# Lecture 1

## Hamiltonian mapping techniques and applications to fusion plasmas

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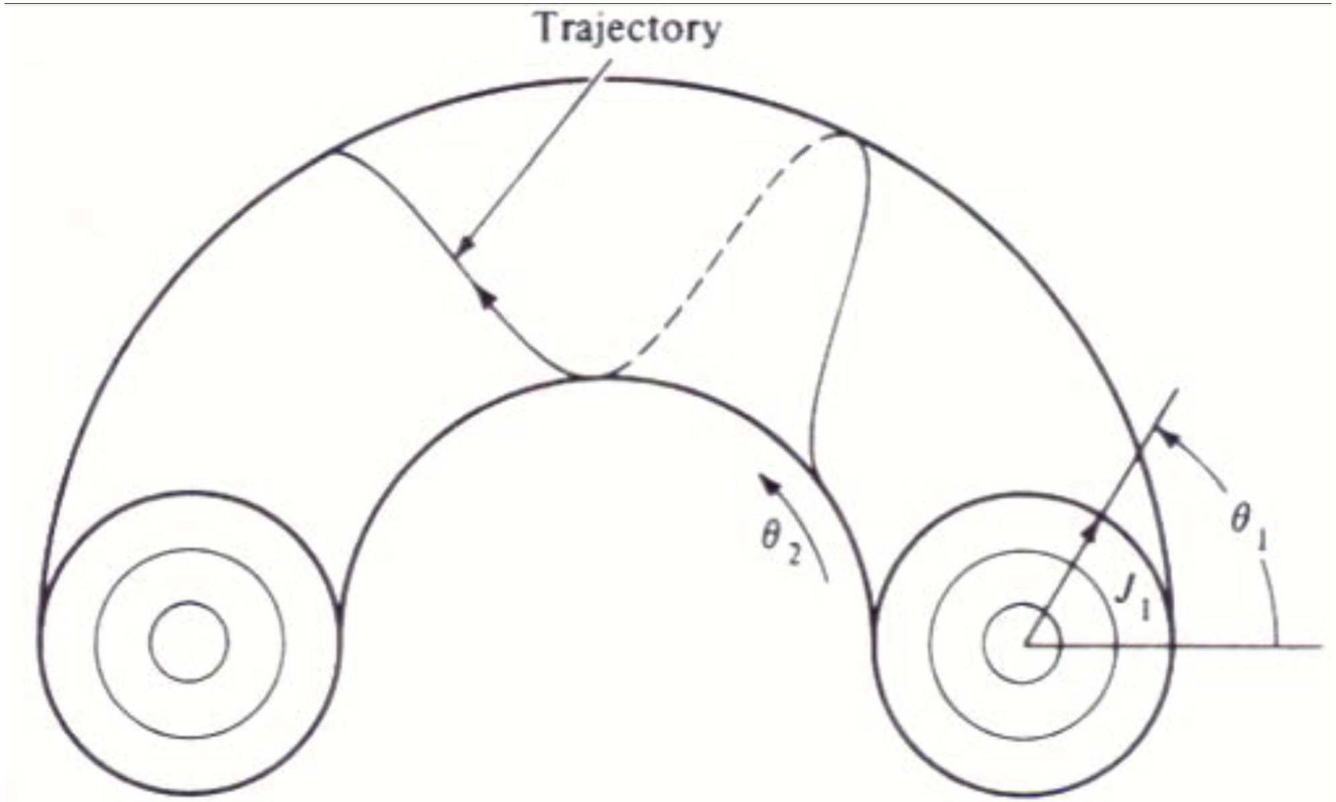
# General notion of mapping

- Toroidal fusion plasmas are characterized by double periodicity.




- Visualization of multiply periodic motions is important for increased understanding.

- Topological considerations that can be made, naturally yield to a set of difference equations, that can be seen as *mapping* of the dynamical trajectory onto a subset of the system phase space.
- Not only visualization (and understanding) are often simpler via mapping equations. Physics properties, such as numerical calculation of stochastic behaviors, are often simpler when considered for mappings.
- Conversely, regular trajectories are very conveniently described in terms of differential equations.
- All this belongs to the broad subject of Hamiltonian mechanics. This Lecture is concerned with providing a brief recap of general concepts and examples of applications to fusion plasmas.

# Integrable systems

- From Spring 2011 Lectures, we know that – given a system with $N$ degrees of freedom – the system is integrable (also completely integrable or completely separable) if the Hamilton-Jacobi equation is separable into $N$ independent equations.

- The constants $\alpha_i$

$$H_i\left(\frac{\partial S_i}{\partial q_i}, q_i\right) = \alpha_i \quad ; \quad \bar{H} = \sum_i \alpha_i = H_0$$

are called isolating integrals or global invariants of the motion, for they isolate each degree of freedom by the property

$$\frac{\partial H}{\partial p_i} = f(q_i)$$

in some canonical coordinate system.

- A Hamiltonian with $N$ degrees of freedom is integrable if and only if $N$ independent isolating integrals exist.
- A sufficient condition for integrability is that the $N$ independent integrals are in involution, i.e. their Poisson brackets with each other vanish

$$[\alpha_i, \alpha_j] = 0$$

- This condition ensures that $\boldsymbol{\alpha}$ is a complete set of new momenta in some transformed coordinate system. Any complete set of $N$ functions of $\boldsymbol{\alpha}$, such as the action variables $\boldsymbol{J}$, are a set of isolating integrals, for their Poisson brackets automatically vanish.

# The extended phase space

- The Least Action Principle, underlying the derivation of the equations of motion, can be rewritten introducing a parameter $\zeta$, with respect to which all variations are independent; i.e.

$$\delta \left[ \int \left( \boldsymbol{p} \frac{d\boldsymbol{q}}{d\zeta} - H(\boldsymbol{p}, \boldsymbol{q}, t) \frac{dt}{d\zeta} \right) d\zeta \right] = 0$$

- From the form of the variational principle, it is intuitive that we can define an extended phase space setting in $2N + 2$ dimensions by letting $\bar{p}_i = p_i$, $\bar{q}_i = q_i$ for $i = 1, N$ and $\bar{p}_{N+1} = -H$, $\bar{q}_{N+1} = t$.

- This corresponds to introducing the generating function

$$F_2 = \sum_{i=1}^{N} \bar{p}_i q_i - Ht \;\; ; \;\; \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = H(\boldsymbol{p}, \boldsymbol{q}, t) - H$$

- Equations of motion in the Hamiltonian form are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} \;\; ; \;\; \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i}$$

- $\bar{H}$ is independent of the parameter (time-like) $\zeta$, so that $\bar{H} = \text{const}$, while $t(\zeta) = \zeta + \zeta_0 = \zeta$, for $\zeta_0$ can be arbitrarily set to zero.

# The reduced phase space

- Given the concept of the extended phase space, every system can be described as autonomous system, i.e. with an Hamiltonian that does not explicitly depend on time; i.e.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0$$

- Conversely, one could use this equation to solve one of the momenta, say $p_N$, as a function of $\bar{p}_i = p_i$ and $\bar{q}_i = q_i$ for $i = 1, N-1$ and $\zeta = q_N$ considered as a parameter; $p_N = p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$. Letting $\bar{H} = -p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$, the equations of motion in the $2N-2$ dimensional reduced phase space are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} = -\frac{1}{\dot{q}_N}\frac{\partial H}{\partial q_i} \quad ; \quad \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i} = \frac{1}{\dot{q}_N}\frac{\partial H}{\partial p_i}$$

- If the momentum $p_N$ is a constant, since the original $H(\boldsymbol{p}, \boldsymbol{q})$ did not explicitly depend on $q_N$, then $\bar{H}$ is constant and the reduced phase space can be further reduced.

E: Comment this last statement in the light of the remark given in Lecture 3 of the Spring 2010 intensive course on *Resonant and non-resonant interactions in beam-plasma systems*; i.e. that the general problem of fluctuations induced transport by gyrokinetic turbulence in toroidal systems is equivalent to an autonomous Hamiltonian with 3 degrees of freedom. $\Rightarrow$ Lecture 2 (GK transport theory)

- A very useful application of the notion of reduced phase space is the definition of a Poincaré surface of section. (example of Hamiltonian mapping)

- For a 2D system, the motion is bound to occur within the energy surface $H(p_1, p_2, q_1, q_2) = H_0$, which can be written as

$$p_2 = p_2(p_1, q_1, q_2)$$

- If the motion is bounded in the phase space, the motion can repeatedly cross the plane $q_2 =$ const, which is a convenient choice of the surface of section, coinciding with the reduced phase space of the original Hamiltonian system.

- In general, the subsequent crossings of the motion in the $(p_1, q_1)$ surface of section can occur everywhere. However, if a constant of motion exist, in addition to $H_0$, then $I(p_1, p_2, q_1, q_2) =$ const in addition to $p_2 = p_2(p_1, q_1, q_2)$. Therefore

$$p_1 = p_1(q_1, q_2)$$

  and the subsequent crossings of the motion in the $(p_1, q_1)$ surface of section must lie on one curve.

- Vice-versa, the fact that the motion in the $(p_1, q_1)$ surface of section lies on a curve can be used as evidence of the existence of a constant of motion in addition to $H_0$.

# Near-integrable systems

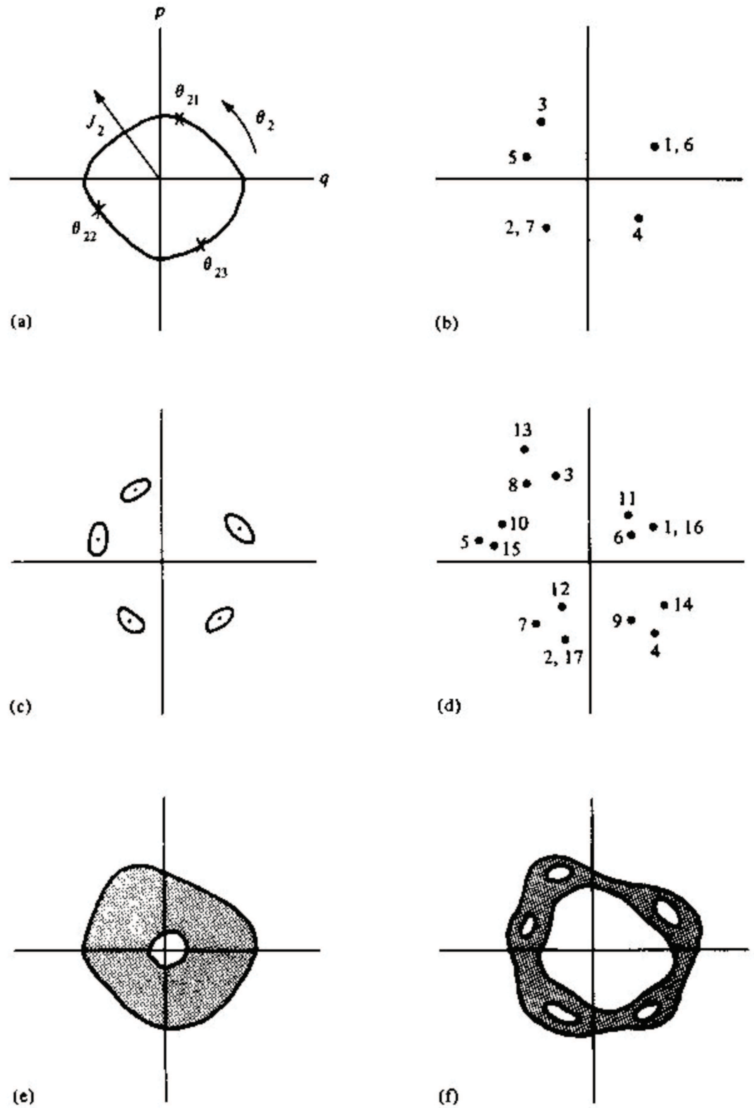
- From the definition of integrability given earlier, we can define a system as near-integrable if it can be described as perturbation of an integrable system.
- The peculiar nature of near-integrable systems is to be characterized by both regular trajectories and regions of stochasticity, which are all consequence of the deterministic solutions of Hamilton's equations.
- For systems with two degrees of freedom, stochastic regions are separated from regular trajectories by regular curves.
- In systems with more than two degrees of freedom, regular trajectories no longer separate stochastic regions and the systems are characterized by the phenomenon of *Arnold diffusion*.

# Systems with two degrees of freedom: resonances

- The Hamiltonian of a near integrable periodic system with two degrees of freedom, in the action-angle variables of the unperturbed motion $\boldsymbol{J}, \boldsymbol{\theta}$, can be written as

$$H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \epsilon H_1(J_1, J_2, \theta_1, \theta_2)$$

- The trajectories lie in the 3-dimensional surface $H =$ const. of the four dimensional phase space.
- Regular trajectories depend discontinuously on initial conditions: their presence does not imply the existence of an isolating integral (global invariant) of the motion (or a corresponding symmetry).
- Still, where regular trajectories exist, they represent exact invariants of the motion.




Surface of section plots showing type of trajectories:
(a) regular trajectory of a conditionally periodic system
(b) fixed (or periodic) points of a primary resonance for a periodic system
(c) trajectories form primary islands near fixed points
(d) secondary resonance due to the coupling of periodic motion around a primary island and the unperturbed motion
(e) annular layer of stochasticity between two invariant curves
(f) stochastic layer bounded by primary and secondary invariant curves
(*Lieberman and Lichtenberg*, 2010).

- Regular trajectories are either

  - conditionally periodic: they densely cover a toroidal surface of constant action, where the angle coordinates run around the two directions of the surface with frequencies, whose ratio is not a rational number

  - closed periodic curves, winding around the torus an integral number of turns. This is the case of a resonance, with $k\omega_1(\boldsymbol{J}) + l\omega_2(\boldsymbol{J}) = 0$

- Primary resonances give rise to primary islands, which give rise to secondary resonances and their islands and so on.

- Regions of stochastic trajectories fill a finite portion of the energy surface in phase space. Periodic trajectories exist in these regions, but trajectories near them do not move on stable islands near them.

- Stochastic motion always occurs near separatrices between invariant curves and their islands. This occurs because:

  - the island oscillation frequency $\omega$ goes to zero with respect to the frequency of the unperturbed motion $\omega_0$

  - the separation between actions of neighboring resonances ($k$ and $k \pm 1$), $k\omega - \omega_0$, tends to zero as the separatrix is approached

  - the stochasticity region formed near a separatrix is called the *resonance layer*

- For small $\epsilon$ and for systems with two degrees of freedom, resonance layers are thin and separated by invariant curves.

- For increasing $\epsilon$, invariant curves separating different island chains are strongly perturbed and eventually destroyed.

- Merging of primary resonance layers leads to the appearance of global or strong stochasticity .

# Systems with more than two degrees of freedom

- In systems with two degrees of freedom, resonance layers are divided and isolated from each other by two dimensional KAM surfaces.

E: Show that two dimensional KAM surfaces divide the three dimensional space $H(\boldsymbol{J}, \boldsymbol{\theta}) = E$ into a set of closed volumes.

- For systems with three degrees of freedom or more, the three dimensional (or more) KAM surfaces cannot divide the five-dimensional (or more) $H(\boldsymbol{J}, \boldsymbol{\theta}) = E$ into closed volumes.
- For $N > 2$ degrees of freedom, the $N$-dimensional KAM surfaces do not divide the $2N-1$-dimensional constant-energy volume into distinct regions and all stochastic layers of the energy surface in the phase space are connected into a single complex network, the *Arnold web*.
- The Arnold web permeates the entire phase space intersecting or lying arbitrarily close to every point.

- For initial conditions within the web, the subsequent motion will eventually intersect every finite region of the constant-energy volume even for $\epsilon \to 0$.
- The merging of stochastic trajectories into a single web was demonstrated by Arnold for a specific nonlinear Hamiltonian (*Arnold*, 1964). A general proof for a general Hamiltonian does not exist yet, but many numerical examples are available.
- The diffusion rate along resonant layers has been given by Chirikov (*Chirikov*, 1979) for the case of three resonances.
- For the coupling among many resonances, a rigorous upper bound of the diffusion rate has been obtained by Nekhoroshev (*Nekhoroshev*, 1977), which however generally overestimates the actual rate by orders of magnitude.

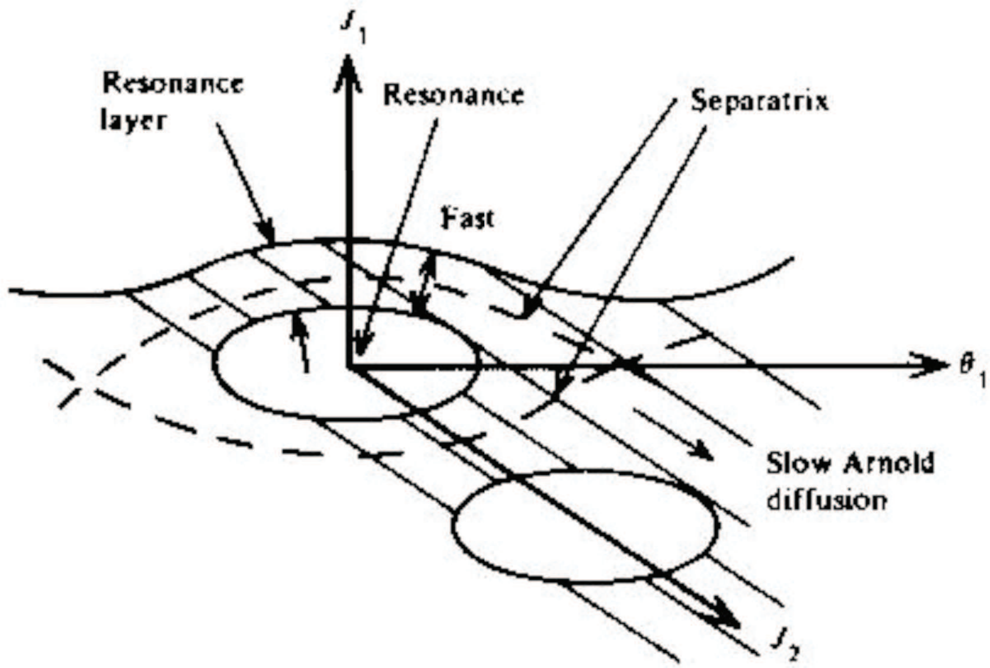



Illustration of processes of fast diffusion across a resonance layer and slow diffusion along the resonance layer (*Lichtenberg and Lieberman*, 2010).

# General notion of mapping

- All this belongs to the broad subject of Hamiltonian mechanics. This Lecture is concerned with providing a brief recap of general concepts and examples of applications to fusion plasmas. (back to p.3)

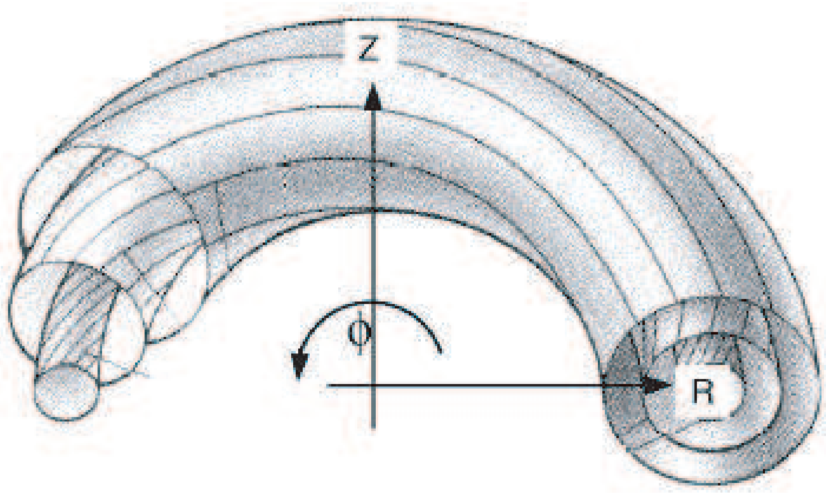
- Transport phenomena in collisionless fusion plasmas are naturally described in phase space (Lecture 2)

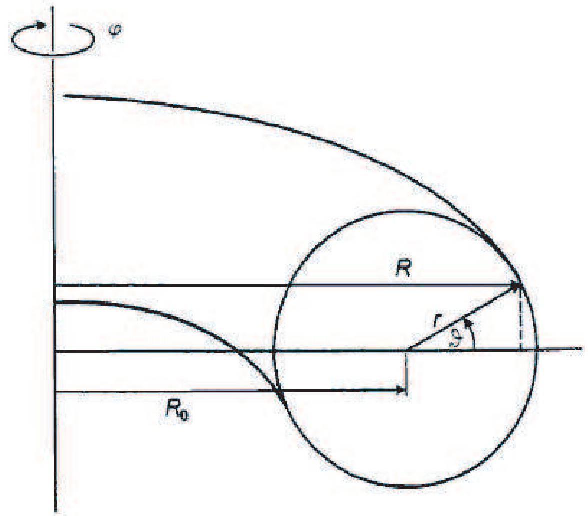
  ⇒ Identify particle resonances in tokamak plasmas

  ⇒ Construct mapping of physics related quantities

  ⇒ PEANUT is a Python Ep ANalysis sUiTe
  - suite for EPs analysis (M. V. Falessi, Y. Li, Z. Qiu, G. Wei)
  - DAEPS, EQUIPE, FALCON, are part of this suite
  - PEANUT can run on the EP workflow
  - fully integrated in IMAS (Lauber et al)

# Tokamak: 2D axisymmetric equilibrium




Jacobian $J = (\nabla\psi \times \nabla\theta \cdot \nabla\phi)^{-1}$

$\mathbf{B}\cdot\nabla\psi = 0 \Rightarrow \mathbf{B} = B_1\nabla\psi\times\nabla\theta + B_2\nabla\phi\times\nabla\psi$

$\nabla\cdot\mathbf{B} = 0 \Rightarrow \partial_\theta B_2 = 0 \Rightarrow B_2 = B_2(\psi)$

Choose $\psi = \Phi_p/2\pi \Rightarrow B_2 = 1$;
$\Phi_p = \int J\mathbf{B}\cdot\nabla\theta d\psi d\phi = 2\pi\int B_2(\psi)d\psi$

$\mathbf{B} = F(\psi,\theta)\nabla\phi + \nabla\phi\times\nabla\psi$

$F = F(\psi)$ only if there are no toroidal forces on the system!

$$(4\pi/c)\mathbf{J} = \nabla F\times\nabla\phi + R^2\nabla\phi\nabla\cdot(R^{-2}\nabla\psi)$$

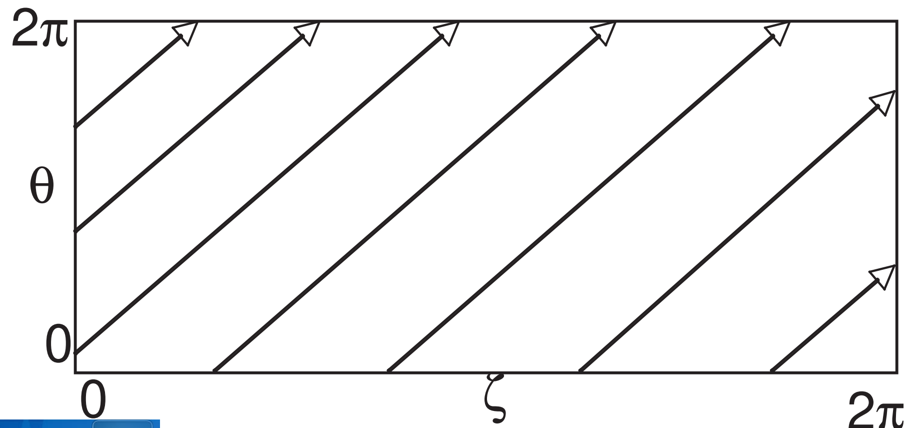
$$(4\pi/c)\mathbf{J}\times\mathbf{B} = -\nabla\psi\nabla\cdot(R^{-2}\nabla\psi) - R^{-2}F\nabla F + J^{-1}\nabla\phi\partial_\theta F$$

$F = F(\psi)$ only if there are no toroidal forces on the system!

E: What is the effect of rotation? And of anisotropic pressure tensor (still diagonal)?

## Straight Field line flux coordinates

2D with Generic $\zeta = \phi - \nu(\theta,\psi)$




$\vdots$

Choose $\nu(\theta,\psi)$ such that $q = \mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta = q(\psi)$

Clebsch representation for $\mathbf{B}$

$$\mathbf{B} = \nabla\alpha(\psi,\theta,\zeta)\times\nabla\psi = \nabla(\zeta - q\theta)\times\nabla\psi$$

E: Show that magnetic field lines are intersections of surfaces $\alpha(\psi,\theta,\zeta) = \alpha_0$ and $\psi = \psi_0$.

Further freedom of choosing $\theta$ is used to select $J = (\nabla\psi\times\nabla\theta\cdot\nabla\zeta)^{-1}$

Examples are:

$J = J_H(\psi)$: Hamada coordinates

$J = \hat{J}_B(\psi)/B^2$: Boozer coordinates

# Equilibrium particle motion in toroidal geometry

- More information following Lecture 1 of Spring 2012 Lecture Notes.
- Note $\boldsymbol{b}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\psi = -(\mathcal{J}B_0)^{-1}F(\psi)$. Then, given $B_\parallel^* = B_0 + (c/e)\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\bar{p}_\parallel$ and $\Omega = eB_0/(mc)$,

$$
\begin{aligned}
\dot{\psi} &= -\frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{\partial G}{\partial\theta} \;, \\
\dot{\theta} &= \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{\partial G}{\partial\psi} \;.
\end{aligned}
$$

$$
G = \psi - F(\psi)v_\parallel/\Omega \simeq -(c/e)P_\phi
$$

which is connected with the lowest order expression (in the drift parameter expansion) for the toroidal canonical angular momentum.

- Consider also

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\psi = B_0 - \frac{q(\psi)F(\psi)}{\mathcal{J}B_0} \quad ,$$

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\theta = \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F}{\mathcal{J}B_0}\frac{\partial \nu(\psi,\theta)}{\partial \psi} \quad .$$

In this way, one obtains

$$\dot{\zeta} = \frac{q v_\parallel}{\mathcal{J}B_\parallel^*} + \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{mc}{e}\left[\frac{\partial}{\partial \psi}\left(\mathcal{J}B_0 v_\parallel - q\frac{F v_\parallel}{B_0}\right) + \frac{\partial}{\partial \theta}\left(\mathcal{J}v_\parallel \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F v_\parallel}{B_0}\frac{\partial \nu(\psi,\theta)}{\partial \psi}\right)\right] \quad ,$$

$$\begin{aligned} \dot{\xi} \quad &= \quad \dot{\zeta} - q\dot{\theta} - q_\psi \theta \dot{\psi} = \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{mc}{e}\left\{\frac{\partial}{\partial \psi}\left(\mathcal{J}B_0 v_\parallel\right)\right. \\ &\left. + \frac{\partial}{\partial \theta}\left[\mathcal{J}v_\parallel \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F v_\parallel}{B_0}\left(\frac{\partial \nu(\psi,\theta)}{\partial \psi} + q_\psi \theta\right)\right]\right\} \quad . \end{aligned}$$

# Integral invariants and action angle coordinates

- The quantities $(\bar{\mu}, \bar{\zeta})$ and $(P_\phi, \phi)$ are action-angle coordinates for the unperturbed particle motion. For fully characterizing the equilibrium particle motion in action-angle coordinates, we are still missing $(J, \eta)$, with

$$J = m \oint v_\| d\ell = m \oint \left| \dot{\boldsymbol{X}} \right|^2 dt = m \oint v_\|^2 dt$$

being the *second invariant* and $\eta$ the respective conjugate canonical angle, defined as

$$\eta = \omega_b \int_0^\theta \frac{d\theta'}{\dot{\theta}'} \quad .$$

E: Justify the definition of $\eta$ as the canonical angle conjugate to $J$ and discuss the implications of interpreting $v_\|$ in the definition of $J$ as parallel velocity along $\boldsymbol{b}$ or along the guiding center trajectory arc-length.

- With the definitions above, $J$ can be computed once for all given the reference magnetic equilibrium, i.e.

$$J = J(H_0, \bar{\mu}, P_\phi) \quad \leftrightarrow \quad H_0 = H_0(\bar{\mu}, J, P_\phi) \quad .$$

- Meanwhile, from the definition of $J$ and $\omega_b$

$$\frac{\partial J}{\partial H_0} = \frac{2\pi}{\omega_b} \quad \leftrightarrow \quad \omega_b = 2\pi \frac{\partial H_0}{\partial J} \quad .$$

E: Demonstrate this last statement. Hint: refer to the solution of the take-home exam of the Spring 11 Lectures.

Note: The distinction between $\mu$ and $\bar{\mu}$ (guiding center and gyrocenter magnetic moment) is unneeded in the absence of fluctuations. We maintain it here for consistency of notation.

# Resonance conditions and equilibrium particle motion in toroidal geometry

- Particle motions in toroidal geometry depend on constants of motion and reflect the equilibrium magnetic configuration. Meanwhile, resonance conditions also depend on geometry and breaking of constants of motion by wave-particle resonances reflects fluctuation induced transport.

- Consider a generic fluctuation $f(r,\theta,\xi\equiv\zeta-q(r)\theta)$ in Clebsch coordinates $(r,\theta,\xi)$ [Z.X. Lu etal 12], leaving time dependences implicit. We can generally write

$$f(r,\theta,\xi)=\sum_n e^{in\xi}F_n(r,\theta) \quad ,$$

- While periodicity in $\xi$ is maintained, periodicity in $\theta$ is substituted by $F_n(r,\theta+2\pi) = e^{2\pi inq}F_n(r,\theta)$. In $(r,\vartheta)$ mapping space, the transform corresponding to using Clebsch coordinates is obtained by the periodization operator

$$
\begin{aligned}
F_n(r,\theta) &= 2\pi \sum_\ell e^{2\pi i\ell nq}\hat{F}_n(r,\theta-2\pi\ell) \\
&= \sum_m e^{i(nq-m)\theta}\int e^{i(m-nq)\vartheta}\hat{F}_n(r,\vartheta)d\vartheta \quad .
\end{aligned}
$$

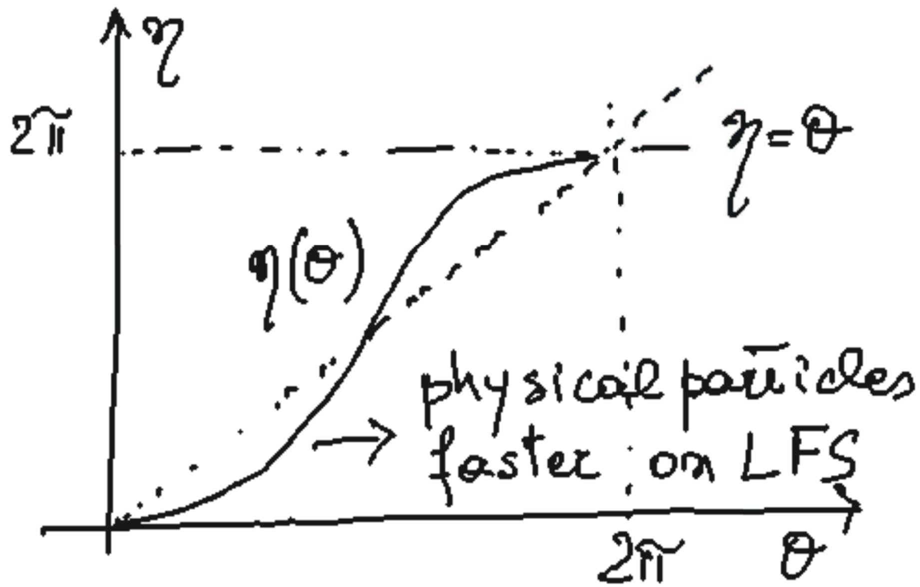
The governing equations are generally expressed either for $F_n(r,\theta)$ or for $\hat{F}_n(r,\vartheta)$, as discussed by [Z.X. Lu etal 12]. All together,

$$
\begin{aligned}
f(r,\theta,\xi) &= \sum_{m,n} e^{in\xi}e^{i(nq-m)\theta}F_{m,n}(r) \quad , \\
F_{m,n}(r) &= \int e^{i(m-nq)\vartheta}\hat{F}_n(r,\vartheta)d\vartheta \quad .
\end{aligned}
$$

- Recall $\eta$ is the canonical angle conjugate to $J$ (second invariant), obtained from integration along the particle orbit as

$$J = m \oint v_\parallel d\ell \;\;, \qquad \eta = \omega_b \int_0^\theta \frac{d\theta'}{\dot\theta'} \;\;, \qquad \omega_b = \frac{2\pi}{\oint d\theta / \dot\theta} .$$

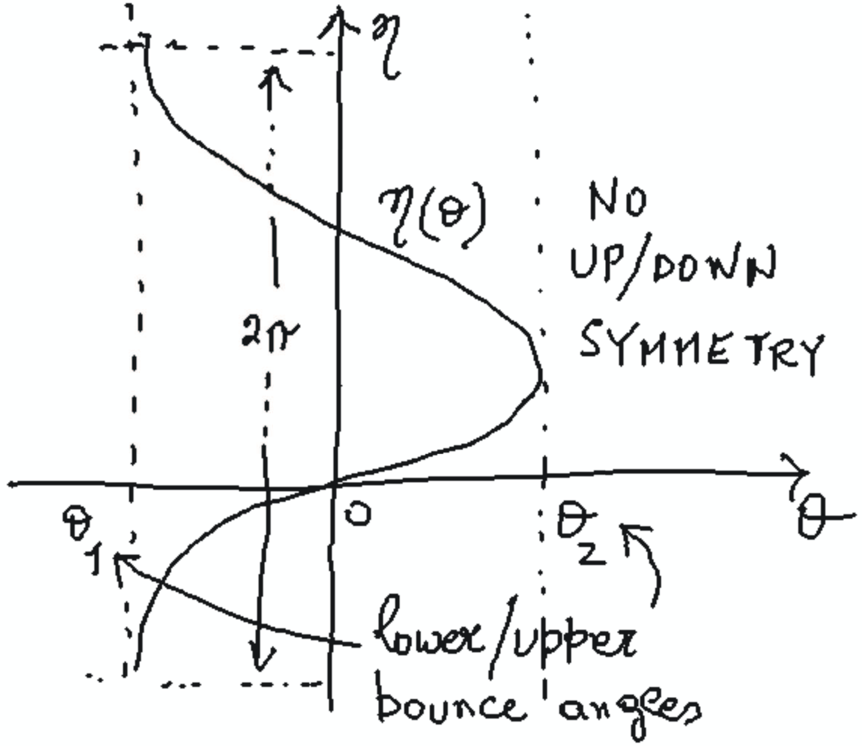
- For circulating particles, recalling that $\omega_b$ is the transit/bounce frequency,




$$\theta = \eta + \Theta_C(\eta) = \omega_b \tau + \Theta_C(\eta) \;\;,$$

where $\Theta_C(\eta)$ is a periodic functions of $\eta$, which depends also on $H_0, \mu, P_\phi$ or, equivalently, of $\mu, J, P_\phi$; and $\tau$ is a time-like parameter describing the particle motion along the particle trajectory.

- For magnetically trapped particles




$$\theta = \Theta_T(\eta) \quad ,$$

where $\Theta_T(\eta)$ is a periodic function of $\eta$, which depends also on $H_0, \mu, P_\phi$ or, equivalently, of $\mu, J, P_\phi$.

- From the periodicity of the radial particle motion in the equilibrium fields, we also have

$$r = \bar{r} + \rho(\eta) \quad ,$$

with $\rho(\eta)$ a periodic function of $\eta$, different for trapped and circulating particles, which depends also on $H_0, \mu, P_\phi$ or, equivalently, of $\mu, J, P_\phi$.

E: From the equations of motions in the equilibrium magnetic field show that the particle trajectories in the $(\psi, \theta)$ plane are closed and discuss how this implies the results above for the radial particle motion.

- The connection of $\xi$ with $\eta = \omega_b \tau$ is more subtle to obtain. Recall the definition
$$\xi = \zeta - q(\psi)\theta = \phi - \nu(\psi, \theta) - q(\psi)\theta$$
and that the particle orbit follows the magnetic field line and is accompanied by a periodic motion about it, of period $2\pi/\omega_b$, which is therefore a periodic function of $\eta$.

- Keeping this in mind we can conclude that, except for periodic dependences in $\eta$ (indicated by the $\simeq$ sign and the $\ldots$)
$$\zeta \simeq \bar{\omega}_d \tau + \int q d\theta + \ldots \simeq \bar{\omega}_d \tau + \bar{q}\theta + \ldots$$
with $q(\bar{r} + \rho(\eta)) = \bar{q}(\bar{r}) + Q(\bar{r} + \rho(\eta))$ and $\oint Q d\theta = \oint (\dot{\theta}/\omega_b) Q d\eta = 0$.

- The representation for $\xi$ is then obtained as

$$\xi = \bar{\omega}_d \tau + (\bar{q} - q)\, \theta + \Xi(\eta) \quad ,$$

with $\Xi(\eta)$ a periodic function of $\eta$, which depends also on $H_0, \mu, P_\phi$ or, equivalently, of $\mu, J, P_\phi$.

- Given the above representation for $\xi$, the definition of the precession frequency is

$$\bar{\omega}_d = \frac{\omega_b}{2\pi} \oint \left( \dot{\zeta} - q\dot{\theta} \right) \frac{d\theta}{\dot{\theta}} \quad ,$$

E: Demonstrate the last statement. Hint: take the total time derivative of the $\xi$ parametric expression and integrate on one full periodic orbit in $\eta$.

- Summarizing: introduce the periodic functions $\Theta(\eta)$, $\Xi(\eta)$ and $\rho(\eta)$, i.e. without distinction between trapped and circulating particles but keeping in mind that they are different for each class of particles, which depend also on $H_0, \mu, P_\phi$ or, equivalently, of $\mu, J, P_\phi$

| Circulating particles | Trapped particles |
|---|---|
| $\theta = \eta + \Theta(\eta) = \omega_b \tau + \Theta(\eta)$ | $\theta = \Theta(\eta)$ |
| $r = \bar{r} + \rho(\eta)$ | $r = \bar{r} + \rho(\eta)$ |
| $\xi = \bar{\omega}_d \tau + (\bar{q} - q)\,\theta + \Xi(\eta)$ | $\xi = \bar{\omega}_d \tau + (\bar{q} - q)\,\theta + \Xi(\eta)$ |

with the definition

$$q(\bar{r} + \rho(\eta)) \equiv \bar{q}(\bar{r}) + Q(\bar{r} + \rho(\eta)) \qquad \oint Q d\theta = \oint (\dot{\theta}/\omega_b) Q d\eta = 0$$

- Here, $\tau$ is a time-like parameter tracking the position along the trajectory ($\eta \equiv \omega_b \tau$), but no time dependence has been assumed so far.

- With the above parametrization of particle trajectories in terms of action-angle variables and characteristic particle frequencies in the equilibrium magnetic configuration, we can systematically decompose the fluctuating field in action-angle coordinates, for given constants of motion.

- This procedure, which is not the usual practice, corresponds to decompose a mode structure in elements that are actually describing the wave-particle interactions, once the Fourier decomposition is known. This is somewhat tedious, but systematic, and has the advantage of:
  - identifying the resonance condition in general geometry and suggesting a practical way of computing it
  - describing wave-particle resonances and their effect on the particle motion in the most natural coordinates describing the unperturbed particle motion
  - lifting of $f$ to the particle phase space

- Recall that (see p.26)

$$f(r,\theta,\xi) = \sum_{m,n} e^{in\xi} e^{i(nq-m)\theta} F_{m,n}(r)$$

- Using the trajectory parametrization summarized on p.32, we have

$$f(r,\theta,\xi) = \sum_{m,n,\ell} \lambda_{n,m} e^{i(n\bar{\omega}_d + \ell\omega_b)\tau} \mathcal{F}_{m,n,\ell}(\bar{r}) \quad ,$$

$$\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi) = \frac{1}{2\pi} \oint \exp\left\{ in\Xi(\eta) + i\left[n\bar{q}(\bar{r}) - m\right]\Theta(\eta)\right\} F_{m,n}(\bar{r}+\rho(\eta)) e^{-i\ell\eta} d\eta \quad .$$

- Note that, here, we have made explicit the dependences of $\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi)$ on $\mu$, $J$, $P_\phi$. Note also that no specific time dependence has been assumed here, so far, and that the time-like parameter $\tau$ parameterizes $(r,\theta,\xi)$ along the particle trajectory, as noted before.

- Note also that, because of the finite orbit width $\rho(\eta)$, the effective mode width – that of $\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi)$ – can be significantly larger than that of the Fourier harmonic $F_{m,n}(r)$.

E: Explain qualitatively why the effective mode width can be larger than that of the Fourier components and discuss which physics this may impact. Explain what is the physical meaning of the dependence of the fluctuation strength on phase space variables such as $\mu$, $J$, $P_\phi$.

- The mode structure decomposition in action angle variables corresponds to a further Fourier decomposition of the amplitudes $F_{m,n}(r)$ in bounce harmonics ($\eta \equiv \omega_b \tau$), as effectively experienced by the particle moving on the phase space surface given by its constants of motion.

E: Can you recognize this further Fourier decomposition? Can you explain why it is induced by the periodic functions $\Theta(\eta)$, $\Xi(\eta)$ and $\rho(\eta)$? $\Rightarrow$ push-forward representation in drift/banana center (one can construct corresponding pull-back)

- The quantity $\lambda_{n,m}$ is $\lambda_{n,m} = 1$ for trapped particles, while, for circulating particles, it is given by

$$\lambda_{n,m} = \exp\left[i\left(n\bar{q}(\bar{r}) - m\right)\omega_b\tau\right] \quad .$$

E: Derive this last equation step by step.

- The resonance condition is obtained when, for some partial amplitudes in the mode structure decomposition, the wave-particle phase is constant along the unperturbed orbits, so that, assuming fluctuations $\propto \exp(-i\omega t)$

$$\omega = \omega(\mu, J, P_\phi) = n\bar{\omega}_d + \ell\omega_b$$

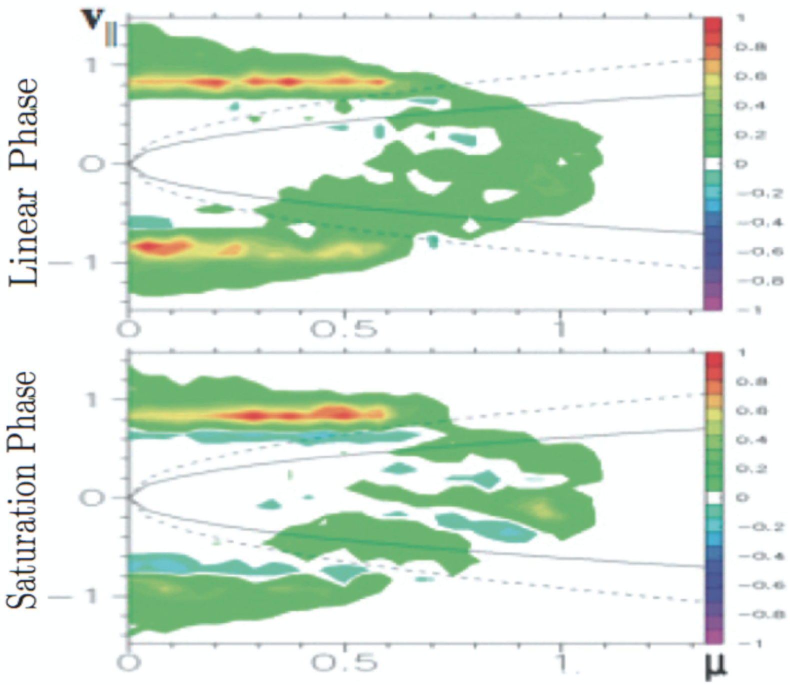
for trapped particles, while, for circulating particles,

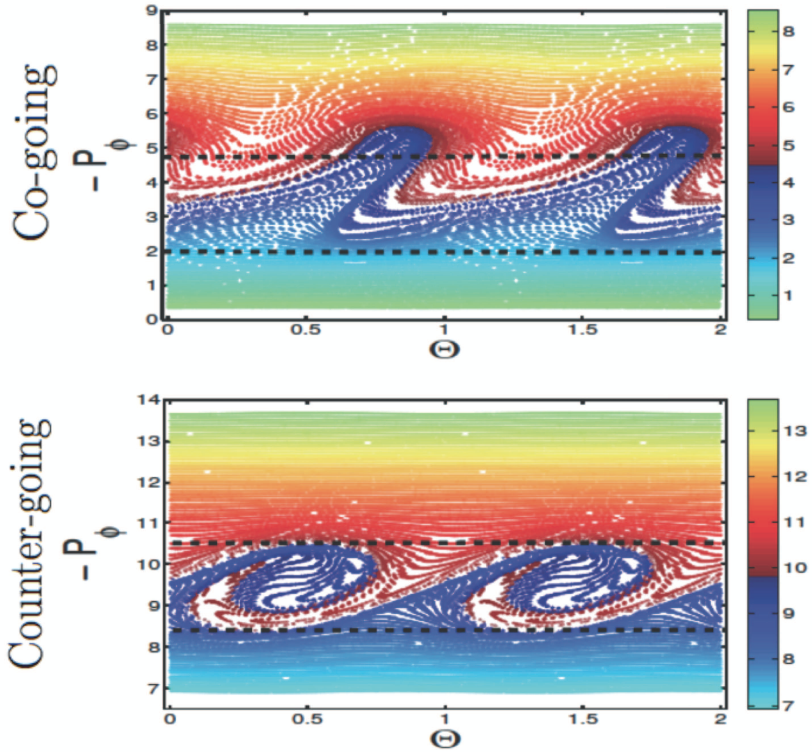
$$\omega = \omega(\mu, J, P_\phi) = n\bar{\omega}_d + \ell\omega_b + \left(n\bar{q}(\bar{r}) - m\right)\omega_b \quad .$$

E: Discuss the physics underlying each term in the resonance conditions above.

# Nonlinear dynamics of BAEs

- Hybrid MHD-Gyrokinetic simulations of Beta induced Alfvén Eigenmodes (BAE) [Wang PRE12] with XHMGC [$\dot{\omega} > 0$, $\dot{\omega}_{\rm res} \propto {\rm sgn} v_{\parallel}$]



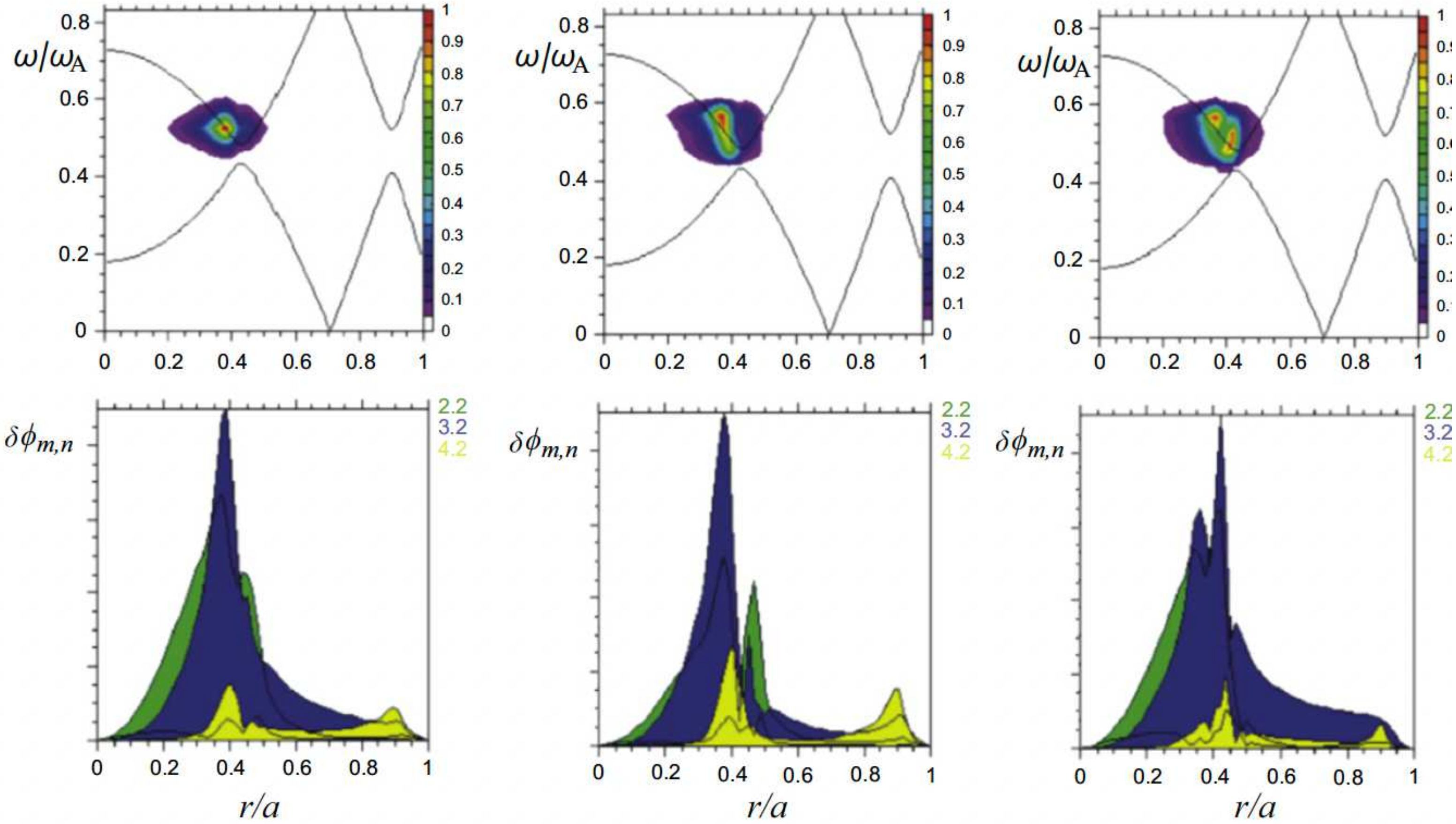

- Evidence of phase locking and radial decoupling being more important than resonance detuning in nonlinear mode dynamics ⇒ confirmed by nonlinear GKE simulations with GTC [Zhang PRL12] (Spring 2014 Lecture 2).

# Numerical implementation of Hamiltonian mappings

- Assume to deal with drift-Alfvén fluctuations, so that $\mu$ is conserved.
- We assume that transport processes take place on time scales longer than the inverse characteristic frequencies of particle motions, i.e. the bounce/transit time. This is consistent with the derivations provided above.
- With these assumptions, we can construct the Hamiltonian mapping from surface of section plots with the plane $\theta = 0$, choosing a precise sign for $v_{\parallel}$ in the case of trapped particles.
- Surface of section plot (Hamiltonian mapping) is carried out plotting particles (kinetic Poincaré plot) in the $(\Theta, P_\phi)$ plane, with $\Theta$ the wave-particle phase and $\dot{\Theta} = 0$ identifying the resonance condition.
- Marker particles colored according to physics information: original position, wave particle power exchange...

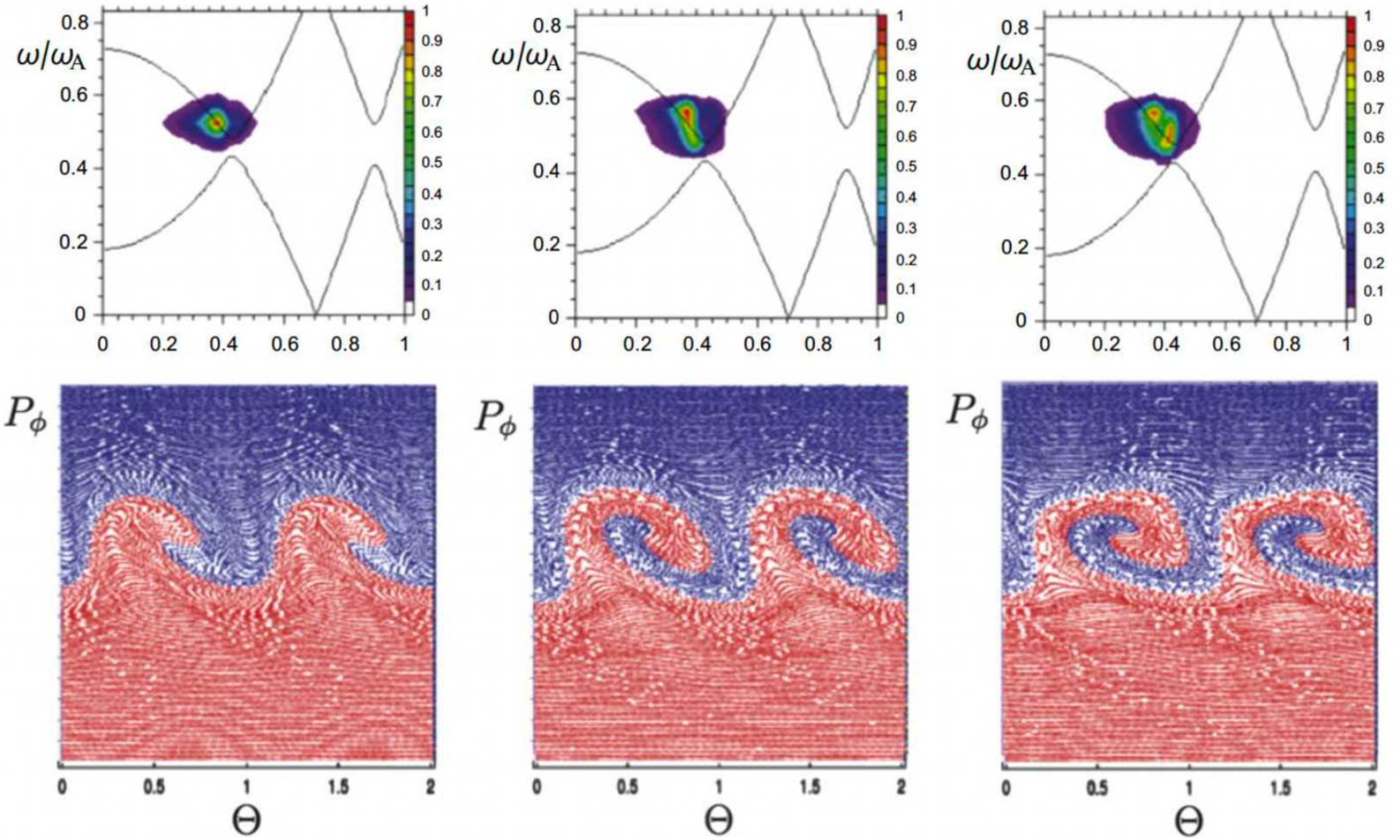
# Nonlinear dynamics of (localized) EPMs

- Numerical Experiment: Hybrid MHD-Gyrokinetic simulations of Energetic Particle Modes (EPM) with XHMGC [Briguglio EPL13]



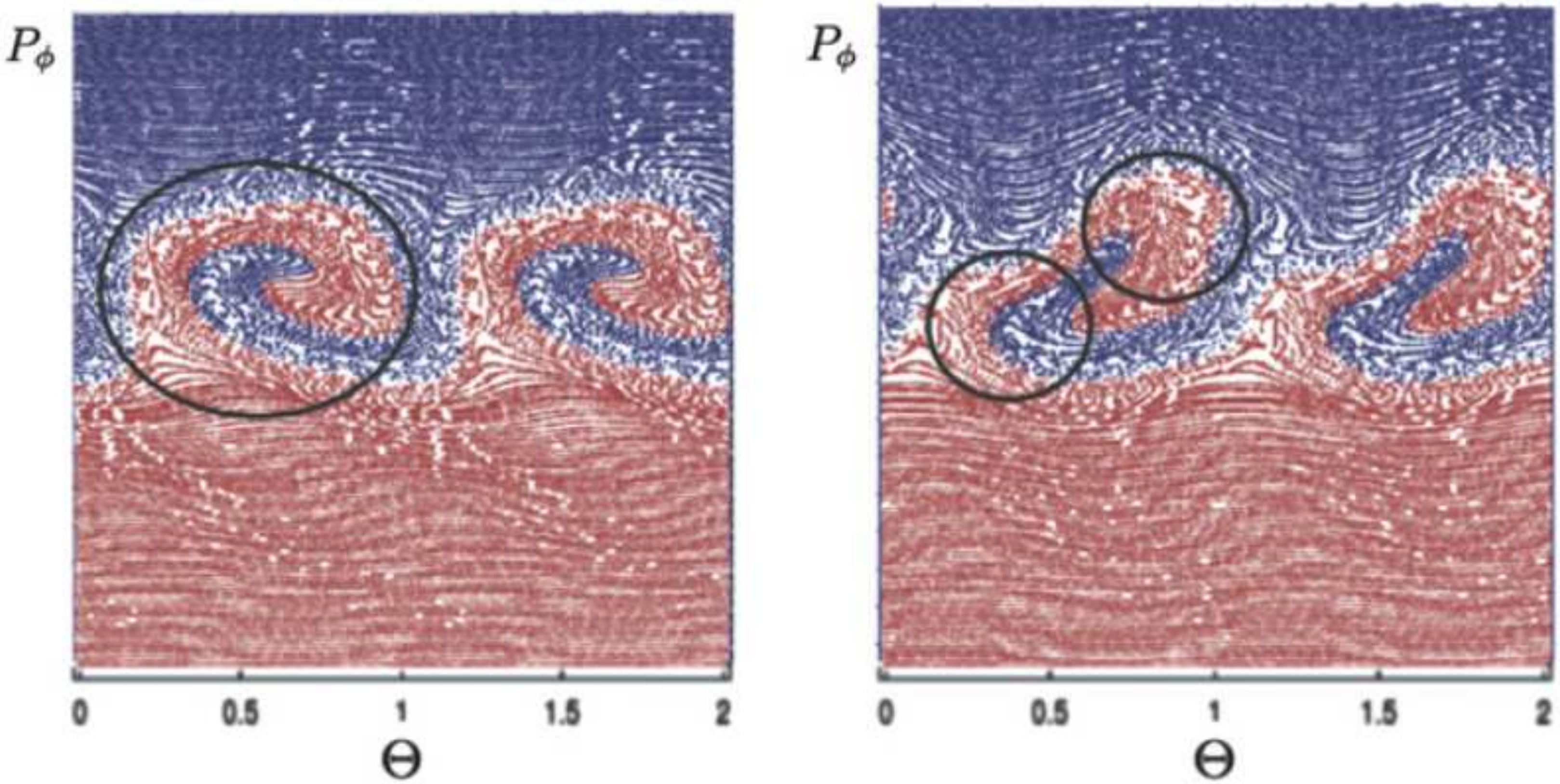

- Evidence of mode/frequency splitting-merging.

- Evidence of mode/frequency splitting-merging $\Leftrightarrow$ phase locking and $\omega_B t \sim 1$ $\Rightarrow$ non-perturbative $\oplus$ non-adiabatic dynamics [Briguglio EPL13].




- Crucial role played by the readiness of EPM wave-packets (continuous spectrum) to respond to nonlinearly modified phase-space zonal structures $\Rightarrow$ Crucial roles of non-uniformities and geometries [Chen RMP16].

□ Evidence of structure formation in particle phase space on $\omega_B t \sim 1$ [Briguglio EPL13] $\Rightarrow$ non-perturbative $\oplus$ non-adiabatic dynamics.




$\omega_A t = 1320$ $\omega_A t = 1353$

- The numerical Hamiltonian mapping consists in recording the values of the particle actions (parameterized by $\mu$) at each transit $p = 0, 1, 2, ...$ through $\theta = 0$ Therefore:

$$\begin{aligned} J_{p+1} &= J_p + \Delta J_p \\ \Theta_{p+1} &= \Theta_p + \Delta\Theta_p \end{aligned} \quad .$$

Here, the value of $\Delta J_p$, determined numerically, reflects fluctuation induced transport, since $\Delta J_p = 0$ at equilibrium.

- Similarly, we can have the numerical Hamiltonian mapping for $P_\phi$, i.e.

$$\begin{aligned} P_{\phi,p+1} &= P_{\phi,p} + \Delta P_{\phi,p} \\ \Theta_{p+1} &= \Theta_p + \Delta\Theta_p \end{aligned} \quad .$$

- Meanwhile, the wave-particle phase is

$$\dot{\Theta} \equiv n\dot{\zeta} - m\dot{\theta} - \omega$$

and, thus,

$$\Theta \equiv n\zeta - m\theta - \omega t \quad .$$

- This definition is based on the mode structure decomposition in Fourier harmonics.

- Due to the periodic oscillatory motions of the particles about the magnetic field line, it is useful to consider the wave-particle phase as defined for the mode structure decomposition in action-angle variables.

- For the unperturbed particle motion,

$$\dot{\theta} \;=\; \epsilon\omega_b + \dot{\tilde{\theta}} \;, \qquad \begin{cases} \epsilon = 0 \text{ trapped particles} \\ \epsilon = 1 \text{ circulating particles} \end{cases}$$

$$\dot{\zeta} \;=\; \bar{\omega}_d + \bar{q}\dot{\theta} + \dot{\tilde{\zeta}} \;,$$

with $\oint \dot{\tilde{\zeta}} dt = \oint \dot{\tilde{\theta}} dt = 0$ and $\bar{q} = \oint q\dot{\theta} d\eta / \oint \dot{\theta} d\eta$; so that,

$$\Delta\Theta_p = \int_{t_p}^{t_p+2\pi/\omega_b} \dot{\Theta} dt = (2\pi/\omega_b)\left[n\bar{\omega}_d + \left((n\bar{q} - m)\epsilon + \ell\right)\omega_b - \omega\right] \;.$$

- The condition $\Delta\Theta_p = 0$ clearly gives the resonance condition for both trapped and circulating particles.

- Note that $\ell$ identifies the transit/bounce harmonic resonance for circulating/trapped particles, due to the periodic oscillatory motions of the particles about the magnetic field line.

- By definition and for practical calculations, at each passage of the particle through the midplane, with $\eta_p = 2\pi p$ and $t_p = \eta_p/\omega_b$

$$\Theta_p \equiv n\zeta_p + (\ell - m)\,\omega_b t_p - \omega t_p = n\zeta_p - \int^{t_p} \omega(t')dt' = n\zeta_p - \omega t_p$$

- The first part in the above expression can be used also for frequency chirping fluctuations, while the last one assumes constant frequency modes. Note that, for large aspect ratio, shifted circular surfaces, $\nu = 0$ (see p. 21), so that $\zeta_p \to \phi_p$ in the expression above.

E: Show that the resonance condition remains the same for frequency sweeping modes. Particularly, show that resonance structure is expected to remain unchanged for $|\dot\omega| \lesssim |\gamma\omega|$, with $\gamma$ the characteristic growth rate.

- More advanced forms of mapping can be derived, using analytic expressions for $\Delta J_p$ and $\Delta P_{\phi,p}$, by means of the definitions

$$\Delta J_p = \int_{t_p}^{t_{p+1}} dt' \dot J \; ,$$
$$\Delta P_{\phi,p} = \int_{t_p}^{t_{p+1}} dt' \dot P_\phi \; .$$

with

$$\Delta \Theta_p = \int_{t_p}^{t_{p+1}} dt' \dot\Theta \; .$$

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