}^\dagger(r_0, \vartheta) \partial_\vartheta \hat{\phi}_{sn}(r_0, \vartheta) \right]_{\vartheta \rightarrow 0^-}^{\vartheta \rightarrow 0^+} .$$

- Recall that the $\sum_{r_0}$ is due to the presence of singular contributions at $nq - \ell - \text{Re}\nu$, due to the Poisson summation formula

$$\sum_{j \in \mathbb{Z}} e^{i2\pi j(nq - \nu^*)} = \sum_{\ell \in \mathbb{Z}} \delta(nq - \nu^* - \ell) = \sum_{\ell \in \mathbb{Z}} |sk_\vartheta|^{-1} \delta(r - r_0 - \ell/nq')$$

- As the fine radial structures are accounted for by integration in ϑ (the parallel mode structure) the remaining radial variations must occur on the mesoscales (Lecture 6). We can thus replace

$$\sum_{r_0} |sk_{\vartheta}|^{-1} \{...\} (r_0) = \int_0^a \{...\} (r) dr .$$

- Using this method, the [general inertia expression](#) specialized to short wavelength modes becomes

$$\delta I = 2\pi^2 \int_0^a dr \left[\frac{k_{\vartheta}^2 c^2 (d\psi/dr)}{\mathcal{J} B_0^2} \right]_{\vartheta=0} \left| \hat{\phi}_{sn0+}(r) \right|^2 i\Lambda(r) ,$$

with the expression for Λ remaining the same and being calculated at a given flux surface r .

E: Comment the previous equation as intuitive form for the general inertia expression and compare it with the result used in Lecture 3. What are the differences?

- The same procedure can be adopted for the potential energy expression. Thus, from [Lecture 3](#) (p.30)

$$\delta W = (2\pi)^3 \sum_{j \in \mathbb{Z}} \int_0^a dr \frac{d\psi/dr}{2} e^{i2\pi n q j} \int_{-\infty}^{\infty} \mathcal{J} d\vartheta \left[\frac{1}{4\pi} \nabla_{\perp} \hat{\Psi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \cdot \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) + \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \nabla \cdot \delta \hat{\mathbf{J}}_{\perp n}(r, \vartheta) \right] .$$

- Because of the phase factor $\propto e^{i2\pi n q j}$, the integral will rapidly vanish for $|nq| \gg 1$ unless $j = 0$. Thus, the leading order expression is

$$\begin{aligned} \delta W &= 2\pi^2 \int_0^a dr \frac{d\psi/dr}{2} \int_{-\infty}^{\infty} \mathcal{J} d\vartheta \left[\left| \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) \right|^2 + 4\pi \hat{\Phi}_{sn}^{\dagger}(r, \vartheta) \nabla \cdot \delta \hat{\mathbf{J}}_{\perp n}(r, \vartheta) \right] \\ &= 2\pi^2 \int_0^a dr \left[\frac{k_{\vartheta}^2 c^2 (d\psi/dr)}{\mathcal{J} B_0^2} \right]_{\vartheta=0} \left| \hat{\phi}_{sn0+}(r) \right|^2 \delta \bar{W}_f(r) . \end{aligned}$$

- The equation above is the [definition of the local contribution to the fluctuation potential energy \$\delta \bar{W}_f\$](#) .

- The final expression for $\delta\bar{W}_f$ is

$$\delta\bar{W}_f(r) = \frac{[\mathcal{J}B_0^2]_{\vartheta=0}/(k_{\vartheta}^2 c^2)}{2 \left| \hat{\phi}_{sn0+}(r) \right|^2} \int_{-\infty}^{\infty} \mathcal{J} d\vartheta \left[\left| \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) \right|^2 + 4\pi \hat{\Phi}_{sn}^{\dagger}(r, \vartheta) \nabla \cdot \delta \hat{\mathbf{J}}_{\perp n}(r, \vartheta) \right] .$$

Compare this expression with the result given in Spring 2017 Lecture 3. Can you recover the result using the general expression?

- Collecting results, we can rewrite the SAW energy functional as

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = \delta W - \delta I = 2\pi^2 \int_0^a dr \left[\frac{k_{\vartheta}^2 c^2 (d\psi/dr)}{\mathcal{J}B_0^2} \right]_{\vartheta=0} \left| \hat{\phi}_{sn0+}(r) \right|^2 [\delta\bar{W}_f(r) - i\Lambda(r)] .$$

- This equation suggests that **global SAW at short wavelength** can be viewed as superposition of radial wave packets, satisfying the **WKB dispersion relation** (GFLDR) [Z & C 2014, Chen RMP 2016]

$$D(r, k_r, \omega) = \delta\bar{W}_f(r, k_r, \omega) - i\Lambda(r, \omega) = 0 .$$

- Note that here, explicit dependences on mode frequency and radial wave number have been shown explicitly.

E: In general $\Lambda(r, \omega)$ does not depend on k_r . Why? Why can $\delta\bar{W}_f$ depend on ω ? What about the limiting case of low-frequency ideal MHD?

E: Reconsider the expression for δW as a weighted integral over $\delta\bar{W}_f$ and the expression for δW introducing $\delta\hat{W}_f$ in Lecture 3. Show that $\delta\hat{W}_f = |s|\delta\bar{W}_f$ for a radially localized mode, where equilibrium is slowly varying and the mode is considered constant over the mode width $|nq'|^{-1} = |sk_\vartheta|^{-1}$.

- The general result obtained so far (GFLDR) [Z & C 2014, Chen RMP 2016], allows us to study the propagation and stability properties of long as well as short wavelength SAW in toroidal geometry.
- The remaining part of this series of lectures will be focused on discussing the properties of short wavelength SAW, using the general theory of [wave propagation in slowly varying, weakly non-uniform media](#); with application to the description of Alfvén waves in tokamaks.

Ballooning formalism and radial envelope

- For high toroidal mode numbers, $|n| \gg 1$, the MSD approach (Lecture 3 and Spring 2017 Lecture 2) reduces to the well-known ballooning-mode representation [Lu 12].
- More specifically, one may define $\Delta_0 = 1/(nq')$, $r_1 \sim \mathcal{O}(a)$ and $|\Delta_0 r_1|^{1/2}$ as, respectively, the microscopic, macroscopic, and mesoscopic radial scale lengths (Lecture 6).
- The poloidal harmonics $f_n(r, nq - m)$ (Lecture 3) can then be expressed as

$$f_n(r, nq - m) = A_n(r) f_{0n}(r_1; nq - m) ,$$

with $A_n(r)$ representing the meso-scale radial envelope.

- $f(r, \theta, \zeta)$ can then be expressed in the following ballooning-mode representation

$$\begin{aligned} f(r, \theta, \zeta) &= \sum_{m \in \mathbb{Z}} e^{in\zeta - im\theta} f_{m,n}(r) = \sum_{m \in \mathbb{Z}} A_n(r) e^{in\zeta - im\theta} f_{0n}(r_1; nq - m) \\ &= \sum_{m \in \mathbb{Z}} A_n(r) e^{in\zeta - im\theta} \int e^{i(m-nq)\vartheta} \hat{f}_{0n}(r_1, \vartheta) d\vartheta \\ &= \sum_{m \in \mathbb{Z}} A_n(r) e^{in\zeta - im\theta} \int e^{i(m-nq)\vartheta} \mathcal{P}_{Bn}(r, \vartheta) [f_{0n}] d\vartheta . \end{aligned}$$

- Here, the functions $f_{0n}(r_1; nq - m)$ are nearly invariant under radial translations by multiples of Δ_0 . Furthermore, due to the meso-scale nature of $A_n(r)$, it is possible to use the eikonal Ansatz

$$A_n(r) \sim \exp i \int^r nq' \theta_k(r_1) dr_1 .$$

- This notation was introduced originally by [Dewar 81, 82] in theoretical analyses of short wavelength ballooning modes. In this way, the expression for $\nabla_{\perp}$ in the $(r, \theta) \mapsto (r, \vartheta)$ mapping becomes

$$\nabla_{\perp} \mapsto [ik_{\vartheta} \nabla r (s\vartheta - s\theta_k) + in \nabla \zeta + ik_{\vartheta} r \nabla \theta] [1 + \mathcal{O}(1/nq)] ,$$

with $k_{\vartheta} \equiv -(nq/r)$ and the magnetic shear defined as $s = rq'/q$. Thus θ_k plays the role of a normalized radial wave vector and, by definition, $k_r = nq'\theta_k = -sk_{\vartheta}\theta_k$.

- To proceed further analytically, let us adopt the so-called (s, α) model equilibrium for a local description of tokamaks with shifted circular magnetic surfaces. Here, $\alpha = -R_0 q^2 d\beta/dr$ is the dimensionless ballooning pressure gradient parameter.
- In this case, Boozer flux coordinates are easily constructed (Spring 2017 [Lecture 2](#)), with $\mathcal{J} = qR_0(\bar{B}_0/B_0^2)$ and $\bar{B}_0$ the equilibrium magnetic field on the magnetic axis.

□ With the above expression of $\nabla_{\perp}$, we find, defining $k_{\vartheta}^2 \hat{\kappa}_{\perp}^2 = -\nabla_{\perp}^2$,

$$\begin{aligned} \hat{\kappa}_{\perp}^2 = & [s(\vartheta - \theta_k) - \alpha \sin \vartheta]^2 (1 + 2\Delta' \cos \vartheta) \\ & - 2s(\vartheta - \theta_k)\Delta' \sin \vartheta + 1 - 2(r/R_0 + \Delta') \cos \vartheta , \end{aligned}$$

where Δ' is the radial derivative of the Shafranov shift and, consistent with the large aspect-ratio tokamak ordering, $|\Delta'| \sim r/R_0 \ll 1$.

□ Introducing (Spring 2017 [Lecture 3](#)) $\hat{\phi}_s \equiv \hat{\kappa}_{\perp} \hat{\Phi}_s$, $\omega_A = v_A/(qR_0)$ with v_A computed at the magnetic field value on axis, $\omega_S = c_S/(qR_0)$, $g(\vartheta, \theta_k) = [s(\vartheta - \theta_k) - \alpha \sin \vartheta] \sin \vartheta + \cos \vartheta$, and neglecting the kink term for $|nq| \gg 1$, the [SAW vorticity equation](#) becomes

$$\begin{aligned} \hat{\mathcal{L}}_A \left(\hat{\phi}_s \right) = & \left[\left(\frac{\partial^2}{\partial \vartheta^2} + \Delta' \cos \vartheta \right) + \frac{\omega^2}{\omega_A^2} (1 + 4(r/R_0) \cos \vartheta) \right. \\ & \left. + \frac{\alpha \cos \vartheta}{\hat{\kappa}_{\perp}^2} - \frac{(s - \alpha \cos \vartheta)^2}{\hat{\kappa}_{\perp}^4} \right] \hat{\phi}_s = \beta q^2 \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_{\perp}} \left(i \frac{B_0 R_0}{c k_{\vartheta} P_0} \right) \delta \hat{P}_{\text{comp}} . \end{aligned}$$

- Meanwhile, the SSW equation can then be written as

$$\hat{\mathcal{L}}_S \left(\delta \hat{P}_{\text{comp}} \right) = \left(1 + \frac{\omega_S^2}{\omega^2} \frac{\partial^2}{\partial \vartheta^2} \right) \delta \hat{P}_{\text{comp}} = -2\Gamma \left(i \frac{ck_\vartheta P_0}{B_0 R_0} \right) \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_\perp} \hat{\phi}_s .$$

- In the equations above, the subscript n from field variables is dropped, since we are treating toroidal mode numbers independently, consistent with the linear analysis.
- With the [theoretical framework](#) presented in this Lecture series, the [above equations](#) can be used to explore the SAW and SSW spectral properties in [axisymmetric tokamaks](#) within the single-fluid MHD approximation.
- Note that, once the solutions $\hat{\Phi}_s(r, \vartheta) = \hat{\phi}_s(r, \vartheta)/\hat{\kappa}_\perp(r, \vartheta)$ and $\delta \hat{P}_{\text{comp}}(r, \vartheta)$ in the (r, ϑ) mapping space are known, the corresponding real space solutions are readily obtained from the MSD definition ([Lecture 3](#), p. 24).

□ Thus,

$$\begin{aligned}\Phi_s(r, \theta, \zeta) &= 2\pi \sum_{\ell \in \mathbb{Z}} e^{in\zeta - inq(\theta - 2\pi\ell)} \hat{\Phi}_s(r, \theta - 2\pi\ell) , \\ \delta P_{\text{comp}}(r, \theta, \zeta) &= 2\pi \sum_{\ell \in \mathbb{Z}} e^{in\zeta - inq(\theta - 2\pi\ell)} \delta \hat{P}_{\text{comp}}(r, \theta - 2\pi\ell) .\end{aligned}$$

□ In the high toroidal mode number limit $|nq| \gg 1$, we further have

$$\begin{aligned}\hat{\Phi}_s(r, \vartheta) &= A(r) \hat{\Phi}_{s0}(r_1, \vartheta) , \\ \delta \hat{P}_{\text{comp}}(r, \vartheta) &= A(r) \delta \hat{P}_{\text{comp}0}(r_1, \vartheta) ,\end{aligned}$$

with the envelope $A(r)$ given by the eikonal representation at p. 9.

□ After suitable normalization, which can be arbitrarily adopted for properly rendering the fluctuating plasma response in dimensionless form, we may rewrite the governing equations as follows:

$$\begin{pmatrix} \frac{\hat{\phi}_s}{(\beta q^2)^{1/2} B_0 R_0} \\ i \frac{\delta \hat{P}_{\text{comp}}}{(2\Gamma)^{1/2} P_0} \end{pmatrix} \equiv A(r, t) e^{iS(r, t)} \begin{pmatrix} e_1(r, t) y_1(\vartheta; r, t) \\ e_2(r, t) y_2(\vartheta; r, t) \end{pmatrix} .$$

□ Here, the functions $y_1(\vartheta; r, t)$ and $y_2(\vartheta; r, t)$ are solution of

$$\begin{aligned} \hat{\mathcal{L}}_A e_1 y_1 - (2\Gamma \beta q^2)^{1/2} \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_\perp} e_2 y_2 &= 0 , \\ \hat{\mathcal{L}}_S e_2 y_2 - (2\Gamma \beta q^2)^{1/2} \frac{g(\vartheta, \theta_k)}{\hat{\kappa}_\perp} e_1 y_1 &= 0 , \end{aligned}$$

which are obtained from SAW vorticity and SSW equations (pp. 11-12) by direct substitution of the definition above.

- The functions $y_1(\vartheta; r, t)$ and $y_2(\vartheta; r, t)$ describe the parallel mode structures in the variable ϑ and can be normalized to unity

$$\int_{-\infty}^{\infty} |y_{1,2}(\vartheta; r, t)|^2 d\vartheta = 1 .$$

E: Demonstrate that, with the above definitions, we have just defined the function space for the relevant fields $\hat{\phi}_s$ and $\delta\hat{P}_{\text{comp}}$ as a normed linear space, and, even further, as a Hilbert space.

- Furthermore, $k_r(r, t) = nq'\theta_k(r, t) = \partial_r S(r, t)$, $\omega(r, t) = -\partial_t S(r, t)$, and

$$\hat{e}(r, t) \equiv \begin{pmatrix} e_1(r, t) \\ e_2(r, t) \end{pmatrix} ,$$

is the **unit polarization vector**, defined such that $\hat{e}^+ \cdot \hat{e} = 1$, with $\hat{e}^+$ the adjoint of $\hat{e}(r, t)$.

Eikonal formulation of wave equations

- The WKB eigenvalue problem, generated by the field equations above with outgoing wave boundary conditions for $|\vartheta| \rightarrow \infty$ (no energy source at $|k_r| \rightarrow \infty$), with $\mathbf{A}(r, t) \equiv \hat{\mathbf{e}}(r, t)A(r, t)$, can be written as

$$\mathbf{D}(r, t, k_r, \omega) \cdot \mathbf{A}(r, t)e^{iS(r, t)} = 0 ,$$

$$D_{11}(r, t, k_r, \omega) = \int_{-\infty}^{\infty} y_1^* \mathcal{L}_A y_1 d\vartheta ,$$

$$D_{12}(r, t, k_r, \omega) = D_{21}^*(r, t, k_r, \omega) = (2\Gamma\beta q^2)^{1/2} \int_{-\infty}^{\infty} \frac{g(\vartheta, \theta_k)}{\hat{k}_\perp} y_1^* y_2 d\vartheta ,$$

$$D_{22}(r, t, k_r, \omega) = \int_{-\infty}^{\infty} y_2^* \mathcal{L}_S y_2 d\vartheta .$$

- Note that we generally assume that the **plasma equilibrium does not depend explicitly on time**. However, we maintain explicit time dependence in the notation adopted here for the sake of clarity and generality.

- Note that, in the present case, $y_1(\vartheta; r, t)$ and $y_2(\vartheta; r, t)$ can be assumed as real functions, but the complex notation has been maintained for completeness.
- The $\mathbf{D}(r, t, k_r, \omega)$ operator in is generally a linear integral operator, and the corresponding wave equation can be cast as

$$\int \mathbf{D}(r, t, r - s, t - \tau) \cdot \mathbf{A}(r - s, t - \tau) e^{iS(r-s, t-\tau)} ds d\tau = \mathbf{F}(r, t) .$$

- For the sake of generality, we have added a $\mathbf{F}(r, t)$ term on the r.h.s., which can describe external forcing as well as nonlinear interactions [Chen RMP 16].
- The $\mathbf{D}(r, t, k_r, \omega)$ operator is computed from the **integral kernel** $\mathbf{D}(r, t, r', t')$ by

$$\mathbf{D}(r, t, k_r, \omega) = \int \mathbf{D}(r, t, r - s, t - \tau) e^{-ik_r s + i\omega \tau} ds d\tau .$$

- This form of $\mathbf{D}(r, t, k_r, \omega)$, originally introduced by [Kravtsov 69], is called **non-symmetric** in contrast to the **symmetric form** later adopted by [Bernstein 75].
- There is a profound implication in this **subtle mathematical twist**. In fact, by definition of adjoint operator (see also **Lecture 3**)

$$[\mathcal{D}^+(r, t, r', t')]_{\mu\nu} = [\mathcal{D}(r', t', r, t)]_{\nu\mu}^* ,$$

with μ, ν indexes running over the polarization states; *i.e.*, $\mu, \nu \in (1, 2)$ in the present case.

- Thus, from the above definition of the integral operator, we have

$$\begin{aligned} \mathbf{D}^+(r, t, k_r, \omega) &= \int \mathcal{D}^+(r, t, r-s, t-\tau) e^{-ik_r s + i\omega\tau} ds d\tau \\ &= \int \mathcal{D}^{*T}(r-s, t-\tau, r, t) e^{-ik_r s + i\omega\tau} ds d\tau . \end{aligned}$$

- Meanwhile, using the inverse formula of the integral operator definition on p. 18,

$$\begin{aligned} \mathbf{D}^{*T}(r-s, t-\tau, r, t) &= \frac{1}{4\pi^2} \int \mathbf{D}^{*T}(r-s, t-\tau, k'_r, \omega') e^{ik'_r s - i\omega' \tau} dk'_r d\omega' \\ &= \frac{1}{4\pi^2} \int e^{ik'_r s - i\omega' \tau} e^{-s\partial_r - \tau\partial_t} \mathbf{D}^{*T}(r, t, k'_r, \omega') dk'_r d\omega' . \end{aligned}$$

E: Substitute the definition in the integral operator definition in the expression above and verify that it gives you an identity.

- The expression for the adjoint operator finally becomes

$$\begin{aligned} \mathbf{D}^+(r, t, k_r, \omega) &= \frac{1}{4\pi^2} \int e^{i(k'_r - k_r)s - i(\omega' - \omega)\tau} e^{-i\partial_{k'_r} \partial_r + i\partial_{\omega'} \partial_t} \mathbf{D}^{*T}(r, t, k'_r, \omega') dk'_r d\omega' ds d\tau \\ &= e^{-i\partial_{k_r} \partial_r + i\partial_{\omega} \partial_t} \mathbf{D}^{*T}(r, t, k_r, \omega) . \end{aligned}$$

- A rigorous and formal analysis of this equation is linked with the phase space representation of wave equations (see also [Lecture 5](#)) and the construction of [Wigner-Weyl symbols](#) [McDonald 1988] and [pseudo-differential operators](#) [Hörmander 1997].
- Below, we summarize the most important results and provide governing equations that are needed for our analyses of mode structures, stability properties and nonlinear dynamics.

Wave-packet amplitude evolution equation

- In a slowly varying and weakly non-uniform plasma, we may identify typical time and spatial scales, respectively T and L , characterizing the spatiotemporal evolution of the background medium.
- Introducing a small dimensionless parameter $\varepsilon \ll 1$, we consider **wave packets** in the form of $\mathbf{A}(r, t)e^{iS(r, t)}$, with $\mathbf{A}(r, t) \equiv \hat{\mathbf{e}}(r, t)A(r, t)$, and

$$\begin{aligned} |k_r(r, t)|L &= |\partial_r S(r, t)|L \sim \varepsilon^{-1} \gg 1, \\ |\omega(r, t)|T &= |\partial_t S(r, t)|T \sim \varepsilon^{-1} \gg 1. \end{aligned}$$

- Furthermore, we impose that the $\mathbf{F}(r, t)$ term on the r.h.s. of the wave equation (p. 17) is $\mathcal{O}(\varepsilon)$; and, for now, we also assume

$$\begin{aligned} |\partial_r \ln \mathbf{A}| &\sim |\partial_r \ln k_r| \sim |\partial_r \ln \omega| \sim L^{-1}, \\ |\partial_t \ln \mathbf{A}| &\sim |\partial_t \ln k_r| \sim |\partial_t \ln \omega| \sim T^{-1}. \end{aligned}$$

- As $\mathbf{D}(r, t, k_r, \omega)$ is generally complex, we separate real and imaginary parts

$$\mathbf{D}(r, t, k_r, \omega) = \mathbf{D}_R(r, t, k_r, \omega) + i\mathbf{D}_I(r, t, k_r, \omega) ,$$

and assume that

$$\mathbf{D}_R(r, t, k_r, \omega) = \mathbf{D}_R^0(r, t, k_r, \omega) + \varepsilon \mathbf{D}_R^1(r, t, k_r, \omega) + \dots ,$$

$$\mathbf{D}_I(r, t, k_r, \omega) = \varepsilon \mathbf{D}_I^1(r, t, k_r, \omega) + \dots .$$

- Thus, $\mathbf{D}_I^0(r, t, k_r, \omega) = 0$ and $\mathbf{D}_R^0(r, t, k_r, \omega)$ is symmetric, consistent with definitions on p. 16.
- We can also separate Hermitian from anti-Hermitian plasma response noting that, consistent with [ordering of spatio-temporal scales](#), the definition of Hermitian conjugate on p. 19 yields

$$\mathbf{D}^+(r, t, k_r, \omega) \simeq \left[1 + i \left(\frac{\partial^2}{\partial \omega \partial t} - \frac{\partial^2}{\partial k_r \partial r} \right) \right] \mathbf{D}^{*T}(r, t, k_r, \omega) .$$

□ Hence, from definition of Hermitian and anti-Hermitian operators,

$$\begin{aligned}
 \mathbf{D}_H(r, t, k_r, \omega) &\equiv \frac{1}{2} (\mathbf{D}(r, t, k_r, \omega) + \mathbf{D}^+(r, t, k_r, \omega)) \\
 &= \mathbf{D}_H^0(r, t, k_r, \omega) + \varepsilon \mathbf{D}_H^1(r, t, k_r, \omega) + \dots \\
 &\simeq \left[1 + \frac{i}{2} \left(\frac{\partial^2}{\partial \omega \partial t} - \frac{\partial^2}{\partial k_r \partial r} \right) \right] \mathbf{D}_R^0(r, t, k_r, \omega) \\
 &\quad + \frac{\varepsilon}{2} (\mathbf{D}^1(r, t, k_r, \omega) + \mathbf{D}^{1*T}(r, t, k_r, \omega)) ,
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{D}_A(r, t, k_r, \omega) &\equiv \frac{-i}{2} (\mathbf{D}(r, t, k_r, \omega) - \mathbf{D}^+(r, t, k_r, \omega)) \\
 &= \varepsilon \mathbf{D}_A^1(r, t, k_r, \omega) + \dots \\
 &\simeq \frac{-i\varepsilon}{2} (\mathbf{D}^1(r, t, k_r, \omega) - \mathbf{D}^{1*T}(r, t, k_r, \omega)) \\
 &\quad - \frac{1}{2} \left(\frac{\partial^2}{\partial \omega \partial t} - \frac{\partial^2}{\partial k_r \partial r} \right) \mathbf{D}_R^0(r, t, k_r, \omega) .
 \end{aligned}$$

- Evolution equations for the wave packet are obtained expanding $\mathbf{A}(r-s, t-\tau)$ and $e^{iS(r-s, t-\tau)}$ in the wave equation (p. 17) as

$$\begin{aligned} \mathbf{A}(r-s, t-\tau) &= \mathbf{A}(r, t) - s\partial_r \mathbf{A} - \tau\partial_t \mathbf{A} \\ &\quad + \left[(s^2/2)\partial_r^2 + s\tau\partial_r\partial_t + (\tau^2/2)\partial_t^2 \right] \mathbf{A} + \dots, \end{aligned}$$

$$\begin{aligned} e^{iS(r-s, t-\tau)} &= e^{iS(r, t)} e^{-ik_r s + i\omega \tau} \left[1 + i(s^2/2)\partial_r k_r + i(s\tau/2) \right. \\ &\quad \left. \times (\partial_t k_r - \partial_r \omega) - i(\tau^2/2)\partial_t \omega + \dots \right]. \end{aligned}$$

- At the lowest order, $\mathcal{O}(1)$, the wave equation (p. 17) yields

$$\mathbf{D}_R^0(r, t, k_r, \omega) \cdot \mathbf{A}(r, t) = 0.$$

- In the example considered here, $\mathbf{D}_R^0(r, t, k_r, \omega)$ is a 2×2 matrix that admits two independent orthonormal eigenvectors, $\hat{\mathbf{e}}_i(r, t, k_r, \omega)$ [$i \in (1, 2)$], and can, thus, be reduced to diagonal form.

□ In particular,

$$\mathbf{D}_R^0(r, t, k_r, \omega) = \sum_{i=1,2} D_{Ri}^0(r, t, k_r, \omega) \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i^+ ,$$

with

$$D_{Ri}^0(r, t, k_r, \omega) \equiv \hat{\mathbf{e}}_i^+(r, t, k_r, \omega) \cdot \mathbf{D}_R^0(r, t, k_r, \omega) \cdot \hat{\mathbf{e}}_i(r, t, k_r, \omega) .$$

□ The two independent solutions represent SAW and SSW polarizations, whose local dispersion relation can be written as

$$D_{Ri}^0(r, t, k_r, \omega) = 0 ,$$

which yields

$$\omega = \Omega(k_r, r, t) .$$

E: In the light of the results of the present Lecture (see p. 6), can you say what is the dispersion relation for SAW and SSW?

- The local dispersion relation generates the **Hamilton - Jacobi** equation

$$D_{Ri}^0(r, t, \partial_r S(r, t), -\partial_t S(r, t)) = 0 ,$$

and $S(r, t)$ is the **Hamilton principal function**, which can be constructed in terms of the characteristics

$$\begin{aligned} \dot{r} &= -\frac{\partial_{k_r} D_{Ri}^0}{\partial_{\omega} D_{Ri}^0} = \partial_{k_r} \Omega(k_r, r, t) , \\ \dot{k}_r &= \frac{\partial_r D_{Ri}^0}{\partial_{\omega} D_{Ri}^0} = -\partial_r \Omega(k_r, r, t) , \\ \dot{\omega} &= -\frac{\partial_t D_{Ri}^0}{\partial_{\omega} D_{Ri}^0} = \partial_t \Omega(k_r, r, t) . \end{aligned}$$

- For a stationary plasma equilibrium, the solution for $S(r, t)$ can be written as an integral along the characteristics.

- With $\omega = \Omega(r_0, k_{r0})$ and $k_{r0} = \partial_{r_0} S(r_0)$:

$$S(r, t) = S(r_0) - \omega t + \int_{r_0}^r k_r(r') dr' .$$

- At the next order, considering terms up to $\mathcal{O}(\varepsilon)$, the wave equation becomes

$$\begin{aligned} & (\mathbf{D}_I^1 - i\mathbf{D}_R^1) \cdot \mathbf{A} + \frac{\partial \mathbf{D}_R^0}{\partial \omega} \cdot \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \mathbf{D}_R^0}{\partial k_r} \cdot \frac{\partial \mathbf{A}}{\partial r} + \frac{1}{2} \left[-\partial_r k_r \frac{\partial^2 \mathbf{D}_R^0}{\partial k_r^2} \right. \\ & \left. + (\partial_t k_r - \partial_r \omega) \frac{\partial^2 \mathbf{D}_R^0}{\partial k_r \partial \omega} + \partial_t \omega \frac{\partial^2 \mathbf{D}_R^0}{\partial \omega^2} \right] \cdot \mathbf{A} = -ie^{-iS} \mathbf{F} . \end{aligned}$$

- Combining terms, this equation can be cast as

$$\begin{aligned} & \left[\mathbf{D}_I^1 + \frac{1}{2} \left(\frac{\partial^2 \mathbf{D}_R^0}{\partial k_r \partial r} - \frac{\partial^2 \mathbf{D}_R^0}{\partial \omega \partial t} \right) \right] \cdot \mathbf{A} + \left[\frac{\partial \mathbf{D}_R^0}{\partial \omega} \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{1}{2} \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{D}_R^0}{\partial \omega} \right) \cdot \mathbf{A} \right] \\ & - i\mathbf{D}_R^1 \cdot \mathbf{A} - \left[\frac{\partial \mathbf{D}_R^0}{\partial k_r} \cdot \frac{\partial \mathbf{A}}{\partial r} + \frac{1}{2} \frac{\partial}{\partial r} \left(\frac{\partial \mathbf{D}_R^0}{\partial k_r} \right) \cdot \mathbf{A} \right] = -ie^{-iS} \mathbf{F} . \end{aligned}$$

- Taking the scalar product with $\hat{\mathbf{e}}^+(r, t)$, dropping the subscript i denoting either SAW or SSW polarizations for simplicity, and noting $\mathbf{A}(r, t) \equiv A(r, t)\hat{\mathbf{e}}(r, t)$, one derives [Kravtsov 69, Bernstein 75]

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\partial D_R^0}{\partial \omega} A^2 \right) - \frac{\partial}{\partial r} \left(\frac{\partial D_R^0}{\partial k_r} A^2 \right) + 2D_A^1 A^2 - 2iD_R^1 A^2 \\ & + \left(\hat{\mathbf{e}}^+ \cdot \frac{d}{dt} \hat{\mathbf{e}} - \frac{d}{dt} \hat{\mathbf{e}}^+ \cdot \hat{\mathbf{e}} \right) \frac{\partial D_R^0}{\partial \omega} A^2 - \left(\frac{\partial \hat{\mathbf{e}}^+}{\partial \omega} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial t} - \frac{\partial \hat{\mathbf{e}}^+}{\partial t} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial \omega} \right) A^2 \\ & + \left(\frac{\partial \hat{\mathbf{e}}^+}{\partial k_r} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial r} - \frac{\partial \hat{\mathbf{e}}^+}{\partial r} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial k_r} \right) A^2 = -2ie^{-iS} A \hat{\mathbf{e}}^+ \cdot \mathbf{F} . \end{aligned}$$

- Here, we have introduced the notation

$$\begin{aligned} D_R^1(r, t, k_r, \omega) &= \varepsilon \hat{\mathbf{e}}^+(r, t) \cdot \mathbf{D}_R^1(r, t, k_r, \omega) \cdot \hat{\mathbf{e}}(r, t) , \\ D_A^1(r, t, k_r, \omega) &= \varepsilon \hat{\mathbf{e}}^+(r, t) \cdot \mathbf{D}_A^1(r, t, k_r, \omega) \cdot \hat{\mathbf{e}}(r, t) . \end{aligned}$$

- Furthermore, we have taken into account that

$$\begin{aligned}\frac{\partial \mathbf{D}_R^0}{\partial \omega} \cdot \hat{\mathbf{e}} &= \frac{\partial D_R^0}{\partial \omega} \hat{\mathbf{e}} - \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial \omega}, \\ \hat{\mathbf{e}}^+ \cdot \frac{\partial \mathbf{D}_R^0}{\partial \omega} &= \hat{\mathbf{e}}^+ \frac{\partial D_R^0}{\partial \omega} - \frac{\partial \hat{\mathbf{e}}^+}{\partial \omega} \cdot \mathbf{D}_R^0,\end{aligned}$$

with similar identities holding for ∂_ω replaced by ∂_{k_r} .

- Letting $A(r, t) = |A(r, t)| \exp i\varphi(r, t)$, the equation on p. 28 is usually **split into two equations**, one describing **transport of the envelope intensity $|A|^2$** , and another accounting for the **evolution of the envelope phase**; that is

$$\frac{\partial}{\partial t} \left(\frac{\partial D_R^0}{\partial \omega} |A|^2 \right) - \frac{\partial}{\partial r} \left(\frac{\partial D_R^0}{\partial k_r} |A|^2 \right) = -2 \left[D_A^1 - \mathbb{Im} \left(\frac{\hat{\mathbf{e}}^+ \cdot \mathbf{F}}{A e^{iS}} \right) \right] |A|^2,$$

$$\begin{aligned} \frac{\partial D_R^0}{\partial \omega} \frac{\partial \varphi}{\partial t} - \frac{\partial D_R^0}{\partial k_r} \frac{\partial \varphi}{\partial r} = & \left[D_R^1 - \mathbb{R}e \left(\frac{\hat{\mathbf{e}}^+ \cdot \mathbf{F}}{A e^{iS}} \right) \right] + \frac{i}{2} \left(\hat{\mathbf{e}}^+ \cdot \frac{d}{dt} \hat{\mathbf{e}} - \frac{d}{dt} \hat{\mathbf{e}}^+ \cdot \hat{\mathbf{e}} \right) \frac{\partial D_R^0}{\partial \omega} \\ & - \frac{i}{2} \left(\frac{\partial \hat{\mathbf{e}}^+}{\partial \omega} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial t} - \frac{\partial \hat{\mathbf{e}}^+}{\partial t} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial \omega} \right) \\ & + \frac{i}{2} \left(\frac{\partial \hat{\mathbf{e}}^+}{\partial k_r} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial r} - \frac{\partial \hat{\mathbf{e}}^+}{\partial r} \cdot \mathbf{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial k_r} \right) . \end{aligned}$$

- These equations are consistent with any of the well-known existing asymptotic analyses of wave equations, in particular with those based on the **non-symmetric** form of the integral operator proposed by [Kravtsov 69], as well as those adopting the **symmetric** form assumed by [Bernstein 75].
- Although this point may seem obvious and also a necessary prerequisite for any acceptable theoretical description of wave propagation in space and time varying media, there are some subtleties related with it that may be worth exploring.

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