Lecture 6

Toroidal Alfvén Eigenmodes and Energetic Particle Modes

Fulvio Zonca

http://www.afs.enea.it/zonca

ENEA C.R. Frascati, C.P. 65 - 00044 - Frascati, Italy.

Institute for Fusion Theory and Simulation, Zhejiang University, Hangzhou 310027, P.R.C.

June 20.th, 2019

Kinetic theory and global dispersion relation
of Alfvén waves in tokamaks
June 4 – 20, 2019, IFTS – ZJU, Hangzhou

Micro- and meso- scale excitation of low-frequency AE/EPM

- From Lecture 5, the general expression of Λ reduces to fluid limit and incorporates Landau damping (esp. of Alfvén-acoustic branch)

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right)+q^2\frac{\omega\omega_{ti}}{\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)F(\omega/\omega_{ti})-\frac{\omega_{*Ti}}{\omega}G(\omega/\omega_{ti})\right.$$
$$\left.-\frac{N_m(\omega/\omega_{ti})}{2}\left(\frac{N_{m+1}(\omega/\omega_{ti})}{D_{m+1}(\omega/\omega_{ti})}+\frac{N_{m-1}(\omega/\omega_{ti})}{D_{m-1}(\omega/\omega_{ti})}\right)\right]$$

low frequency AE/EPM can be excited by both thermal ions (micro-scale) and energetic ions (meso-scales)

$$i\Lambda = \delta\bar{W}_f + \delta\bar{W}_k \qquad \text{(fast ions included)}$$

- Excitation of low-frequency AE by thermal ions is most easily seen using the simplified expression of Λ derived earlier: AITG excitation mechanism

$$\Lambda^2 = \frac{1}{\omega_A^2}\left[\omega^2 - \left(\frac{7}{4}+\frac{T_e}{T_i}\right)q^2\omega_{ti}^2\right] + i\sqrt{\pi}q^2 e^{-\omega^2/\omega_{ti}^2}\frac{\omega^2}{\omega_A^2}\left(\frac{\omega_{ti}}{\omega}-\frac{\omega_{*Ti}}{\omega_{ti}}\right)\left(\frac{\omega^2}{\omega_{ti}^2}+\frac{T_e}{T_i}\right)^2 .$$

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ accumulation point becomes unstable! The unstable continuum is not a concern
(E: compute how much the mode can grow before mode converting to KAW. Hint: compute how long takes for a wave-packet born at small ϑ-ballooning to reach large ϑ where KAW physics is important).

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ and equilibrium effects localize the AE, the Alfvénic ITG mode is excited (AITG) $\Leftarrow$ Lecture 5 $\Rightarrow$ More in the following.

- Energetic particle contribution to potential energy from Lecture 5 (p. 21) and Lecture 3 & Lecture 4 (p. 23)

$$\delta \bar{W}_k = \left(\delta\hat{\Phi}_{0+}^{(0)\dagger}\delta\hat{\Phi}_{0+}^{(0)}\right)^{-1}\left(-\frac{1}{2}\right)\int_{-\infty}^{\infty}\delta\hat{\Phi}^{(0)\dagger}\left[\frac{4\pi R_0 q^2}{B_0}\frac{g(\vartheta,\theta_k)\omega}{ck_\vartheta\hat{\kappa}_\perp}\left\langle m_E\left(\mu B_0+v_\parallel^2\right)J_0\delta\hat{K}_E\right\rangle_v\right]d\vartheta$$

E: Derive this expression by using the form of $\delta\bar{W}_k$ of Lecture 4.

- This expression is ready for numerical implementation, once $\delta\hat{K}_E$ is provided. Complete analytical expressions are also available and given in [POP 2014; Chen RMP 2016].
- For simplicity, neglect finite orbit width effects, and assume $\omega_b \gg \bar{\omega}_d$.
- Since the thermal plasma has infinite conductivity, we can take $\delta\phi = \delta\psi$ at the lowest order.

- The GKE can be written as (p.19 Lecture 5)

$$\left(\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp\right) \delta K = i\frac{e}{m} \left[Q\bar{F}_0 \langle \delta\phi_g - \delta\psi_g \rangle \right.$$
$$\left. -\boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp Q\bar{F}_0 \partial_t^{-1} \langle \delta\psi_g \rangle \right] \quad .$$

- Applying present approximations and letting $-i\boldsymbol{v}_d \cdot \boldsymbol{\nabla} \equiv \omega_d \equiv \tilde{\omega}_d + \bar{\omega}_d$ (incorporating the toroidal mode number into the definition of $\bar{\omega}_d$)

$$\left(\omega + iv_\parallel \nabla_\parallel - \tilde{\omega}_d - \bar{\omega}_d\right) \delta K = -\frac{e}{m}\frac{\omega_d}{\omega} Q\bar{F}_0 \langle \delta\psi_g \rangle$$
$$\Rightarrow \qquad \left(\omega + iv_\parallel \nabla_\parallel - \bar{\omega}_d\right) \delta K_b \simeq -\frac{e}{m}\frac{\omega_d}{\omega} Q\bar{F}_0 e^{iQ_b} \langle \delta\psi_g \rangle$$

- Here, we have noted that $\delta K = e^{-iQ_b}\delta K_b$, and $v_\parallel \nabla_\parallel Q_b - \tilde{\omega}_d = 0$.

- The pull back operator e^{-iQ_b} from bounce-banana center to guiding center is nearly an identity for negligible orbit width and $\delta K \simeq \delta K_b$. Similarly $\langle \delta\psi_g \rangle \simeq \delta\psi \simeq \delta\phi$.

- Since $\bar{\omega}_d \ll \omega_b$, at the lowest order $\nabla_\parallel \delta K \simeq \partial_\vartheta \delta \hat{K} \simeq 0$. To avoid secularities in ϑ and introducing bounce averaging on unperturbed particle motions

$$\overline{[\ldots]} \equiv \left(\oint \frac{d\ell}{v_\parallel} \right)^{-1} \oint \frac{d\ell}{v_\parallel} [\ldots]$$

it is possible to obtain

$$\delta \hat{K} = \frac{e}{m} \frac{Q \bar{F}_0}{\omega} e^{-inq\vartheta} \left(\frac{\overline{e^{inq\vartheta} \omega_d \delta \hat{\phi}}}{\bar{\omega}_d - \omega} \right) ,$$

which represents the integral particle response.

E: Discuss why this response of energetic particles is integral, even if we assumed negligible orbit width. What type of approximation could you introduce to further simplify the problem?

- In general, the integral particle response reflect the "memory" of the system in collisionless fusion plasmas

$$e^{-inq\vartheta}\overline{e^{inq\vartheta}\omega_d\delta\hat{\phi}} \equiv \hat{\hat{\omega}}_d\delta\hat{\phi}$$

- Here, note that $\hat{\omega}_d$ is a notation for recalling that we need to treat ∇B_0 and curvature drifts in the same fashion, for correctly accounting for equilibrium and perturbed parallel pressure balance. Meanwhile $\hat{\hat{\omega}}_d$ is defined by the above equation.
- Further approximation is generally possible either assuming deeply trapped particles and/or fixed mode structure; e.g., one single (m, n).

- Assuming $\bar{F}_{0E}$ to be symmetric in $v_\parallel$, and $\bar{\omega}_d = \Omega_d = -(\mu B_0 + v_\parallel^2)k_\vartheta/(\Omega R_0)$ for deeply trapped particles, with $\tau_b = 2\pi/\omega_b = 2\pi q R_0/[(r/R_0)\mathcal{E}]^{1/2}$

$$\delta \bar{W}_k = \frac{\pi^2}{|s|}\frac{e^2}{mc^2} q R_0 B_0 \sum_{v_\parallel/|v_\parallel|=\pm 1} \int d\mathcal{E} \int d\mu \left(\frac{\Omega_d}{k_\vartheta}\right)^2 \tau_b \frac{Q\bar{F}_0}{\bar{\omega}_d - \omega} \; .$$

- Consider EPs with an isotropic slowing down distribution function and injection energy E_F much larger than the critical energy E_c, so that for $E_F/m_E > \mathcal{E} > E_c/m_E$ the EP energy is predominantly transferred to thermal electrons by collisional friction (as for α-particles in fusion plasmas); *i.e.*,

$$\bar{F}_0 = \frac{3P_{0E}}{4\pi E_F} \frac{\mathrm{H}(E_F/m_E - \mathcal{E})}{(2\mathcal{E})^{3/2} + (2E_c/m_E)^{3/2}} \; ,$$

where H denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is $(3/2)P_{0E}$ for $E_F \gg E_c$.

- Substituting into the expression for $\delta \bar{W}_k$, we obtain

$$\delta \bar{W}_k = \frac{3\pi (r/R_0)^{1/2}\alpha_E}{8\sqrt{2}|s|}\left[1 + \frac{\omega}{\bar{\omega}_{dF}} \ln\left(\frac{\bar{\omega}_{dF}}{\omega} - 1\right) + i\pi\frac{\omega}{\bar{\omega}_{dF}}\right] \quad ,$$

with $\bar{\omega}_{dF} = \bar{\omega}_d(\mathcal{E} = E_F/m_E)$ and having noted that $|\omega_{*E}| \gg |\omega|$ for SAW resonantly excited by EPs.

E: Derive this expression step by step filling in the missing algebra.

E: Use the simplified expression for Λ on p. 3 to show that BAE can be excited. Can you discuss the condition for exciting EPM?

RP: Use the same dispersion relation to show that there is a transition from EP driven modes at long wavelength (BAE) to thermal ion driven modes (AITG) for increasing mode number.

RP: Adopt the fluid expression of Λ derived in Lecture 5 and demonstrate the properties of the EP excitations of the low-frequency fluctuation spectrum. What can you conclude from the comparison of BAE vs BAAE excitation?

TAE frequency gap and symmetry breaking

- In the large-$|\vartheta|$ limit Eigenmode Equation is the Schrödinger equation of a free particle moving in a small perturbing periodic potential ($\epsilon_0 = 2(r/R_0 + \Delta'))$ (E: show this!
Hint: use $\Omega^2 \simeq 1/4$ and Eqs. at p.23 Lecture 5).

$$\left[\frac{\partial^2}{\partial \vartheta^2} + \Omega^2 \left(1 + 2\epsilon_0 \cos\vartheta\right) \right] \hat{\phi}_s = 0 \ .$$

- For $\epsilon_0 = 0$, $\hat{\Phi}$ would be just plane waves $\hat{\phi}_s = \exp(\pm i k \vartheta)$ with $k = \Omega = \omega/\omega_A$.

- The traveling waves can be reflected by the background perturbation, but, in a 1-D periodic lattice of period L, they constructively combine in standing waves only when the Bragg reflection condition is met:

$$2L = l\lambda \ , \qquad \lambda \equiv (2\pi/k) \ , \qquad l = 1, 2, 3, \dots \ .$$

- In this case $L = 2\pi$. Then we must have $k = \Omega = l/2$ for standing waves $\approx \sin(l\vartheta/2)$ and $\approx \cos(l\vartheta/2)$ to form
- For the two lowest energy states, introducing $\langle f|g|f\rangle = (1/4\pi)\int_0^{4\pi} d\vartheta f g f^*$ the eigenfrequency is given by

$$\sin\left(\frac{\vartheta}{2}\right) : \quad \Omega^2 = \frac{1}{4} - \frac{\left\langle \sin\left(\frac{\vartheta}{2}\right) \middle| 2\epsilon_0\Omega^2\cos\vartheta \middle| \sin\left(\frac{\vartheta}{2}\right)\right\rangle}{\left\langle \sin\left(\frac{\vartheta}{2}\right) \middle| \sin\left(\frac{\vartheta}{2}\right)\right\rangle} = \frac{1}{4}(1+\epsilon_0) \quad ,$$

$$\cos\left(\frac{\vartheta}{2}\right) : \quad \Omega^2 = \frac{1}{4} - \frac{\left\langle \cos\left(\frac{\vartheta}{2}\right) \middle| 2\epsilon_0\Omega^2\cos\vartheta \middle| \cos\left(\frac{\vartheta}{2}\right)\right\rangle}{\left\langle \cos\left(\frac{\vartheta}{2}\right) \middle| \cos\left(\frac{\vartheta}{2}\right)\right\rangle} = \frac{1}{4}(1-\epsilon_0) \quad ,$$

- It is not surprising that upper and lower edges of the shear Alfvén continuous spectrum are found in this way. The $\approx \sin(\vartheta/2)$ and $\approx \cos(\vartheta/2)$ eigenfunctions are, however, not normalizable. In other words, they are still eigenstates of the continuum.

- This draws a perfect analogy between Alfvén waves in a toroidal plasma and wave packets of Bloch electrons in a 1-D crystal lattice. The role of magnetic shear and α is to create a 'local' potential well, which can transform TAE modes into discrete modes, with well behaved normalizable eigenfunction.

- These equilibrium non-uniformities are the "defects" [Chen 2007,2008]. The potential well is local because it causes bound states to exist in a narrow radial region (smaller than the minor radius); furthermore, it varies with the radial position.

- **Effect of geometries**: the general form of the SAW vorticity equation for $\hat{\kappa}_\perp \gg 1$ is

$$\left[\left(\frac{\partial^2}{\partial \vartheta^2} - \frac{\partial^2_\vartheta |\boldsymbol{\nabla} r|}{|\boldsymbol{\nabla} r|} \right) + \frac{\omega^2 \mathcal{J}^2 B_0^2}{v_A^2} \right] \hat{\phi}_s = 0 \ .$$

E: Verify that this equation reproduces that on p. 10 for the simple case of a large aspect-ratio tokamak with shifted circular surfaces; that is, using the $s - \alpha$ model equilibrium given in Lecture 2.

RP: Derive the equation above from the proper limit of the SAW vorticity equation in general geometry, given in Lecture 2. (Hint: follow the derivation of [Z&C POP 14]).

- The general Floquet solution can be written as

$$\hat{\phi}_s = F_\nu(\vartheta) = e^{i\nu\vartheta} P(\vartheta) \quad ,$$

where ν is the Floquet "characteristic exponent" and $P(\vartheta)$ is a 2π periodic function.

- Floquet solutions are characterized by stable and unstable regions. In stable regions, where ν is real for real coefficients of the corresponding equation; therefore, $F_\nu(\vartheta)$ in non-normalizable, the (r,θ) space solutions are characterized by radial singular structures and correspond to the shear Alfvén continuous spectrum.

- Meanwhile, unstable regions with complex ν and exponentially growing or decaying $F_\nu(\vartheta)$ yield sharply varying but regular solutions in the frequency gaps of the Alfvén continuum.

- Boundaries between stable and unstable solutions are obtained for

$$\nu^2 = \frac{\ell^2}{4} \quad , \qquad \ell \in \mathbb{N} \quad ,$$

which corresponds to the Bragg reflection condition discussed previously.

- In ϑ space, physical boundary conditions are obtained by imposing outgoing waves. This is due to the fact that $|\vartheta| \to \infty$ corresponds to vanishing wavelengths, where it is possible to assume that no energy sources are present.

- With ω_0 the typical (linear) mode frequency, physical boundary conditions are obtained for [Z&C POP14]

$$\nu = \mathrm{sgn}(\vartheta) \left(\mathrm{sgn}(\omega_0) |\mathbb{R}\mathrm{e}\nu| + i |\mathbb{I}\mathrm{m}\nu| \right) \quad .$$

- For $0 \leq |\mathbb{R}\mathrm{e}\nu| \leq 1/2$, we have the Floquet first stable region; *i.e.*, the shear Alfvén continuous spectrum at frequencies lower than those of the toroidal Alfvén eigenmode (TAE) frequency gap. For $1/2 \leq |\mathbb{R}\mathrm{e}\nu| \leq 1$, the Floquet second stable region then corresponds to the continuum between TAE and ellipticity induced Alfvén eigenmode (EAE) frequency gaps. Etc.

- Identifying as $y_1(\vartheta)$ the Floquet solution of the SAW vorticity equation for $\hat{\kappa}_\perp \gg 1$ with $[y_1(2k\pi), y_1'(2k\pi)] = [1,0]$, and as $y_2(\vartheta)$ that with $[y_2(2k\pi), y_2'(2k\pi)] = [0,1]$, one has (up-down symm. equil.) [Z&C POP14]

$$\cos(2\pi\nu) = y_1\left(2(k+1)\pi\right) \qquad \text{and} \qquad \frac{P'(2k\pi)}{P(2k\pi)} + i\nu = i\frac{\sin(2\pi\nu)}{y_2\left(2(k+1)\pi\right)} \quad , \qquad k \in \mathbb{N} \ .$$

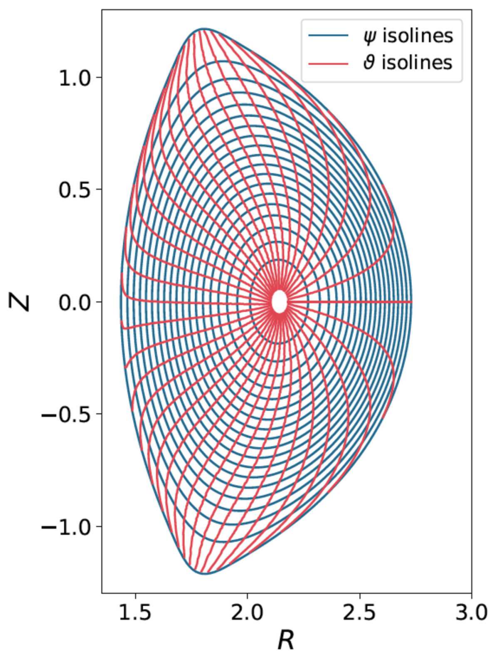
- These expressions further demonstrate that the solutions of the shear Alfvén continuum are traveling waves, which are absorbed at $|\vartheta| \to \infty$. (E: Why?)

E: Show that with these expressions you can implement the physical boundary condition in the SAW vorticity equation for general geometry. What are the connection formulae that you must apply at the origin? Show that this results in the local mode (TAE) dispersion relation.

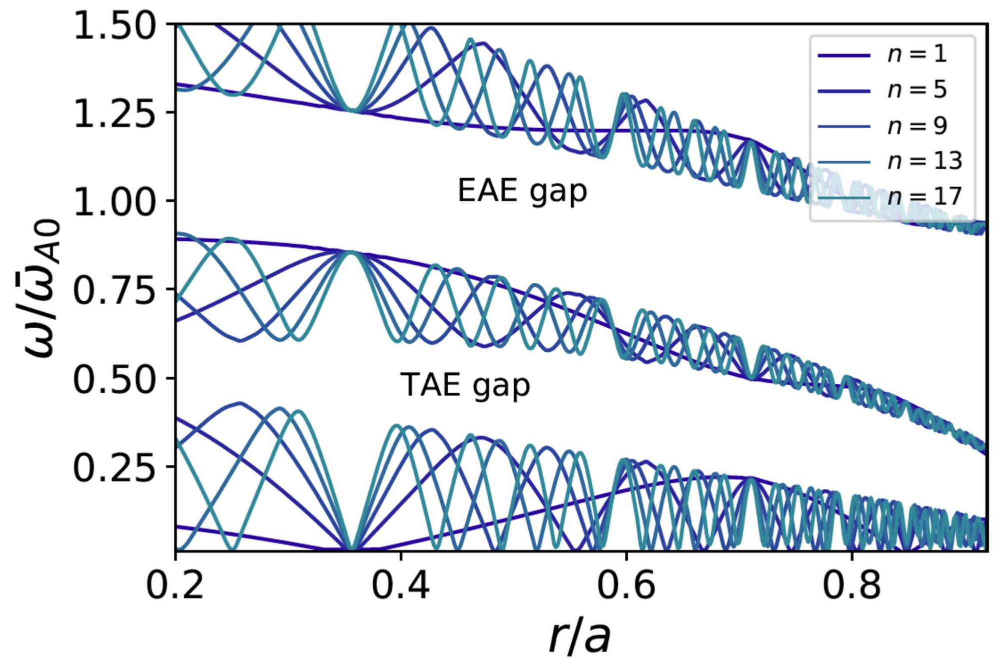
RP: Using the SAW vorticity equation in general geometry, given in [Z&C POP 14], write your own 1D numerical code and apply physical boundary conditions together with regularity conditions at the origin, and obtain the general local mode dispersion relation numerically.

- With the continuous spectrum and asymptotic solution, we can easily construct the TAE dispersion relation from the GFLDR

$$\frac{P'(2k\pi)}{P(2k\pi)} + i\nu = i\Lambda = \delta\bar{W}_f + \delta\bar{W}_k$$



DTT Boozer coordinates



TAE/EAE frequency continuum in DTT

- Solution along field lines yields $\delta\bar{W}_f$ and $\delta\bar{W}_k$ [Falessi, Carlevaro et al. 2019].

TAE linear dispersion relation

- Take $\hat{\phi}_s = A(\vartheta)\cos(\vartheta/2) + B(\vartheta)\sin(\vartheta/2)$. The relevant equations are:

$$-A' + \Gamma_- B = 0 \,, \qquad B' + \Gamma_+ A = 0 \;.$$

$$\Gamma_\pm = (\Omega^2 - 1/4) \pm \epsilon_0 \Omega^2$$

- Lowest order solution is:

$$A(\vartheta) = \exp(-\Gamma|\vartheta|)$$

$$\Gamma = (-\Gamma_+ \Gamma_-)^{1/2}$$

E: Show that this is the solution. Can you write the solution for $B(\vartheta)$?

- Condition for localized modes in ϑ space (nonsingular radial structures) is:

$$\Gamma^2 = \epsilon_0^2 \Omega^4 - (\Omega^2 - 1/4)^2 > 0$$

$$|\Omega^2 - 1/4| < \epsilon_0 \Omega^2 \quad \Rightarrow \text{FREQUENCY GAP}$$

Asymptotic matching with internal region

- Internal region equation is

$$\left[\frac{\partial^2}{\partial \vartheta^2} + \frac{1}{4} + \frac{\alpha \cos \vartheta}{\hat{\kappa}_\perp} - \frac{\left(s - \alpha \cos \vartheta \right)^2}{\hat{\kappa}_\perp^2} \right] \hat{\phi}_{sI} = 0$$

- For $|s|, |\alpha| \ll 1$, the external region solution $\hat{\phi}_{sE} = A(\vartheta) \cos(\vartheta/2) + B(\vartheta) \sin(\vartheta/2)$ applies in the internal region as well
- For $|s|, |\alpha| \ll 1$, the mode is located very near the lower continuum accumulation point (verify a posteriori), so $|\Gamma_+| \ll -\Gamma_- \simeq \epsilon_0/2$

$$\begin{array}{ll} \hat{\phi}_{sI} \simeq (-\Gamma_-)^{1/2} \cos \vartheta/2 & \text{internal region} \\ \hat{\phi}_{sE} \simeq \left[(-\Gamma_-)^{1/2} \cos \vartheta/2 \pm (\Gamma_+)^{1/2} \sin \vartheta/2 \right] e^{\mp \Gamma \vartheta} & \text{external region} \end{array}$$

- Apply perturbation theory and method of variation of arbitrary constant to derive the asymptotic behavior of internal solution (E: fill in the algebra!)

$$\hat{\phi}_{sI} = (-\Gamma_-)^{1/2}\left\{1 - \int_0^{\vartheta}\left[\frac{(s-\alpha\cos\vartheta')^2}{\hat{\kappa}_\perp^2} - \frac{\alpha\cos\vartheta'}{\hat{\kappa}_\perp}\right]\sin\vartheta' d\vartheta'\right\}\cos(\vartheta/2)$$

$$+(-\Gamma_-)^{1/2}\left\{\int_0^{\vartheta}\left[\frac{(s-\alpha\cos\vartheta')^2}{\hat{\kappa}_\perp^2} - \frac{\alpha\cos\vartheta'}{\hat{\kappa}_\perp}\right](1+\cos\vartheta')d\vartheta'\right\}\sin(\vartheta/2)$$

- Matching the $\sin\vartheta/2$ terms one find the TAE dispersion relation as [Cheng, Chen, Chance 1985]

$$\frac{(\Gamma_+)^{1/2}}{(-\Gamma_-)^{1/2}} = \delta T_f = \frac{s\pi}{4}\left(1 - \alpha\frac{1+s}{s^2}\right)$$

- δT_f plays the role of effective potential energy. That the mode exists for $\delta T_f > 0$ (ideal MHD stability) is consistent with the theoretical framework of the GFLDR (Lecture 4) [Z&C POP 14].

E: Show that for $\alpha \to \alpha_c = s^2/(1+s)$ the TAE mode merges into the lower continuum accumulation point. Discuss what happens to the mode damping rate when this occurs.

RP: What other relevant limits can you solve analytically by asymptotic matching of internal and external region solutions? Compare your results with the outcome of numerical simulations from RP at p. 16.

E: Show that $i\Lambda = (1/2)(\Gamma_+)^{1/2}/(-\Gamma_-)^{1/2}$ for the TAE mode. And thus that $\delta T_f = 2\delta\bar{W}_f$. (Hint: read [Z&C POP 14]).

RP: Given the RP at p. 16 and above, can you introduce finite θ_k dependence in the vorticity equation? Can you use the resultant equation to generate the radial envelope equation (Lecture 4) and solve the global dispersion relation and mode structure?

E: Compute the real frequency shift due to kinetic particle compressions and radiative damping using perturbation theory.

EP excitation of TAE/KTAE

- Illustrate the possible role of EP in the layer by EP excitation of TAE/KTAE [Chen RMP 2016] assuming small EP drift orbits.

- Relevant equations for TAE's (KAW's) as a vorticity equation. Derived assuming small Finite Larmor radius (FLR) correction to ideal Ohm's Law (Lecture 5)

$$\left[\partial_\vartheta^2 + \Omega^2\left(1 + 2\epsilon_0\cos\theta\right) + \frac{\alpha\cos\vartheta}{\hat{\kappa}_\perp^2} - \frac{(s-\alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^4} - \right.$$
$$\left. - \frac{s^4\rho_K^2}{\hat{\kappa}_\perp^2}(\vartheta-\theta_k)^4\right]\hat{\Phi} - \sum_j \frac{4\pi\omega e_j}{c^2}\frac{q^2R^2}{\hat{\kappa}_\perp k_\vartheta^2} < \omega_d\delta\hat{K} >_{vj} = 0 \ .$$

- Here, $\hat{\Phi} = \hat{\kappa}_\perp\delta\phi$, $\hat{\kappa}_\perp(\vartheta) = 1+[s(\vartheta-\theta_k)-\alpha\sin\vartheta]^2$, $\alpha = -Rq^2\beta'$, $\beta = 8\pi P/B^2$, $\Omega^2 = \omega^2/\omega_A^2$, $\epsilon_0 = 2(r_o/R+\Delta')$, $k_\vartheta = -nq/r$, and Δ' is the derivative of the Shafranov shift.

$$\omega_{dj}(\vartheta) = \Omega_{dj} \left\{ \cos\vartheta + [s(\vartheta - \theta_k) - \alpha \sin\vartheta] \sin\vartheta \right\}$$

$$\Omega_{dj} = -k_\vartheta \frac{cm_j}{e_j BR} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) \text{ and } < \ldots >_v = \int d^3v(\ldots)$$

- Kinetic effects associated to finite ion Larmor orbits and electron collisions as well as resistivity are accounted for by (Lecture 5)

$$\rho_K^2 = (k_\vartheta^2 \rho_i^2/2)[3/4 + (T_e/T_i)(1 - i\delta)] - i k_\vartheta^2 c^2 \eta / 16\pi\omega$$

$$\delta = \left(\frac{r}{R_0} \right)^{3/2} \sqrt{\frac{\nu_e}{\omega}} \left[\frac{1 + \ln\left(8 \frac{r}{R_0} \frac{\omega}{\nu_e}\right)}{(r/R_0)^{3/2} + (\nu_e/\omega)^{3/2}} \right] .$$

$$\nu_e = \frac{4\pi e^4 n_e \ln\Lambda}{m_e^{1/2} (2T_e)^{3/2}}$$

□ The distribution function $\delta \hat{K}_j$ is derived from the gyrokinetic equation (p. 22 Lecture 5; with $\omega_t = v_\parallel / qR$ the transit frequency)

$$[\omega_t \partial_\vartheta \; - \; i \; (\omega - \omega_d)]_j \; \delta \hat{K}_j \; = \; i \left(\frac{e}{m} \right)_j \; Q F_{0j} \, \frac{1}{\hat{\kappa}_\perp} \left(\frac{\omega_d}{\omega} \right)_j \hat{\Phi} \; ,$$

E: Note that here fast ion finite Larmor radius is neglected. Is that consistent with the assumption of thermal FLR? Why? What is fast ion FLR producing?

□ All characteristics of Alfvén waves are described:

- $\epsilon_0 = 0, \; \rho_K^2 = 0, \; \Rightarrow$ ideal MHD continuum
- $\epsilon_0 \neq 0, \; \rho_K^2 = 0, \; \Rightarrow$ TAE's
- $\epsilon_0 = 0, \; \rho_K^2 \neq 0, \; \Rightarrow$ KAW's
- $\epsilon_0 \neq 0, \; \rho_K^2 \neq 0, \; \Rightarrow$ KTAE's
- Energetic particle effects $\propto < \omega_d \delta \hat{K} >_h, \; \Rightarrow$ EPM's (ϵ_0 not crucial; E: can you tel why?)

□ Short scale mode structure from the large $\hat{\kappa}_\perp >> 1$ behavior

Eigenmode analysis

- Eigenmode analysis follows p. 18 with $\hat{\Phi} \simeq A(\vartheta)\cos(\vartheta/2) + B(\vartheta)\sin(\vartheta/2)$.
- Results remain the same: coupled differential equations for $A(\vartheta)$ and $B(\vartheta)$, with the addition of FLR effects and energetic particles.
- FLR effects are $\propto \hat{\kappa}_\perp^2 \sim s^2\vartheta^2$ as $\hat{\kappa}_\perp \gg 1$: no coupling between $A(\vartheta)$ and $B(\vartheta)$.
- Energetic particle response is modulated by geodesic curvature coupling: $\omega_d \simeq \Omega_d s\vartheta \sin\vartheta$.

$$
\begin{aligned}
\left[\omega_t \partial_\vartheta - i\left(\omega - \omega_d\right)\right]_j \delta\hat{K}_j \;&=\; i\left(\frac{e}{m}\right)_j QF_{0j}\frac{s\vartheta}{|s\vartheta|}\left(\frac{\Omega_d}{\omega}\right)_j\left[\frac{A}{2}\left(\sin\frac{\vartheta}{2} + \sin\frac{3\vartheta}{2}\right)\right. \\
&\left. + \frac{B}{2}\left(\cos\frac{\vartheta}{2} - \cos\frac{3\vartheta}{2}\right)\right]
\end{aligned}
$$

- Solution is straightforward, using $|\omega_t| \gg |\omega_d|$ and $|A'/A|, |B'/B| \ll 1$.

$$
\begin{aligned}
\delta \hat{K}_j \; = \; & - \left(\frac{e}{m}\right)_j QF_{0j} \frac{s\vartheta}{|s\vartheta|} \left(\frac{\Omega_d}{\omega}\right)_j \left[\left(\frac{\omega B/2 - i\omega_t A/4}{\omega^2 - \omega_t^2/4}\right) \cos\frac{\vartheta}{2} \right. \\
& + \left(\frac{\omega A/2 + i\omega_t B/4}{\omega^2 - \omega_t^2/4}\right) \sin\frac{\vartheta}{2} - \left(\frac{\omega B/2 + 3i\omega_t A/4}{\omega^2 - 9\omega_t^2/4}\right) \cos\frac{3\vartheta}{2} \\
& \left. + \left(\frac{\omega A/2 - 3i\omega_t B/4}{\omega^2 - 9\omega_t^2/4}\right) \sin\frac{3\vartheta}{2} \right]
\end{aligned}
$$

- Note that, due to geodesic curvature coupling, the particle response has both $\sim \cos(\vartheta/2)$ and $\sim \sin(\vartheta/2)$ dependences, as well as $\sim \cos(3\vartheta/2)$ and $\sim \sin(3\vartheta/2)$. This is called sideband generation and is typical of toroidal geometry where magnetic drifts induce $k_\parallel \pm 1/(qR_0)$ behaviors in the parallel mode structure.

E: Derive the above equation step by step and discuss the physics implication of sideband generation.

- When substituted back into the vorticity equation, the particle response $\propto \omega_t$ vanishes if one assumes even particle distribution function in $v_\parallel$ space.

E: Convince yourself that this is indeed the case. Why should the particle distribution be even in $v_\parallel$ space? What happens if this is not the case?

- Compute now the coupling of kinetic particle compression via geodesic curvature in the vorticity equation, neglecting odd terms $\propto \omega_t$ and modulations that are not modifying the leading order AE eigenfunction

$$
\begin{aligned}
-\sum_j \frac{4\pi\omega e_j}{c^2}\frac{q^2R^2}{\hat{\kappa}_\perp k_\vartheta^2} < \omega_d \delta\hat{K} >_{vj} \quad &= \quad \frac{\pi q^2}{B^2}\sum_j m_j \left\langle \left(\frac{v_\perp^2}{2}+v_\parallel^2\right)^2 QF_{0j} \right. \\
&\left. \times \left(\frac{\omega}{\omega^2-\omega_t^2/4}+\frac{\omega}{\omega^2-9\omega_t^2/4}\right)\right\rangle_{vj} \left(A\cos\frac{\vartheta}{2}+B\sin\frac{\vartheta}{2}\right) + \ldots
\end{aligned}
$$

E: Derive the above equation step by step. Is it valid for thermal particles as well, in addition to energetic particles? What does this response describe?

- Collecting terms and recalling p.18, the relevant equations are:

$$
\begin{aligned}
-A' + \left(\Gamma_- - s^2\rho_K^2\vartheta^2\right) B &= 0 \\
B' + \left(\Gamma_+ - s^2\rho_K^2\vartheta^2\right) A &= 0 \; .
\end{aligned}
$$

$$
\Gamma_\pm = (\Omega^2 - 1/4 - \beta_1) \pm \epsilon_0 \Omega^2
$$

$$
\begin{aligned}
\beta_1 = \omega \frac{\pi q^2}{B^2} \Bigg\langle \sum_j m_j \left(\frac{v_\perp^2}{2} + v_\parallel^2\right)^2 Q F_{0j} \Bigg[\left(\frac{\omega_t^2}{4} - \omega^2\right)^{-1} \\
+ \left(\frac{9\omega_t^2}{4} - \omega^2\right)^{-1} \Bigg] \Bigg\rangle \; .
\end{aligned}
$$

- Two relevant limits:

  - Small (perturbative) kinetic effects $\Rightarrow$ TAE's
  - Kinetic effects $\approx$ Toroidal effects $\Rightarrow$ KTAE's

TAE Linear Stability

- Treat kinetic effects perturbatively
- Lowest order solution is (p. 18):

$$A(\vartheta) = \exp(-\Gamma|\vartheta|)$$
$$\Gamma = (-\Gamma_+\Gamma_-)^{1/2}$$

E: Show that this is the solution. Can you write the solution for $B(\vartheta)$?

- Condition for localized modes in ϑ space (nonsingular radial structures) is:

$$\Gamma^2 = \epsilon_0^2\Omega^4 - (\Omega^2 - 1/4)^2 > 0$$
$$|\Omega^2 - 1/4| < \epsilon_0\Omega^2 \quad \Rightarrow \text{FREQUENCY GAP}$$

- Expression for the *local* growth rate, including thermal electron and ion Landau damping:

$$\frac{\gamma}{\omega} = \frac{4\pi^2 q^3 R}{B^2} \left\langle \sum_j m_j \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)^2 Q F_{0j} \left[\delta(v_\parallel - v_A) + \frac{1}{3}\delta(v_\parallel - v_A/3) \right] \right\rangle + \frac{\mathrm{Im}(s^2 \rho_K^2)}{\Gamma^2} \; .$$

RADIATIVE DAMPING

$\approx k_\theta^2 \rho_i^2$

E: Use F_0 as Maxwellian distribution function to compute explicitly thermal electron and ion Landau damping. Comment on the importance of ion Landau damping at high-β.

- Energy flow to $|\vartheta| >> 1$ (small scale) $\Rightarrow$ radiative damping [Mett and Mahajan '92]

- Typical scale: $|\vartheta| \approx 1/\Gamma$. $\Gamma_{\pm} = (\Omega^2 - 1/4 - \beta_1) \pm \epsilon_0 \Omega^2 \Rightarrow$ perturbative treatment of energetic particles may overestimate R.D.
  E: Can you show why this is the case?

- Consistency condition:
  - $(\epsilon_0 s^2 \rho_K^2)/\Gamma^4 << 1$
  - In general: $|\gamma/\omega| << \epsilon_0 \omega_A$

  E: Comment both these conditions and explain what they mean

- It may break down because of *local* effects (large R.D.) or *global* effects $\Rightarrow$
  CONTINUUM
  DAMPING

KTAE dispersion relation and linear stability

- Approach the lower continuum accumulation point:
  - $\Gamma_+ \simeq 0$ and $\Gamma_- \simeq -\epsilon_0/2$
- Relevant equations:

$$A'' + \Gamma_- \left(\Gamma_+ - s^2 \rho_K^2 \vartheta^2\right) A = 0$$

- Dispersion relation:

$$\Gamma_+ = -(2N+1) \left(\frac{s^2 \rho_K^2}{\Gamma_-}\right)^{1/2}$$

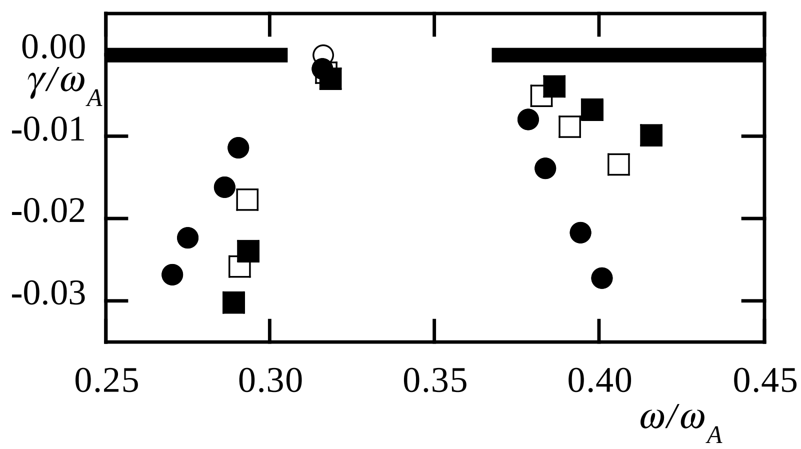
- At the upper continuum $\Gamma_+ \simeq \epsilon_0/2$ and $\Gamma_- \simeq 0$:

$$\Gamma_- = (2N+1)\left(\frac{s^2\rho_K^2}{\Gamma_+}\right)^{1/2}$$

- Radial width $\Delta r_{KTAE} \approx (k_\theta s)^{-1}(\epsilon_0 s^2 \rho_K^2)^{1/4}$. $\Delta r_{KTAE} \ll \Delta r_{TAE} \approx (k_\theta s)^{-1}\Gamma \Rightarrow$ $\Rightarrow \epsilon_o s^2 \rho_K^2/\Gamma^4 << 1$.

E: Show that the derived KTAE dispersion relations explain this figure (shown before)

E: Show that for $\rho_i^2 = 0, \eta \neq 0$ lower and upper KTAE are on stable lines at 45^o departing from accumulation points [Cheng, Chen, Chance 1985]. How about $\rho_i^2 \neq 0, \eta = 0$? [Mett and Mahajan 1992]



References and reading material

L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

Z. X. Lu, F. Zonca and A. Cardinali, Phys. Plasmas **19**, 042104 (2012).

F. Zonca, L. Chen, S. Briguglio, G. Fogaccia, G. Vlad and X. Wang, New J. Phys. **17**, 013052 (2015).

T. M. Antonsen and B. Lane, Phys. Fluids **23**, 1205 (1980).

P. J. Catto and W. M. Tang and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).

E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).

A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).

L. Chen and A. Hasegawa, J. Geophys. Res. **96**, 1503 (1991).

H. Qin, W. M. Tang and G. Rewoldt, Phys. Plasmas **5**, 1035 (1998).

G. Chew, F. Goldberger and F. Low, Proc. Roy. Soc. Ser. A **236**, 112 (1956).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072120 (2014).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072121 (2014).

L. Chen, R. B. White, and M. N. Rosenbluth, Phys. Rev. Lett. **52**, 1122 (1984).

A. Hasegawa and L. Chen, Phys. Rev. Lett. **35**, 370 (1975).

A. Hasegawa and L. Chen, Phys. Fluids **19**, 1924 (1976).

R. R. Mett and S. M. Mahajan, Phys. Fluids B **4**, 2885 (1992).

G. Vlad, F. Zonca and S. Briguglio, Riv. Nuovo Cimento **22**, 1 (1999).

X. Wang, F. Zonca and L. Chen, Plasma Phys. Control. Fusion **52**, 115005 (2010).

L. Chen and A. Hasegawa, J. Geophys. Res. **96**, 1503 (1991).

F. Zonca and L. Chen, Plasma Phys. Control. Fusion **48**, 537 (2006).

G. Y. Fu and C. Z. Cheng, Phys. Fluids B **4**, 3722 (1992).

H. L. Berk, B .N. Breizman and H. Ye, Phys. Lett. A **162**, 475 (1992).

S.-T. Tsai and L. Chen, Phys. Fluids B **5**, 3284 (1993).

L. Chen, Phys. Plasmas **1**, 1519 (1994).

A. Brizard, Phys. Plasmas **1**, 2460 (1994).

M. Falessi, N. Carlevaro, V. Fusco, G. Vlad and F. Zonca, *Shear Alfvén and acoustic continuum in general axisymmetric toroidal geometry*, to be published in PoP.