

Lecture 3

Shear Alfvén waves in tokamaks:
Variational formulation and variational principle.
Integral form of the dispersion relation.

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Physics of Alfvén waves and energetic particles in fusion plasmas
– Mode structures and global dispersion of Alfvén waves in tokamaks –
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Shear Alfvén wave equations in tokamaks: summary and overview.

- We start with a brief summary and overview from [Spring 2017 Lectures](#), i.e. from [Lecture 2](#) and [Lecture 3](#).
- Refer to a high-aspect-ratio $\epsilon = a/R_0 \ll 1$, low- β tokamak equilibrium.
- Noting $|\nabla| \sim \mathcal{O}(1/a)$ and $|\kappa| \equiv |\mathbf{b}_0 \cdot \nabla \mathbf{b}_0| \sim \mathcal{O}(1/R)$, and expanding $\delta \xi_{\perp} = \delta \xi_{\perp 0} + \delta \xi_{\perp 1} + \dots$, we can express $\delta \xi_{\perp 0}$ as

$$\delta \xi_{\perp 0} = \frac{c}{B_0} \mathbf{b}_0 \times \nabla \Phi_s ,$$

with Φ_s being the perturbed stream function.

- Correspondingly, with the perturbed (magnetic) flux $\Psi_s = -c \mathbf{b}_0 \cdot \nabla \Phi_s$,

$$\delta \mathbf{B}_0 = \nabla \times (\mathbf{b}_0 \Psi_s) \simeq \nabla \Psi_s \times \mathbf{b}_0 .$$

- Thus, the magnetic field perturbation is predominantly perpendicular and the parallel magnetic field perturbation satisfies the perpendicular pressure balance ([Lecture 2](#))

$$B_0 \delta B_{\parallel} + 4\pi \delta P \simeq \mathcal{O}(\epsilon) |4\pi \delta P| \ll |4\pi \delta P| ;$$

i.e., the compressional Alfvén wave is suppressed here.

- Meanwhile, the pressure fluctuation can be written as

$$\delta P = -\delta \xi \cdot \nabla P_0 + \delta P_{\text{comp}} ,$$

where δP_{comp} denotes the compressional component of δP , and, in the ideal MHD fluid model,

$$\delta P_{\text{comp}} = -\Gamma P_0 (\nabla \cdot \delta \xi) .$$

- The value of δP_{comp} is given by the perturbed parallel force balance equation

$$\rho_{m0} \partial_t^2 \delta \xi_{\parallel} = -\nabla_{\parallel} \delta P_{\text{comp}} .$$

- In the fluid limit description, SAW are described by two fields Φ_s and δP_{comp} , which are provided by the SAW vorticity equation (Lecture 2, Spring 2017):

$$\begin{aligned}\mathcal{L}_A(\Phi_s) &\equiv B_0 \nabla_{\parallel} \left(\frac{1}{B_0} \nabla_{\perp}^2 \nabla_{\parallel} \Phi_s \right) - \nabla_{\perp} \cdot \left(\frac{\partial_t^2}{v_A^2} \nabla_{\perp} \Phi_s \right) \\ &\quad + \frac{8\pi}{B_0} (\mathbf{b}_0 \times \boldsymbol{\kappa}) \cdot \nabla_{\perp} \left[\frac{\mathbf{b}_0}{B_0} \cdot (\nabla P_0 \times \nabla_{\perp} \Phi_s) \right] \\ &\quad + \frac{4\pi}{cB_0} [\mathbf{b}_0 \times \nabla_{\perp} (\nabla_{\parallel} \Phi_s)] \cdot \nabla J_{0\parallel} \\ &= -\frac{8\pi}{cB_0} (\mathbf{b}_0 \times \boldsymbol{\kappa}) \cdot \nabla_{\perp} \delta P_{\text{comp}} .\end{aligned}$$

- And the parallel force balance (SSW equation; Lecture 2, Spring 2017):

$$\begin{aligned}\mathcal{L}_S(\delta P_{\text{comp}}) &\equiv \partial_t^2 \delta P_{\text{comp}} - c_S^2 B_0 \nabla_{\parallel} \left(\frac{1}{B_0} \nabla_{\parallel} \delta P_{\text{comp}} \right) \\ &= -\frac{2\Gamma P_0 c}{B_0} (\mathbf{b}_0 \times \boldsymbol{\kappa}) \cdot \nabla_{\perp} \partial_t^2 \Phi_s ,\end{aligned}$$

The shear Alfvén wave energy functional.

- The first step in the construction of a variational principle is the definition of a proper functional.

E: Reconsider the Exercise on p. 32 of Lecture 1. That is: Read about the MHD energy principle in your favorite text-book. Discuss it and its implications in the light of the theoretical framework of Lecture 1.

- Consider the following SAW energy functional [Chen and Hasegawa 1991]

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = -\frac{1}{2} \int \nabla \Phi_s^\dagger \cdot \delta \mathbf{J} d\mathbf{r} .$$

- Separating parallel and perpendicular components in $\delta \mathbf{J}$

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = \frac{1}{2} \int \left(\nabla_{\parallel} \Phi_s^\dagger \frac{c}{4\pi} \nabla_{\perp}^2 \Psi_s - \nabla_{\perp} \Phi_s^\dagger \cdot \delta \mathbf{J}_{\perp} \right) d\mathbf{r} .$$

- Here, we have used the parallel Ampère's law

$$\delta J_{\parallel} = -\frac{c}{4\pi} \nabla_{\perp}^2 \Psi_s [1 + \mathcal{O}(\epsilon)]$$

where $\nabla_{\perp}^2 = \nabla^2 - \nabla_{\parallel}^2$ and $\nabla_{\parallel} = \mathbf{b}_0 \cdot \nabla$.

- Recalling that $\Psi_s = -c\mathbf{b}_0 \cdot \nabla \Phi_s$, and integrating by parts,

$$\begin{aligned} \mathcal{F}[\Phi_s, \delta P_{\text{comp}}] &= \frac{1}{2} \int \left(\frac{|\nabla_{\perp} \Psi_s|^2}{4\pi} - \nabla_{\perp} \Phi_s^{\dagger} \cdot \delta \mathbf{J}_{\perp} \right) d\mathbf{r} , \\ &= \frac{1}{2} \int \left(\frac{|\delta \mathbf{B}|^2}{4\pi} - \nabla_{\perp} \Phi_s^{\dagger} \cdot \delta \mathbf{J}_{\perp} \right) d\mathbf{r} , \end{aligned}$$

where we have taken into account that $|\delta B_{\parallel}|^2 \ll |\delta \mathbf{B}_{\perp}|^2 \simeq |\delta \mathbf{B}|^2$, and assumed homogeneous boundary conditions to annihilate surface terms.

- Let's now focus on the second term in the SAW energy functional. From [Lecture 2](#) (Spring 2017), we can write

$$\delta \mathbf{J}_\perp = \frac{c}{B_0} \mathbf{b}_0 \times (\rho_{m0} \partial_t^2 \delta \boldsymbol{\xi}_\perp - \delta \mathbf{F}_\perp) ,$$

where $\delta \mathbf{F}_\perp$ is a perpendicular force density that can be cast as:

$$\delta \mathbf{F}_\perp = -\nabla_\perp \delta P + \frac{\delta B_\parallel}{B_0} \nabla_\perp P_0 + \frac{J_{0\parallel}}{c} \nabla_\perp \Psi_s .$$

E: Derive the above result using the analysis of [Lecture 2](#) (Spring 2017).

- Using this results, we can rewrite the second term in the SAW energy functional as

$$\begin{aligned} -\nabla_\perp \Phi_s^\dagger \cdot \delta \mathbf{J}_\perp &= \delta \boldsymbol{\xi}_\perp^\dagger \cdot (\rho_{m0} \partial_t^2 \delta \boldsymbol{\xi}_\perp - \delta \mathbf{F}_\perp) \\ &= \delta \boldsymbol{\xi}_\perp^\dagger \cdot \left(\rho_{m0} \partial_t^2 \delta \boldsymbol{\xi}_\perp + \nabla_\perp \delta P - \frac{\mathbf{J}_0 \times \delta \mathbf{B}}{c} \right) . \end{aligned}$$

- These results can be collected into the expression of the SAW energy functional

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = \frac{1}{2} \int \left[\frac{|\delta \mathbf{B}|^2}{4\pi} + \delta \boldsymbol{\xi}_{\perp}^{\dagger} \cdot \left(\rho_{m0} \partial_t^2 \delta \boldsymbol{\xi}_{\perp} + \nabla_{\perp} \delta P - \frac{\mathbf{J}_0 \times \delta \mathbf{B}}{c} \right) \right] d\mathbf{r} .$$

- Integrating by parts, the $\propto \delta \boldsymbol{\xi}_{\perp}^{\dagger} \cdot \nabla_{\perp} \delta P$ term can be rewritten as

$$\frac{1}{2} \int \delta \boldsymbol{\xi}_{\perp}^{\dagger} \cdot \nabla_{\perp} \delta P d\mathbf{r} = \frac{1}{2} \int \nabla \cdot \delta \boldsymbol{\xi}_{\perp}^{\dagger} (\delta \boldsymbol{\xi} \cdot \nabla P_0 + \Gamma P_0 \nabla \cdot \delta \boldsymbol{\xi}) d\mathbf{r} ,$$

where, again, boundary terms are dropped and we have used δP expression at p. 3.

- We now use the identity

$$\nabla \cdot \delta \boldsymbol{\xi}_{\perp}^{\dagger} = \nabla \cdot \delta \boldsymbol{\xi}^{\dagger} - B_0 \nabla_{\parallel} \left(\delta \xi_{\parallel}^{\dagger} / B_0 \right) .$$

- In this way, upon integration by parts along the magnetic field line, we obtain

$$\begin{aligned} \frac{1}{2} \int \nabla \cdot \delta \boldsymbol{\xi}_{\perp}^{\dagger} (\Gamma P_0 \nabla \cdot \delta \boldsymbol{\xi}) d\mathbf{r} &= \frac{1}{2} \int \left(\Gamma P_0 |\nabla \cdot \delta \boldsymbol{\xi}|^2 + B_0 \nabla_{\parallel} \left(\delta \xi_{\parallel}^{\dagger} / B_0 \right) \delta P_{\text{comp}} \right) d\mathbf{r} , \\ &= \frac{1}{2} \int \left(\Gamma P_0 |\nabla \cdot \delta \boldsymbol{\xi}|^2 - \delta \xi_{\parallel}^{\dagger} \nabla_{\parallel} \delta P_{\text{comp}} \right) d\mathbf{r} = \frac{1}{2} \int \left(\Gamma P_0 |\nabla \cdot \delta \boldsymbol{\xi}|^2 + \delta \xi_{\parallel}^{\dagger} \rho_{m0} \partial_t^2 \delta \xi_{\parallel} \right) d\mathbf{r} \end{aligned}$$

- Collecting terms, the final form of the SAW energy functional becomes

$$\begin{aligned} \mathcal{F}[\Phi_s, \delta P_{\text{comp}}] &= \frac{1}{2} \int \left[|\delta \mathbf{B}|^2 / (4\pi) - \delta \boldsymbol{\xi}_{\perp}^{\dagger} \cdot (\mathbf{J}_0 \times \delta \mathbf{B}) / c \right. \\ &\quad \left. + (\nabla \cdot \delta \boldsymbol{\xi}_{\perp}^{\dagger}) (\delta \boldsymbol{\xi}_{\perp} \cdot \nabla P_0) + \Gamma P_0 |\nabla \cdot \delta \boldsymbol{\xi}|^2 \right. \\ &\quad \left. + \delta \boldsymbol{\xi}^{\dagger} \cdot \rho_{m0} \partial_t^2 \delta \boldsymbol{\xi} \right] d\mathbf{r} , \\ &= \delta W - \delta I . \end{aligned}$$

Here, δW represents the potential energy of the fluctuation, while δI is the plasma inertia response.

E: Comment about the physical meaning of the various terms in the expression above. Can you recognize the elements of the MHD energy principle? What is the most important difference in the present expression?

- In order to reduce the previous form of the SAW energy functional to the more familiar and **intuitive expression of δW_f in ideal MHD**, we must rewrite

$$-\delta\boldsymbol{\xi}_{\perp}^{\dagger} \cdot (\mathbf{J}_0 \times \delta\mathbf{B})/c + (\nabla \cdot \delta\boldsymbol{\xi}_{\perp}^{\dagger})(\delta\boldsymbol{\xi}_{\perp} \cdot \nabla P_0) = -(J_{0\parallel}/c)\delta\boldsymbol{\xi}_{\perp}^{\dagger} \times \mathbf{b}_0 \cdot \delta\mathbf{B}_{\perp} \\ -2\boldsymbol{\kappa} \cdot \delta\boldsymbol{\xi}_{\perp}^{\dagger}(\delta\boldsymbol{\xi}_{\perp} \cdot \nabla P_0) + \frac{B_0^2}{4\pi} |\nabla \cdot \delta\boldsymbol{\xi}_{\perp} + 2\boldsymbol{\kappa} \cdot \delta\boldsymbol{\xi}_{\perp}|^2 - \frac{\delta B_{\parallel}^2}{4\pi}$$

where we have used the expression

$$\frac{\delta B_{\parallel}}{B_0} = -\nabla \cdot \delta\boldsymbol{\xi}_{\perp} - 2\boldsymbol{\kappa} \cdot \delta\boldsymbol{\xi}_{\perp} + \frac{4\pi \nabla P_0}{B_0^2} \cdot \delta\boldsymbol{\xi}_{\perp}$$

following from the definition $\delta\mathbf{B} = \nabla \times (\delta\boldsymbol{\xi} \times \mathbf{B}_0)$.

E: Derive the expression of $\delta B_{\parallel}$ using Faraday's law (hint: follow Spring 2017 Lectures). Note that this expression depends only on the perpendicular plasma displacement. Can you explain why?

E: Calculate the expression of $\delta B_{\parallel}$ using the leading order expression for the perpendicular plasma displacement on p. 2. Can you comment on the condition under which it is negligible? (hint: use the equilibrium force balance).

□ Using the above results and substituting into the expression of the SAW energy functional, we obtain the equivalent form

$$\begin{aligned}
 \mathcal{F}[\Phi_s, \delta P_{\text{comp}}] &= \frac{1}{2} \int \left[\frac{|\delta \mathbf{B}_{\perp}|^2}{4\pi} + \frac{B_0^2}{4\pi} |\nabla \cdot \delta \boldsymbol{\xi}_{\perp} + 2\boldsymbol{\kappa} \cdot \delta \boldsymbol{\xi}_{\perp}|^2 + \Gamma P_0 |\nabla \cdot \delta \boldsymbol{\xi}|^2 \right. \\
 &\quad \left. - (J_{0\parallel}/c) \delta \boldsymbol{\xi}_{\perp}^{\dagger} \times \mathbf{b}_0 \cdot \delta \mathbf{B}_{\perp} - 2\boldsymbol{\kappa} \cdot \delta \boldsymbol{\xi}_{\perp}^{\dagger} (\delta \boldsymbol{\xi}_{\perp} \cdot \nabla P_0) \right. \\
 &\quad \left. + \delta \boldsymbol{\xi}^{\dagger} \cdot \rho_{m0} \partial_t^2 \delta \boldsymbol{\xi} \right] d\mathbf{r} , \\
 &= \delta W - \delta I .
 \end{aligned}$$

- As noted above, δW represents the potential energy of the fluctuation, while δI is the plasma inertia response.
- There is a direct correspondence between this form of the SAW energy functional and the MHD energy principle. However, contrary to MHD, where low-frequency waves are typically considered, inertia can contribute to the potential energy at finite frequency, e.g. as for Alfvén Eigenmodes and/or more general Alfvénic fluctuations.
- The criterion for separating inertia from potential energy is based on spatial scale separation (cf. later this Lecture): inertia is due to radial singular structures connected with the SAW continuous spectrum, while potential energy is due to regular structures connected with the usual MHD-like fluctuation behavior.

- Classification of terms in the SAW energy functional
- The first three terms are stabilizing and describe, respectively, field line bending stabilization, magnetic field compression stabilization and plasma compression stabilization
 - The two terms on the second line can be destabilizing and account for, respectively, ballooning-interchange drive (due to pressure-curvature coupling), and kink drive ($\propto J_{0\parallel}$)
 - The last term account for plasma inertia response

E: Review the classification of the various contributions in the SAW energy functional in the light of your knowledge of the MHD energy principle. Comment about the stabilizing and/or the destabilizing effect that each contribution can have. Can you qualitatively explain the meaning of bad curvature? Use a picture to support your argument.

The shear Alfvén wave variational principle.

- So far, we have demonstrated that the following SAW energy functional [Chen and Hasegawa 1991]

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = -\frac{1}{2} \int \nabla \Phi_s^\dagger \cdot \delta \mathbf{J} d\mathbf{r} = \frac{1}{2} \int \Phi_s^\dagger \nabla \cdot \delta \mathbf{J} d\mathbf{r} ,$$

reduces identically to the MHD energy principle in the low frequency limit, when the leading order asymptotic expressions for plasma displacement and magnetic fluctuation are used for SAW (Lecture 2 and Lecture 3 of Spring 2017 Intensive Course).

- Here, integrating by parts and using homogeneous boundary conditions (ideal conducting wall) at the plasma surface, we note that the SAW energy functional is obtained by multiplication of the vorticity equation ($\nabla \cdot \delta \mathbf{J} = 0$) by $\Phi_s^\dagger$ and integrating over the plasma domain [Chen and Hasegawa 1991].

- That the SAW energy functional is a proper variational principle, thus, logically follows from its equivalence to the MHD energy principle in the proper limit.

E: Review and discuss the MHD energy principle as a variational formulation of magnetohydrodynamics.

- Here, we follow the general methodology introduced in [Lecture 2](#) [Gerjuoy 1983] and show that the SAW energy functional is indeed a proper variational principle [Z & C 2014, C & Z RMP 2016].
- Introduce the following definitions and variational form ([Lecture 2](#)):
 - SAW vorticity equation (p. 4): $\mathcal{B}_A[\Phi_s, \delta P_{\text{comp}}] = 0$
 - SSW parallel force balance (p. 4): $\mathcal{B}_S[\Phi_s, \delta P_{\text{comp}}] = 0$

$$\begin{aligned} \mathcal{F}_v[\Phi_{s,t}, \delta P_{\text{comp},t}, \mathcal{L}_t, \tilde{\mathcal{L}}_t] = & \mathcal{F}[\Phi_{s,t}, \delta P_{\text{comp},t}] + \mathcal{L}_{at}^\dagger \mathcal{B}_A[\Phi_{s,t}, \delta P_{\text{comp},t}] + \mathcal{B}_A^\dagger[\Phi_{s,t}, \delta P_{\text{comp},t}] \mathcal{L}_{bt} \\ & + \tilde{\mathcal{L}}_{at}^\dagger \mathcal{B}_S[\Phi_{s,t}, \delta P_{\text{comp},t}] + \mathcal{B}_S^\dagger[\Phi_{s,t}, \delta P_{\text{comp},t}] \tilde{\mathcal{L}}_{bt} . \end{aligned}$$

- Here, we follow [Chen and Hasegawa 1991] as well as [Z & C 2014, C & Z RMP 2016] and choose the trial functions $[\Phi_{s,t}, \delta P_{\text{comp},t}]$ such that

$$\mathcal{B}_S[\Phi_{s,t}, \delta P_{\text{comp},t}] = 0 .$$

Thus, the auxiliary fields $\tilde{\mathcal{L}}$ are redundant (Lecture 2) and the variational form reduces to

$$\mathcal{F}_v[\Phi_{s,t}, \delta P_{\text{comp},t}, \mathcal{L}_t] = \mathcal{F}[\Phi_{s,t}, \delta P_{\text{comp},t}] + \mathcal{L}_{at}^\dagger \mathcal{B}_A[\Phi_{s,t}, \delta P_{\text{comp},t}] + \mathcal{B}_A^\dagger[\Phi_{s,t}, \delta P_{\text{comp},t}] \mathcal{L}_{bt} .$$

E: Comment about the choice $\mathcal{B}_S[\Phi_{s,t}, \delta P_{\text{comp},t}] = 0$ and discuss why the variational principle thereby obtained is nonetheless general and useful for the intended scope.

- The condition that $\mathcal{F}_v$ is a genuine variational principle reduces to

$$\delta \mathcal{F}_v = \delta \mathcal{F}[\Phi_s, \delta P_{\text{comp}}] + \mathcal{L}_a^\dagger \delta \mathcal{B}_A[\Phi_s, \delta P_{\text{comp}}] + \delta \mathcal{B}_A^\dagger[\Phi_s, \delta P_{\text{comp}}] \mathcal{L}_b = 0 .$$

E: Reconsider Lecture 2 (p. 6) and show that the above condition actually ensures that the considered functional is a proper variational principle. What can you say about variations of Φ_s and δP_{comp} ? Are they independent?

- Explicitating the various terms, and noting $\mathcal{B}_A[\Phi_s, \delta P_{\text{comp}}] = \nabla \cdot \delta \mathbf{J}$, we can rewrite

$$\delta \mathcal{F}_v = \delta \frac{1}{2} \int \Phi_s^\dagger \nabla \cdot \delta \mathbf{J} d\mathbf{r} + \int \mathcal{L}_a^\dagger \delta (\nabla \cdot \delta \mathbf{J}) d\mathbf{r} + \int \delta (\nabla \cdot \delta \mathbf{J})^\dagger \mathcal{L}_b d\mathbf{r} = 0 .$$

- Imposing independent variations of Φ_s and $\Phi_s^\dagger$, the vanishing of the variation with respect to $\delta \Phi_s$ imposes

$$\mathcal{L}_a^\dagger = -\frac{1}{2} \Phi_s^\dagger .$$

E: Demonstrate the previous result. Hint: impose the vanishing of the variation with respect to $\delta (\nabla \cdot \delta \mathbf{J})$.

- Meanwhile, integrating by parts and imposing the vanishing of the variation with respect to $\delta \nabla \Phi_s^\dagger$, we obtain

$$-\frac{1}{2} \int \delta (\nabla \Phi_s^\dagger) \cdot \delta \mathbf{J} d\mathbf{r} - \int \delta (\nabla_i \Phi_s^\dagger) \left\{ \frac{\delta(\delta J_k^\dagger)}{\delta (\nabla_i \Phi_s^\dagger)} \right\} (\mathbf{r}) \nabla_k \mathcal{L}_b(\mathbf{r}) d\mathbf{r} = 0 .$$

- Thus, the problem of determining $\mathcal{L}_b$ in the most general case involves the definition of variational/functional derivative ([Lecture 1](#), p. 20), which may be difficult to solve.
- The problem has a simple and general solution, when the current $\delta \mathbf{J}$ is related to Φ_s as a linear functional through, e.g., a linear integral operator Σ (see [Lecture 4](#) for more details)

$$\delta \mathbf{J}(\mathbf{r}, t) = \int \Sigma(\mathbf{r}, t, \mathbf{r} - \mathbf{x}, t - \tau) \cdot \nabla \Phi_s(\mathbf{r} - \mathbf{x}, t - \tau) d\mathbf{x} d\tau ,$$

$$\delta \mathbf{J}^\dagger(\mathbf{r}, t) = \int \nabla \Phi_s^\dagger(\mathbf{r} - \mathbf{x}, t - \tau) \cdot \Sigma^\dagger(\mathbf{r} - \mathbf{x}, t - \tau, \mathbf{r}, t) d\mathbf{x} d\tau .$$

using the definition $[\Sigma^\dagger(\mathbf{r}, t, \mathbf{r}', t')]_{\mu\nu} = [\Sigma(\mathbf{r}', t', \mathbf{r}, t)]_{\nu\mu}^*$.

- Thus, using the above definitions of integral operator and its adjoint, we have

$$\begin{aligned} \int \delta (\nabla \cdot \delta \mathbf{J})^\dagger \mathcal{L}_b d\mathbf{r} &= \\ &= - \int \delta (\nabla \Phi_s^\dagger) \cdot \left[\int \Sigma^\dagger(\mathbf{r}, t, \mathbf{r} + \mathbf{x}, t + \tau) \cdot \nabla \mathcal{L}_b(\mathbf{r} + \mathbf{x}, t + \tau) d\mathbf{x} d\tau \right] d\mathbf{r} \end{aligned}$$

- Meanwhile, we also have

$$\begin{aligned} -\frac{1}{2} \int \delta (\nabla \Phi_s^\dagger) \cdot \delta \mathbf{J} d\mathbf{r} &= \\ &= -\frac{1}{2} \int \delta (\nabla \Phi_s^\dagger) \cdot \left[\int \Sigma(\mathbf{r}, t, \mathbf{r} - \mathbf{x}, t - \tau) \cdot \nabla \Phi_s(\mathbf{r} - \mathbf{x}, t - \tau) d\mathbf{x} d\tau \right] d\mathbf{r} \end{aligned}$$

- Summing the RHS of the last two equations and equating to zero gives the equation for $\mathcal{L}_b$.

- Substituting into the variational principle

$$\begin{aligned}
 \mathcal{F}_v &= - \int \delta \mathbf{J}^\dagger \cdot \nabla \mathcal{L}_b d\mathbf{r} \\
 &= - \int \nabla \Phi_s^\dagger \cdot \left[\int \Sigma^\dagger(\mathbf{r}, t, \mathbf{r} + \mathbf{x}, t + \tau) \cdot \nabla \mathcal{L}_b(\mathbf{r} + \mathbf{x}, t + \tau) d\mathbf{x} d\tau \right] d\mathbf{r} \\
 &= \frac{1}{2} \int \nabla \Phi_s^\dagger \cdot \left[\int \Sigma(\mathbf{r}, t, \mathbf{r} - \mathbf{x}, t - \tau) \cdot \nabla \Phi_s(\mathbf{r} - \mathbf{x}, t - \tau) d\mathbf{x} d\tau \right] d\mathbf{r} \\
 &= \frac{1}{2} \int \nabla \Phi_s^\dagger \cdot \delta \mathbf{J} d\mathbf{r} .
 \end{aligned}$$

- This result confirms that the functional on p. 14 is a genuine variational principle ([E: Comment about the sign change. Does it have any impact on the present result?](#)). This was to be expected and coincides with the fact that the MHD energy principle reduces to the SAW energy functional in the proper limit.

E: Assume now that $\delta \mathbf{J}$ is a linear functional depending on $\nabla \Phi_s$ and higher order derivatives (Why? See pp. 6-7). Calculate the functional derivative on p. 18 and derive an equation for $\mathcal{L}_b$. Can you show that the SAW energy functional is still a good variational principle?

- This result is very general and relies on the fact that $\delta \mathbf{J}$ is a linear functional. However, the present method suggest a practical way to construct a variational principle that reduces to the MHD energy principle even in the more general nonlinear case. Doing so is beyond the scope of these Lectures.
- Next, we will show the criterion for separating inertia from potential energy based on spatial scale separation: inertia is due to radial singular structures connected with the SAW continuous spectrum, while potential energy is due to regular structures connected with the usual MHD-like fluctuation behavior.
- We will finally show that the SAW energy functional is an integral form for the fluctuation dispersion relation.

Mode structure decomposition

(from Spring 2017 Lecture 2)

- In axisymmetric tokamaks, a generic fluctuation field $f(r, \theta, \zeta)$, assuming implicitly $\exp(-i\omega t)$ time dependence, can be expressed as

$$f(r, \theta, \zeta) = e^{in\zeta} \sum_{m \in \mathbb{Z}} e^{-im\theta} f_n(m, r) .$$

- Here, in linear physics, only a single- n mode is considered. Note that for low-frequency ($|\omega| \ll |\Omega_i|$) modes of interest here, finite $k_{\parallel} = |\mathbf{b}_0 \cdot \nabla|$ generally plays crucial stabilizing roles; such that instabilities tend to minimize $k_{\parallel}$.
- Noting the mode decomposition above, we have

$$K_{\parallel} = |k_{\parallel} q R_0| \simeq |nq - m| .$$

- Thus, $f_n(m, r)$ tends to be localized about its corresponding mode-rational surface located at $r = r_m$ where $nq(r_m) = m$. In other words, $f_n(m, r)$ generally should have a strong dependence on $(nq - m)$.

- This suggests to explicitly express $f_n(m, r)$ as $f_n(r, nq - m)$, where the first r dependence corresponds to other “weak” dependence on r :

$$f(r, \theta, \zeta) = e^{in\zeta} \sum_{m \in \mathbb{Z}} e^{-im\theta} f_n(r, nq - m) .$$

- Introducing $\hat{f}_n(r, \vartheta)$ as the Fourier conjugate of $f_n(r, nq - m)$; *i.e.*,

$$f_n(r, nq - m) = \int_{-\infty}^{\infty} e^{i(m-nq)\vartheta} \hat{f}_n(r, \vartheta) d\vartheta ,$$

the mode structure decomposition becomes

$$f(r, \theta, \zeta) = e^{in\zeta} \int_{-\infty}^{\infty} e^{-inq\vartheta} \hat{f}_n(r, \vartheta) \sum_{m \in \mathbb{Z}} e^{im(\vartheta - \theta)} d\vartheta .$$

- Applying the Poisson summation formula

$$\sum_{m \in \mathbb{Z}} e^{im(\vartheta - \theta)} = 2\pi \sum_{\ell \in \mathbb{Z}} \delta(\vartheta - \theta + 2\pi\ell) ,$$

we then have [Lu 12]

$$f(r, \theta, \zeta) = 2\pi \sum_{\ell \in \mathbb{Z}} e^{in\zeta - inq(\theta - 2\pi\ell)} \hat{f}_n(r, \theta - 2\pi\ell) .$$

- Mode structure decompositions (MSD) introduced above are valid for general toroidal mode number, n [Lu 12], and introduce and define the projection operator $\mathcal{P}_{Bn}(r, \vartheta) : f(r, \theta, \zeta) \mapsto \hat{f}_n(r, \vartheta)$, such that MSD reads

$$f(r, \theta, \zeta) = e^{in\zeta} \sum_{m \in \mathbb{Z}} e^{-im\theta} \int e^{i(m-nq)\vartheta} \mathcal{P}_{Bn}(r, \vartheta) [f] d\vartheta .$$

- $\mathcal{P}_{Bn}(r, \vartheta)$ properties allow interpreting ϑ as an extended poloidal angle
- Multiplication by a periodic function $p(\theta)$ in (r, θ) space corresponds to multiplication by a periodic function $p(\vartheta)$ in (r, ϑ) space
 - in Clebsch coordinates (r, θ, ξ) with $\xi \equiv \zeta - q\theta$,
 $\mathbf{b} \cdot \nabla = (\mathcal{J}B_0)^{-1} \partial_\theta \mapsto (\mathcal{J}B_0)^{-1} \partial_\vartheta$

• Finally,

$$\begin{aligned} \nabla_\perp &\mapsto \nabla r \left(-inq' \vartheta + \frac{\partial}{\partial r} \right) + in \nabla \zeta + \nabla \theta \left(\frac{\partial}{\partial \vartheta} - inq \right) - \frac{\mathbf{b}}{\mathcal{J}B_0} \frac{\partial}{\partial \vartheta} \\ &\mapsto \nabla r \frac{\partial}{\partial r} + \nabla \xi \frac{\partial}{\partial \xi} + \nabla \theta \frac{\partial}{\partial \vartheta} - \frac{\mathbf{b}}{\mathcal{J}B_0} \frac{\partial}{\partial \vartheta} . \end{aligned}$$

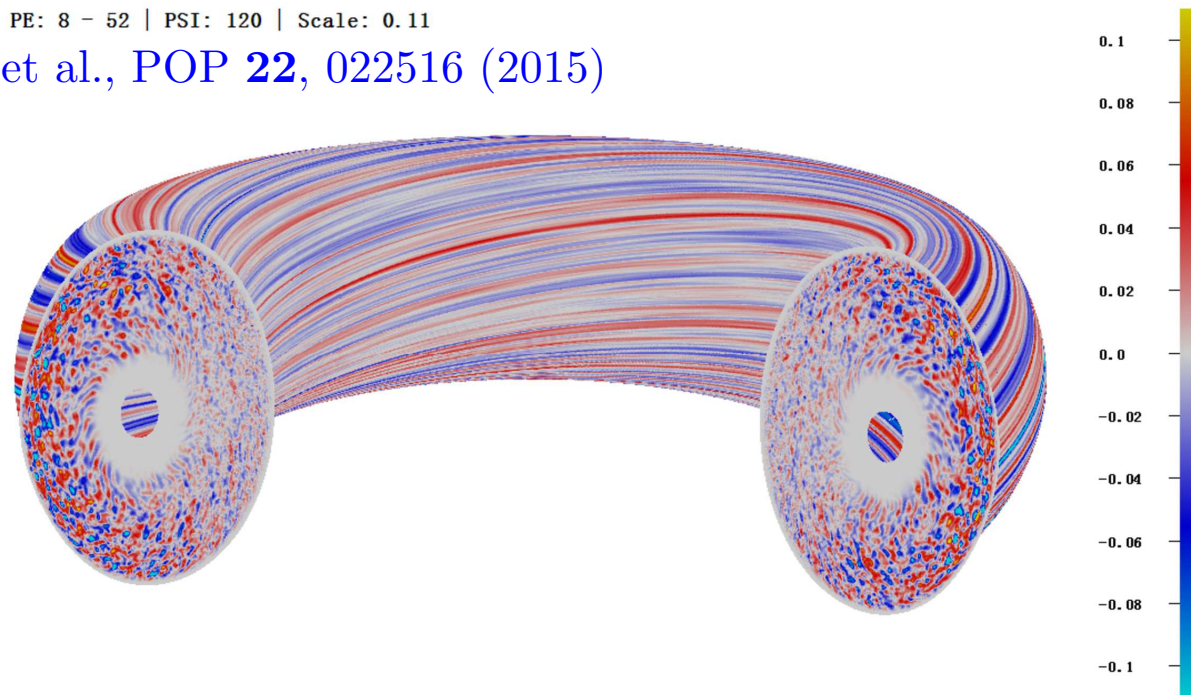
E: Demonstrate all the properties of the projection operator $\mathcal{P}_{Bn}(r, \vartheta)$.

- The MSD approach is valid for general toroidal mode number, n [Lu 12]. For high toroidal mode numbers, $|n| \gg 1$, this approach reduces to the well-known “ballooning-mode” representation.

- Representation is valid for general mode structures in toroidal systems, from macroscopic to microscopic.

$T = 3500$ | $PE: 8 - 52$ | $PSI: 120$ | $Scale: 0.11$

Courtesy of Y. Xiao et al., POP **22**, 022516 (2015)



RP: Exploit this formalism, showing it can recover usual representation and is readily extended to micro-scale (turbulence) and meso-scale fluctuations with its embedded field-aligned flux coordinates. Construct the corresponding algorithms for an elementary problem of your choice.

Application to the SAW energy functional

- The MSD approach, with its general validity [Lu 12], shows that the most natural way to isolate radial singular structures ($|k_r| \rightarrow \infty$) is to take the $|\vartheta| \rightarrow \infty$ limit after transforming from (r, θ, ζ) to (r, ϑ, ζ) space.
- From Spring 2017 [Lecture 2](#) and p. 24, this is achieved by the transformation $\mathcal{P}_{Bn}(r, \vartheta) : f(r, \theta, \zeta) \mapsto \hat{f}_n(r, \vartheta)$,

$$f(r, \theta, \zeta) = e^{in\zeta} \sum_{m \in \mathbb{Z}} e^{-im\theta} \int e^{i(m-nq)\vartheta} \mathcal{P}_{Bn}(r, \vartheta) [f] d\vartheta .$$

- When applied to a functional, such as the SAW energy functional, the [MSD integral transformation](#) is applied to the product of two fluctuations $\Rightarrow$ a convolution integral is expected [Z & C 2014]

$$\mathcal{P}_{Bn}(r, \vartheta) [fg] = 2\pi \sum_{h,j \in \mathbb{Z}} e^{-2i\pi hjq} \mathcal{P}_{Bn-h}(r, \vartheta) [f] \mathcal{P}_{Bh}(r, \vartheta + 2\pi j) [g] .$$

- This definition follows from the definition of the integral transform defining the MSD (**E: Show that this is indeed the case!**).
- Now apply the approach to the SAW energy functional, which, from pp. 5, 14, is given by

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = -\frac{1}{2} \int \nabla \Phi_s^\dagger \cdot \delta \mathbf{J} d\mathbf{r} = \frac{1}{2} \int \Phi_s^\dagger \nabla \cdot \delta \mathbf{J} d\mathbf{r} ,$$

- Applying toroidal flux coordinates

$$\int d\mathbf{r}(\dots) = \int_0^{2\pi} d\zeta \int_0^{2\pi} d\theta \int_0^a \mathcal{J}(d\psi/dr) dr(\dots) .$$

- Using the MSD rule introduce on p. 27, and noting $\mathcal{J}(r, \theta) = \mathcal{J}(r, \vartheta)$ for the Jacobian,

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = (2\pi)^3 \sum_{j \in \mathbb{Z}} \int_0^a dr \frac{d\psi/dr}{2} e^{i2\pi n q j} \int_{-\infty}^{\infty} \mathcal{J} d\vartheta \hat{\Phi}_{sn}^\dagger(r, \vartheta + 2\pi j) \nabla \cdot \delta \hat{\mathbf{J}}_n(r, \vartheta) .$$

E: Derive the above result step by step, using the MSD transformation rule and properties summarized at p. 27. Can you repeat this calculation just based on the definition given on pp. 23-24?

E: Comment about the structure of the SAW energy functional in the MSD representation. Is it more or less complicated than its original form? How many variables/dimensions are involved? Why?

- The natural way to **separate inertia from potential energy**, based on spatial scale separation, is connected with $|k_r| \rightarrow \infty$ as $|\vartheta| \rightarrow \infty$.
- introduce $\vartheta_1 \gg 1$ as a large control parameter
 - recall that in (r, ϑ) space $\nabla_{\parallel} = (\mathcal{J}B_0)^{-1} \partial_{\vartheta}$
 - use the same integration by parts on p. 5 to write

$$\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] = \frac{1}{2} \int \left(\frac{|\nabla_{\perp} \Psi_s|^2}{4\pi} + \Phi_s^{\dagger} \nabla \cdot \delta \mathbf{J}_{\perp} \right) d\mathbf{r} .$$

- Explicitly writing the MSD representation of the SAW energy functional

$$\begin{aligned}\mathcal{F}[\Phi_s, \delta P_{\text{comp}}] &= (2\pi)^3 \sum_{j \in \mathbb{Z}} \int_0^a dr \frac{d\psi/dr}{2} e^{i2\pi n q j} \int_{-\infty}^{\infty} \mathcal{J} d\vartheta \left[\frac{1}{4\pi} \nabla_{\perp} \hat{\Psi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \right. \\ &\quad \left. \cdot \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) + \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \nabla \cdot \delta \hat{\mathbf{J}}_{\perp n}(r, \vartheta) \right] \\ &= \delta W - \delta I .\end{aligned}$$

- By definition, we now divide the $(-\infty, \infty)$ interval in ϑ in three segments: $(-\infty, -\vartheta_1]$, $[-\vartheta_1, \vartheta_1]$, and $[\vartheta_1, \infty)$. Conventionally, and using $\vartheta_1 \gg 1$ as a parameter, we identify [Z & C 2014]

$$\begin{aligned}\delta W &= \lim_{\vartheta_1 \rightarrow \infty} (2\pi)^3 \sum_{j \in \mathbb{Z}} \int_0^a dr \frac{d\psi/dr}{2} e^{i2\pi n q j} \int_{-\vartheta_1}^{\vartheta_1} \mathcal{J} d\vartheta \left[\frac{1}{4\pi} \nabla_{\perp} \hat{\Psi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \right. \\ &\quad \left. \cdot \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) + \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \nabla \cdot \delta \hat{\mathbf{J}}_{\perp n}(r, \vartheta) \right] ,\end{aligned}$$

and $-\delta I$ the remaining contribution due to radial singular structures.

- The integration over $(-\infty, -\vartheta_1]$ and $[\vartheta_1, \infty)$ is carried out as follows, recalling that $\hat{\Psi}_s = -(c/\mathcal{J}B_0)\partial_{\vartheta}\hat{\Phi}_s$ and $\delta\hat{J}_{\parallel} = -(c/4\pi)\nabla_{\perp}^2\hat{\Psi}_s$.
- Let $\nabla_{\perp}^2 \equiv -k_{\vartheta}^2\hat{\kappa}_{\perp}^2$ and $k_{\vartheta} \equiv (-nq/r)$, with $\kappa_{\perp}^2 \simeq s^2\vartheta^2|\nabla r|^2$ in the $|\vartheta| \gg 1$ region of interest (s being the magnetic shear; cf. p. 25).
- Integrating by parts (note the sign and that $k_{\vartheta,n} = -k_{\vartheta,-n}$)

$$\begin{aligned}
 & \left(\int_{-\infty}^{-\vartheta_1} + \int_{\vartheta_1}^{\infty} \right) \left[\frac{\mathcal{J}d\vartheta}{4\pi} \nabla_{\perp} \hat{\Psi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \cdot \nabla_{\perp} \hat{\Psi}_{sn}(r, \vartheta) \right] \\
 & \simeq \left(\int_{-\infty}^{-\vartheta_1} + \int_{\vartheta_1}^{\infty} \right) \left[\frac{k_{\vartheta}^2 c^2 d\vartheta}{4\pi \mathcal{J} B_0^2} s^2 \vartheta^2 |\nabla r|^2 \partial_{\vartheta} \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \partial_{\vartheta} \hat{\Phi}_{sn}(r, \vartheta) \right] \\
 & \simeq - \left[\frac{k_{\vartheta}^2 c^2}{4\pi \mathcal{J} B_0^2} s^2 \vartheta^2 |\nabla r|^2 \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \partial_{\vartheta} \hat{\Phi}_{sn}(r, \vartheta) \right]_{-\vartheta_1}^{\vartheta_1} \\
 & + \left(\int_{-\infty}^{-\vartheta_1} + \int_{\vartheta_1}^{\infty} \right) \left[\mathcal{J} B_0 d\vartheta \hat{\Phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \nabla_{\parallel} \left(\frac{\delta \hat{J}_{\parallel}}{B_0}(r, \vartheta) \right) \right] .
 \end{aligned}$$

- In the last expression, the second integral term adds up exactly to the $\propto \nabla \cdot \delta \hat{\mathbf{J}}_{\perp}$ term and makes the whole integral in $d\vartheta$ vanish, due to the SAW vorticity equation.
- This result shows that the inertial term is due to the first line contribution in the last expression, which is due to the jump of the solution across the region $[-\vartheta_1, \vartheta_1]$.
- For convenience (cf. Spring 2017 [Lecture 2](#), p. 30), it is possible to introduce $\hat{\phi}_s \equiv \hat{\kappa}_{\perp} \hat{\Phi}_s$. In this way

$$\delta I = (2\pi)^3 \sum_{j \in \mathbb{Z}} \int_0^a dr \frac{d\psi/dr}{2} e^{i2\pi n q j} \left[\frac{k_{\vartheta}^2 c^2}{4\pi \mathcal{J} B_0^2} \hat{\phi}_{sn}^{\dagger}(r, \vartheta + 2\pi j) \partial_{\vartheta} \hat{\phi}_{sn}(r, \vartheta) \right]_{\vartheta \rightarrow 0^-}^{\vartheta \rightarrow 0^+}.$$

- Note that, here, the value of θ_1 has been taken as $\rightarrow 0^+$. This is consistent with the present asymptotic matching. In fact, $\vartheta_1 \rightarrow \infty$ in the short length scale limit adopted for computing δW ; and $\vartheta_1 \rightarrow 0^+$ in the large $|\vartheta|$ region, where the radial singular structures are computed [Z & C 2014].

The integral form of the dispersion relation

- From Spring 2017 [Lecture 3](#) (p. 26), we know that the general solution for the radial singular structures of the SAW continuous spectrum is in the form

$$\hat{\phi}_s = e^{i\nu\vartheta} P(\vartheta) ,$$

where ν is the [Floquet characteristic exponent](#) and $P(\vartheta)$ is periodic.

- This means that, using the Poisson summation formula at p. 24, the inertia term picks up [radial singular contributions at \$nq - m \simeq \mathbb{R}\nu\$](#) .
- Assuming that this occurs at one isolated position $r = r_0$, the expression for the inertia becomes

$$\delta I = 2\pi^2 \left[\frac{|k_\vartheta| c^2 (d\psi/dr)}{\mathcal{J} B_0^2 |s|^2} \right]_{r=r_0, \vartheta=0} \left| \hat{\phi}_{sn0+} \right|^2 i\Lambda |s|$$

$$i\Lambda = \left(2 \left| \hat{\phi}_{sn0+} \right|^2 \right)^{-1} \left[\hat{\phi}_{sn}^\dagger(r, \vartheta) \partial_\vartheta \hat{\phi}_{sn}(r, \vartheta) \right]_{\vartheta \rightarrow 0^-}^{\vartheta \rightarrow 0^+} .$$

- Using the same (arbitrary) normalization, one can set

$$\delta W = 2\pi^2 \left[\frac{|k_\vartheta|^2 c^2 (d\psi/dr)}{\mathcal{J} B_0^2 |s|^2} \right]_{r=r_0, \vartheta=0} \left| \hat{\phi}_{sn0+} \right|^2 \delta \hat{W}_f .$$

- This definition was introduced for the analysis of the fishbone instability, using the jump of the MHD radial displacement across the singular layer as normalization condition [CWR 1984].
- In this way, the variational principle obtained for the SAW energy functional becomes an integral form of the dispersion relation, which reads

$$i|s|\Lambda = \delta \hat{W}_f .$$

E: Derive the explicit form of $i\Lambda$ for the fishbone problem. (Hint: look at pp. 9-13 of Spring 2017 Lecture 3).

E: Read the classic paper by [CWR 1984] and discuss the meaning of adding a

$\delta\hat{W}_k$ term on the RHS of the above dispersion relation to describe the fishbone excitation by energetic particles.

- The integral (global) form of the dispersion relation derived above is the simplest form of the general fishbone like dispersion relation (GFLDR) [Z & C 2014], which can be further reduced to a local WKB dispersion relation for short wavelength modes (cf. Lecture 4).

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