Lecture 5

Kinetic Alfvén waves and low-frequency Alfvén Eigenmodes in tokamaks

Fulvio Zonca

http://www.afs.enea.it/zonca

ENEA C.R. Frascati, C.P. 65 - 00044 - Frascati, Italy.

Institute for Fusion Theory and Simulation, Zhejiang University, Hangzhou 310027, P.R.C.

June 18.th, 2019

Kinetic theory and global dispersion relation
of Alfvén waves in tokamaks
June 4 – 20, 2019, IFTS – ZJU, Hangzhou

Kinetic Alfvén Waves

- As a simple example to illustrate the application of the linear gyrokinetic theory, let us analyze the linear wave properties of the kinetic Alfvén wave (KAW).

- KAW is a SAW modified by the finite ion Larmor radius effects.

- When approaching the continuous spectrum, Lecture 1 suggests that the radial wave-vector is

$$|k_x| \cong |\omega'_{\mathcal{A}\ell}(x)t|$$

and, thus, $|k_x| \to \infty$ as $t \to \infty$; *i.e.*, the wave function becomes singular in the asymptotic time limit.

- This indicates the break down of the ideal MHD assumption; i.e., $|k_\perp \rho_i|^2 \ll 1$. Proper treatments of microscopic ρ_i-scale SAW are what require kinetic-theory analysis.

- The most relevant new (with respect to the ideal MHD model) dynamics that appear on short scales are associated with charge separation
  - finite parallel electric field fluctuations ($\delta E_{\parallel}$) due to, *e.g.*, finite ion Larmor radius, small but finite electron inertia and finite plasma resistivity
  - finite $\delta E_{\parallel}$ $\Rightarrow$ additional effects are to be expected also from wave-particle interactions, which yield collisionless wave dissipation (Landau damping).
  - $\Rightarrow$ finite energy propagation across the resonant surfaces $x = x_{R\ell}$ and wave-function singularities are removed on short scales.

Kinetic Alfvén Waves: uniform plasma

- Consider an infinite uniform plasma immersed in a magnetic field $\boldsymbol{B}_0 = B_0\mathbf{e}_z$.
- Furthermore, let the equilibrium distribution functions be isotropic Maxwellians; *i.e.*, for $j = e, i$,

$$\bar{F}_{0j} = n_{0j} F_{Mj} = \frac{n_{0j}}{(\sqrt{\pi} v_{tj})^3} \exp\left(-v^2/v_{tj}^2\right) \ ,$$

  where $v_{tj}^2 = 2T_j/m_j$ and n_{0j} is the equilibrium particle density.
- Taking perturbations of the form $\sim \exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x})$, where $\boldsymbol{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$, we then have, from the linear GKE (Lecture 4),

$$\delta g_{kj} = -\left[\frac{e}{T} F_M J_0 \frac{\omega}{k_\parallel v_\parallel - \omega}\left(\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel\right)\right]_j \ .$$

- By direct substitution into the quasineutrality condition (Lecture 4)

$$\sum_{j=e,i}\left(\frac{n_0e^2}{T}\right)_j\delta\phi_k+\sum_{j=e,i}\left(\frac{n_0e^2}{T}\right)_j\Gamma_{0kj}\left[\xi_{kj}Z_{kj}\delta\phi_k-(1+\xi_{kj}Z_{kj})\delta\psi_k\right]=0\ .$$

- Here, we have defined, for modes with finite $k_\|$, $\delta\psi_k=(\omega\delta A_\|/ck_\|)_k$, $\Gamma_{0kj}=\int J_0^2F_{Mj}d\boldsymbol{v}=I_0(b_{kj})\exp(-b_{kj})$, I_0 is the modified Bessel function, $b_{kj}=k_\perp^2\rho_j^2/2=k_\perp^2(T_j/m_j)/\Omega_j^2$, $\xi_{kj}=\omega/|k_\||v_{tj}$ and

$$Z_{kj}=Z(\xi_{kj})=\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\frac{e^{-y^2}}{(y-\xi_{kj})}dy$$

is the plasma dispersion function.

- Similarly, the vorticity equation (Lecture 4) becomes

$$i\frac{c^2}{4\pi\omega_k}k_\parallel^2 k_\perp^2 \delta\psi_k - i\sum_{j=e,i}\left(\frac{n_0e^2}{T}\right)_j (1-\Gamma_{0kj})\omega_k\delta\phi_k = 0 \; .$$

E: Derive the nonadiabatic particle response from the GKE. Then demonstrate the quasineutrality condition and the gyrokinetic vorticity equation.

- Taking further approximations, $b_{ke} = k_\perp^2\rho_e^2/2 \ll 1$, the gyrokinetic vorticity equation reduces to

$$\delta\phi_k = \frac{k_\parallel^2 v_A^2}{\omega_k^2}\frac{b_k}{(1-\Gamma_k)}\delta\psi_k \; ,$$

with $v_A^2 = B_0^2/(4\pi n_{0i}m_i)$ the Alfvén speed.

- Here, for simplicity of notation, $b_k = b_{ki}$ and $\Gamma_k = \Gamma_{0ki}$.

- Noting that $|\omega/k_\parallel| \sim v_A$ and, for a wide parameter range of interest, $1 \gg \beta \gg m_e/m_i$, we have $|\xi_{ki}| \gg 1 \gg |\xi_{ke}|$.

- Expanding the plasma dispersion function for small/large arguments where appropriate, the quasineutrality condition then becomes (with $\tau = T_e/T_i$; and assuming $e_i = |e|$)

$$\delta\psi_k = [1 + \tau(1 - \Gamma_k)]\, \delta\phi_k \equiv \sigma_k \delta\phi_k \ .$$

- Combining GK vorticity and QN gives the following KAW dispersion relation with wave particle resonances neglected

$$\omega_k^2 = k_\parallel^2 v_A^2 \frac{b_k \sigma_k}{(1 - \Gamma_k)} \ .$$

E: Discuss the properties of the KAW dispersion relation using this result.

- One important feature of KAW is that it possesses a significant component of parallel electric field.

- More specifically, noting that $\delta E_{\parallel k} = -ik_{\parallel}(\delta\phi_k - \delta\psi_k)$ and the QN condition, we have

$$\delta E_{\parallel k} = ik_{\parallel}\tau(1-\Gamma_k)\delta\phi_k \ .$$

- The existence of a finite $\delta E_{\parallel k}$ then leads to finite wave-particle resonance and, thereby, the important implications of charged particle heating as well as transport across $\boldsymbol{B}_0$.

E: Explain why finite KAW heating is connected with the existence of a finite $\delta E_{\parallel k}$. How about cross field transport?

Kinetic Alfvén Waves: 1D non-uniform plasmas

- Consider the one dimensional non-uniform plasma equilibrium introduced in Lecture 1.
- Make same approximations introduced on pp. 6-7 for deriving coupled GK vorticity and QN. Furthermore, for the sake of simplicity, we also assume $(k_x^2 + k_y^2)\rho_i^2 \equiv k_\perp^2 \rho_i^2 \ll 1$.
- It is then possible to show that the vorticity equation (p. 15 Lecture 1) becomes [Hasegawa and Chen 75, 76]

$$\left[\omega^2 \nabla_\perp^2 \rho_K^2 \nabla_\perp^2 + \nabla_\perp \cdot \epsilon_{A\ell} \nabla_\perp\right] \delta\hat{\xi}_{x\ell} = 0 \,,$$

where

$$\rho_K^2 = \left[\frac{3}{4}(1 - i\delta_i) + \frac{T_e}{T_i}(1 - i\delta_e)\right] \rho_i^2 - i\frac{c^2\eta}{4\pi\omega} \,.$$

- Here, δ_i and δ_e indicate, respectively, ion and electron Landau damping contributions, whereas the term proportional to η is due to finite plasma resistivity.
- δ_i and δ_e are "simplified model" expressions that account the finite wave-particle resonant interactions due to the general plasma dispersion function response in the QN condition on p. 5.
- The plasma resistivity response is derived noting that a term $\eta \delta j_{\parallel k}$ should be added on the right hand side of Ohm's law (p. 8).

E: Derive KAW equation from QN condition and GK vorticity equation dropping magnetic curvature and diamagnetic terms in the latter, and adding resistive dissipation in the parallel Ohm's law.

E: Derive the small FLR limit of KAW dispersion relation noting parallel Ampère's law and QN condition.

- In the KAW equation, the singularity at $\epsilon_{A\ell} = 0$ is clearly removed by the term including the 4th order derivative.
- The 4th order derivative is also proportional to $\rho_K^2 \ll 1$, indicating the formation of a boundary layer around the SAW resonant surface.

E: Demonstrate that a boundary layer indeed appears at the SAW resonant surface $\epsilon_{A\ell} = 0$.

- The KAW equation,describes the mode-conversion of a long wavelength MHD mode (the SAW) to a short wavelength kinetic mode, the KAW [Hasegawa and Chen 75, 76].
- The WKB local dispersion relation of KAW's is

$$\omega^2 = \left(1 + k_\perp^2 \rho_K^2\right) \omega_{A\ell}^2 \, ,$$

and indicates that KAW's are propagating for $\epsilon_{A\ell} > 0$ and become cut-off for $\epsilon_{A\ell} < 0$.

- Assuming that $\omega_{\mathcal{A}\ell}(x) \simeq \omega_{\mathcal{A}\ell}(x_{R\ell})(1+\kappa\zeta)$ near the resonance absorption layer, with $\zeta = x - x_{R\ell}$ and $\kappa = (d/dx_{R\ell}) \ln \omega_{\mathcal{A}\ell}(x_{R\ell}) > 0$, one has [Hasegawa and Chen 76]

$$\left(\rho_K^2 \frac{d^2}{d\zeta^2} + \kappa\zeta \right) \delta\hat{\xi}_{x\ell} = 0 \quad .$$

- General solutions are written in terms of Airy functions and have the following form away from the SAW resonant absorption layer

$$\delta\hat{\xi}_{x\ell} = \frac{\delta\hat{\xi}_{x\ell 0}}{\kappa\zeta} \qquad \zeta < 0 \ , \qquad \text{SAW}$$

$$\delta\hat{\xi}_{x\ell} = \frac{\delta\hat{\xi}_{x\ell 0}}{\kappa\zeta} - \frac{\pi^{1/2}\delta\hat{\xi}_{x\ell 0}}{(\kappa\rho_K)^{2/3}} \left(\frac{\rho_K^{2/3}}{\kappa^{1/3}\zeta} \right)^{1/4} \qquad \text{SAW} \oplus \text{KAW}$$

$$\times \exp\left\{ i \left[\frac{2}{3} \left(\frac{\kappa^{1/3}\zeta}{\rho_K^{2/3}} \right)^{3/2} + \frac{\pi}{4} \right] \right\} \qquad \zeta > 0 \quad .$$

Bound states with Kinetic Alfvén Waves

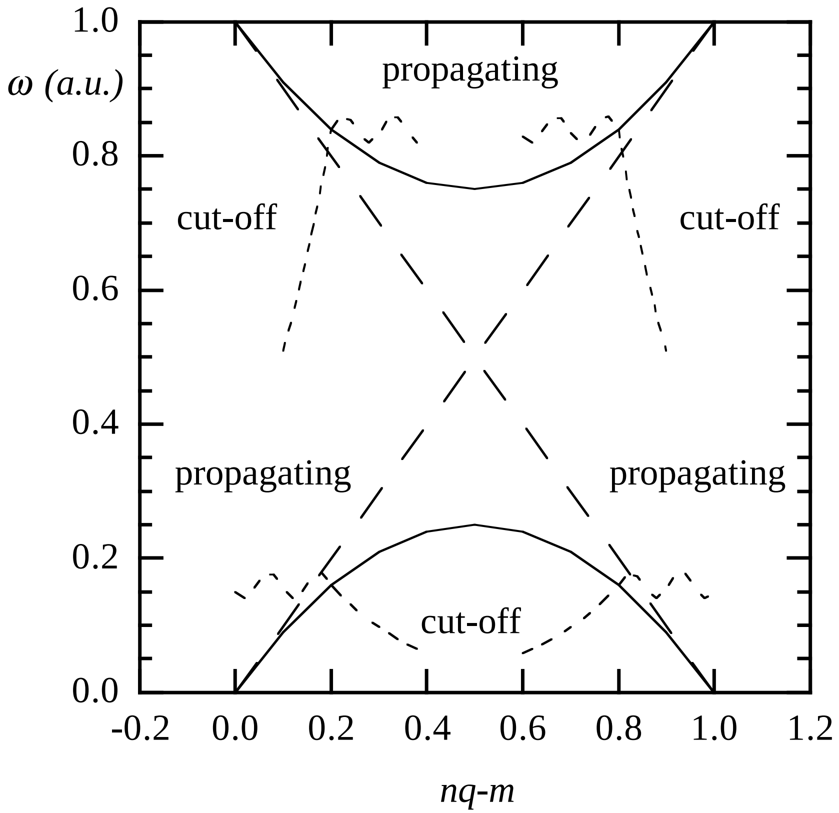
- Kinetic Alfvén Waves (KAW) are characterize by the following dispersion relation

$$\omega^2 = k_\parallel^2 v_A^2 \left(1 + k_\perp^2 \rho_i^2 (3/4 + T_e/T_i)\right)$$

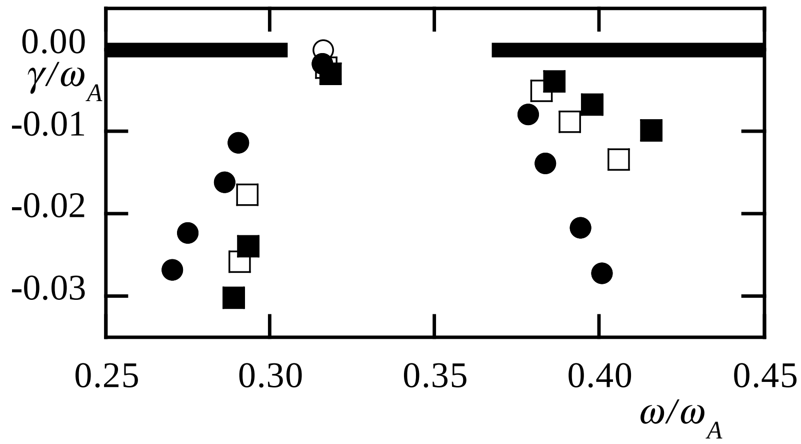
- KAW are propagating on the high-frequency side $(\omega_*/\omega, \omega_{ti}/\omega \to 0)$ and are generic to magnetized plasmas: no toroidal geometry needed
- At SAW continuum resonances, SAW can mode convert to KAW: Alfvén wave heating [Hasegawa and Chen 76]
- In toroidal system, KAW may generate standing waves near the TAE frequency gap. These waves are dubbed Kinetic TAE (KTAE); [Mett and Mahajan 1992].

Lower and Upper KTAE structures

[Vlad et al. 1999]



- Wave structures and strong (lower branch) damping vs. weak (upper branch) damping [Mett and Mahajan 1992]. (Lecture 6)



E: Demonstrate the formation of Kinetic BAE (KBAE) [X. Wang 10]. How about other modes? E.g. KBAAE?

Generalized FL dispersion relation: properties

- From Lecture 2 to Lecture 4, the mode dispersion relation can be always written in the form of a *fishbone-like* dispersion relation [Chen 1984]

$$-i\Lambda + \delta\bar{W}_f + \delta\bar{W}_k = 0 \quad ,$$

where $\delta\bar{W}_f$ and $\delta\bar{W}_k$ play the role of fluid (core plasma) and kinetic (fast ion) contribution to the potential energy, while Λ represents a generalized inertia term.

- The generalized fishbone-like dispersion relation can be derived by asymptotic matching the regular (ideal MHD) mode structure with the general (known) form of the SA wave field in the singular (inertial) region, as the spatial location of the shear Alfvén resonance, $\omega^2 = k_\parallel^2 v_A^2$, is approached.

- Examples are : $\Lambda^2 = \omega(\omega - \omega_{*pi})/\omega_A^2$ for $|k_\parallel q R_0| \ll 1$ and $\Lambda^2 = (\omega_l^2 - \omega^2)/(\omega_u^2 - \omega^2)$ for $|k_\parallel q R_0| \approx 1/2$, with ω_l and ω_u the lower and upper accumulation points of the shear Alfvén continuous spectrum toroidal gap [Chen 94].

- $\delta\bar{W}_f$ is generally real, whereas $\delta\bar{W}_k$ is characterized by complex values, the real part accounting for non-resonant and the imaginary part for resonant wave particle interactions with energetic ions.

- The fishbone-like dispersion relation demonstrates the existence of two types of modes (note: $\Lambda^2 = k_\parallel^2 q^2 R_0^2$ is SAW continuum; see later):
  - a discrete gap mode, or Alfvén Eigenmode (AE), for $\mathbb{R}\mathrm{e}\Lambda^2 < 0$;
  - an Energetic Particle continuum Mode (EPM) for $\mathbb{R}\mathrm{e}\Lambda^2 > 0$.

- For EPM, the iΛ term represents continuum damping. Near marginal stability [Chen 84, Chen 94]

$$\mathbb{R}\mathrm{e}\delta\bar{W}_k(\omega_r) + \delta\bar{W}_f = 0 \quad \text{determines } \omega_r$$

$$\gamma/\omega_r = (-\omega_r\partial_{\omega_r}\mathbb{R}\mathrm{e}\delta\bar{W}_k)^{-1}(\mathbb{I}\mathrm{m}\delta\bar{W}_k - \Lambda) \quad \text{determines } \gamma/\omega_r$$

- For AE, the non-resonant fast ion response provides a real frequency shift, *i.e.* it removes the degeneracy with the continuum accumulation point, while the resonant wave-particle interaction gives the mode drive. Causality condition imposes

  - $\delta \bar{W}_f + \mathbb{R}\mathrm{e}\delta \bar{W}_k > 0$ when AE frequency is above the continuum accumulation point: inertia in excess w.r.t. field line bending

$$\Lambda^2 = \lambda_0^2(\omega_\ell - \omega) \; ; \qquad \omega > \omega_\ell \;\Rightarrow\; \Lambda \to -\mathrm{i}\sqrt{-\Lambda^2}$$

  - $\delta \bar{W}_f + \mathbb{R}\mathrm{e}\delta \bar{W}_k < 0$ when AE frequency is below the continuum accumulation point: inertia in lower than field line bending

$$\Lambda^2 = \lambda_0^2(\omega - \omega_u) \; ; \qquad \omega < \omega_u \;\Rightarrow\; \Lambda \to \mathrm{i}\sqrt{-\Lambda^2}$$

- For AE, $i\Lambda$ represents the shift of mode frequency from the accumulation point
- For both AE and EPM, the SAW accumulation point is the natural gateway through which modes are born at marginal stability
- For EPM, ω is set by the relevant energetic ion characteristic frequency and mode excitation requires the drive exceeding a threshold due to continuum damping. However, the non-resonant fast ion response is crucially important as well, since it provides the compression effect that is necessary for balancing the positive MHD potential energy of the wave.
- Next (Lecture 5 and Lecture 6), consider applications of the GFLDR theoretical framework.

MSD form of reduced drift Alfvén wave equations

- Here and in Lecture 6, we adopt the GFLDR formulation of Alfvénic fluctuations in tokamaks (parallel mode structure).

- It is useful to introduce the following decomposition:

$$\delta g \equiv \delta K + i\frac{e}{m} Q\bar{F}_0 \partial_t^{-1} \langle \delta\psi_g \rangle$$

with $\langle \delta A_{\parallel g} \rangle = -c\partial_t^{-1} \nabla_\parallel \langle \delta\psi_g \rangle$ and the operator $Q\bar{F}_0$ defined as [Chen 84]

$$iQ\bar{F}_0 = -\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} + \frac{\boldsymbol{b} \times \boldsymbol{\nabla} \bar{F}_0}{\Omega} \cdot \boldsymbol{\nabla} \quad .$$

- This subdivision of the particle response is particularly convenient in linear analyses [Chen 84, Chen 91], since $\delta K \to 0$ in the fluid ion ($\omega \gg \bar{\omega}_d, \omega_b$) and massless electron ($\omega_b \gg \omega, \bar{\omega}_d$) limits.

- The GKE can be cast as

$$\left(\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp\right) \delta K = i\frac{e}{m}\left[Q\bar{F}_0 \langle \delta\phi_g - \delta\psi_g \rangle \right.$$
$$\left. -\boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp Q\bar{F}_0 \partial_t^{-1} \langle \delta\psi_g \rangle \right] \quad .$$

- Using the MSD representation (Lecture 1), the field equations are the QN condition for $\delta\phi \to \delta\hat{\Phi} = \hat{\kappa}_\perp \delta\hat{\phi}$, and the GK vorticity equation for $\delta\psi \to \delta\hat{\Psi} = \hat{\kappa}_\perp \delta\hat{\psi}$.

- QN condition (from Lecture 4 p.9, $k_\perp^2 \rho_i^2 \ll 1$), with $k_\perp^2 = k_\vartheta^2 \hat{\kappa}_\perp^2$,

$$\left(1 + \frac{1}{\tau}\right)\left(\delta\hat{\Phi}_n - \delta\hat{\Psi}_n\right) + \left(1 - \frac{\omega_{*pi}}{\omega}\right) k_\vartheta^2 \rho_i^2 \hat{\kappa}_\perp^2 \delta\hat{\Psi}_n$$
$$= \frac{T_i}{n_0 e}\hat{\kappa}_\perp \left\langle \left(1 - k_\vartheta^2 \frac{\mu B_0}{2\Omega^2}\hat{\kappa}_\perp^2\right) \delta\hat{K}_{in} \right\rangle_v \quad .$$

where $\omega_{*pi} = \omega_{*ni} + \omega_{*Ti}$ is the thermal ion diamagnetic frequency and

$$\omega_{*ni} = \left(\frac{T_0 c}{e n_0 B_0}\right)_i (\boldsymbol{b} \times \boldsymbol{\nabla} n_{0i}) \cdot \boldsymbol{k}_\perp \quad ,$$

$$\omega_{*Ti} = \left(\frac{c}{e B_0}\right)_i (\boldsymbol{b} \times \boldsymbol{\nabla} T_{0i}) \cdot \boldsymbol{k}_\perp \quad .$$

- Note that energetic particle density has been neglected, but this approximation may not be good in some of present day expt.
- The GK vorticity equation [Z & C POP14] (from Lecture 4 p.16), with $g(\vartheta, \theta_k) = [s(\vartheta - \theta_k) - \alpha \sin\vartheta] \sin\vartheta + \cos\vartheta$ [(s, α) model equilibrium]

$$\left(\frac{\partial^2}{\partial\vartheta^2} + \Delta' \cos\vartheta\right) \delta\hat{\Psi}_n + \frac{\omega}{\omega_A^2} \left[1 + 4(r/R_0)\cos\vartheta\right] \left[\omega - \omega_{*pi} - \frac{3}{4} k_\vartheta^2 \rho_i^2 \hat{\kappa}_\perp^2 \left(\omega - \omega_{*pi} - \omega_{*Ti}\right)\right] \delta\hat{\Phi}_n$$

$$+ \frac{4\pi R_0 q^2}{B_0} \frac{g(\vartheta, \theta_k)\omega}{c k_\vartheta \hat{\kappa}_\perp} \left[\left\langle m_i \left(\mu B_0 + v_\parallel^2\right) \left(1 - k_\vartheta^2 \frac{\mu B_0}{2\Omega_i^2} \hat{\kappa}_\perp^2\right) \delta\hat{K}_{in} \right\rangle_v + \left\langle m_E \left(\mu B_0 + v_\parallel^2\right) J_0 \delta\hat{g}_{En} \right\rangle_v\right]$$

$$+ \left[\frac{\alpha\cos\vartheta}{\hat{\kappa}_\perp^2} - \frac{(s - \alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^4} + \frac{(\alpha_c - \alpha) g(\vartheta, \theta_k)}{\hat{\kappa}_\perp^2}\right] \delta\hat{\Psi}_n = 0 \quad ,$$

Low frequency Alfvén and acoustic continuum

- Low-frequency limit consists in $\omega^2/\omega_A^2 \ll 1$; i.e., $k_\parallel^2 q^2 R_0^2 \ll 1$.
- SAW and SSW continuous spectra can be readily computed in the limit $\hat{\kappa}_\perp \to \infty$ ($|\nabla_\perp| \to \infty$).
- From the frequency ordering, applied to the governing equations:
  - SAW vorticity: $|\partial_\vartheta^2| \ll 1 \Rightarrow \hat{\phi}_s \sim e^{i\Lambda|\vartheta|}$
  - SSW equation (perturbed parallel force balance vorticity) yields modulated compression response [with $\epsilon_S = 1 - (\omega_S^2/\omega^2)(1+\Lambda^2)$]:

$$\delta \hat{P}_{\rm comp} \simeq \frac{2\Gamma P_0 c}{B_0 R_0}\left(-ik_\vartheta \frac{s\vartheta}{|s\vartheta|}\right)\left[\frac{\epsilon_S}{\epsilon_S^2 - 4\Lambda^2\omega_S^4/\omega^4}\sin\vartheta - i\frac{2\Lambda(\vartheta/|\vartheta|)\omega_S^2/\omega^2}{\epsilon_S^2 - 4\Lambda^2\omega_S^4/\omega^4}\cos\vartheta\right]\hat{\phi}_s$$

E: Derive the expression of the modulated compression response and/or verify it satisfies the SSW equation. Comment about the approximations that are needed to derive this form.

- The expression of the modulated compression response is then substituted back into the SAW vorticity equation, which, because of self-consistency constraints, poses the following condition (that is, definition) of Λ^2

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2} - \frac{\Gamma\beta q^2 \epsilon_S}{\epsilon_S^2 - 4\Lambda^2\omega_S^4/\omega^4} \; .$$

E: Derive this expression step by step. Show that the physical interpretation of Λ^2 is “renormalized” or “enhanced” inertia.

- This expression is consistent with the proper limiting form of the more general kinetic expression that can be derived from kinetic theory (cf. later).

- The “averaged” effect of pressure curvature coupling at $\hat{\kappa}_\perp \to \infty$ (continuous spectrum) results in the equation

$$\left(\partial_\vartheta^2 + \Lambda^2\right)\hat{\phi}_s = 0 \; .$$

- This shows that the continuous spectrum is given by

$$\Lambda^2 = k_\parallel^2 q^2 R_0^2 \ .$$

- **Beta induced Alfvén Eigenmode continuum**: Assuming $|\omega^2| \gg \omega_S^2$ (consistent with $q^2 \gg 1$), $\epsilon_S \simeq 1$ and Λ^2 can be cast as

$$\omega^2 = \omega_{BAE}^2 \left(1 + \frac{\Lambda^2}{\Gamma \beta q^2} \right) \ ,$$

where $\omega_{BAE}^2 \equiv \Gamma \beta q^2 \omega_A^2 = 2 q^2 \omega_S^2$ is the BAE accumulation point.

- Adopting the continuous spectrum expression, $\Lambda^2 = k_\parallel^2 q^2 R_0^2$, we have that the SAW continuum near the BAE accumulation point is

$$\omega^2 = \omega_{BAE}^2 \left(1 + \frac{k_\parallel^2 R_0^2}{\Gamma \beta} \right)$$

Thus, we have a frequency gap for $\omega^2 < \omega_{BAE}^2$.

- **Beta induced Alfvén acoustic Eigenmode continuum**: Consider now the limit $|\epsilon_S^2| \ll |\Lambda^2| \ll 1$. This is the slow sound wave (SSW) approximation and implies $\omega^2 \simeq \omega_S^2$.

- The equation for Λ^2 can be cast as

$$\omega^2 = \omega_S^2 \left[1 + \Lambda^2 \left(1 - \frac{2}{q^2} + \frac{4\Lambda^2}{\Gamma \beta q^2} \right) \right] .$$

- Similar to the BAE case, letting $\Lambda^2 = k_\parallel^2 q^2 R_0^2$, we have that the Alfvén - acoustic continuum near the BAAE accumulation point is

$$\omega^2 = \omega_S^2 \left[1 + k_\parallel^2 q^2 R_0^2 \left(1 - \frac{2}{q^2} + \frac{4 k_\parallel^2 R_0^2}{\Gamma \beta} \right) \right]$$

Thus, for $q^2 > 2$, we have a frequency gap for $\omega^2 < \omega_S^2$.

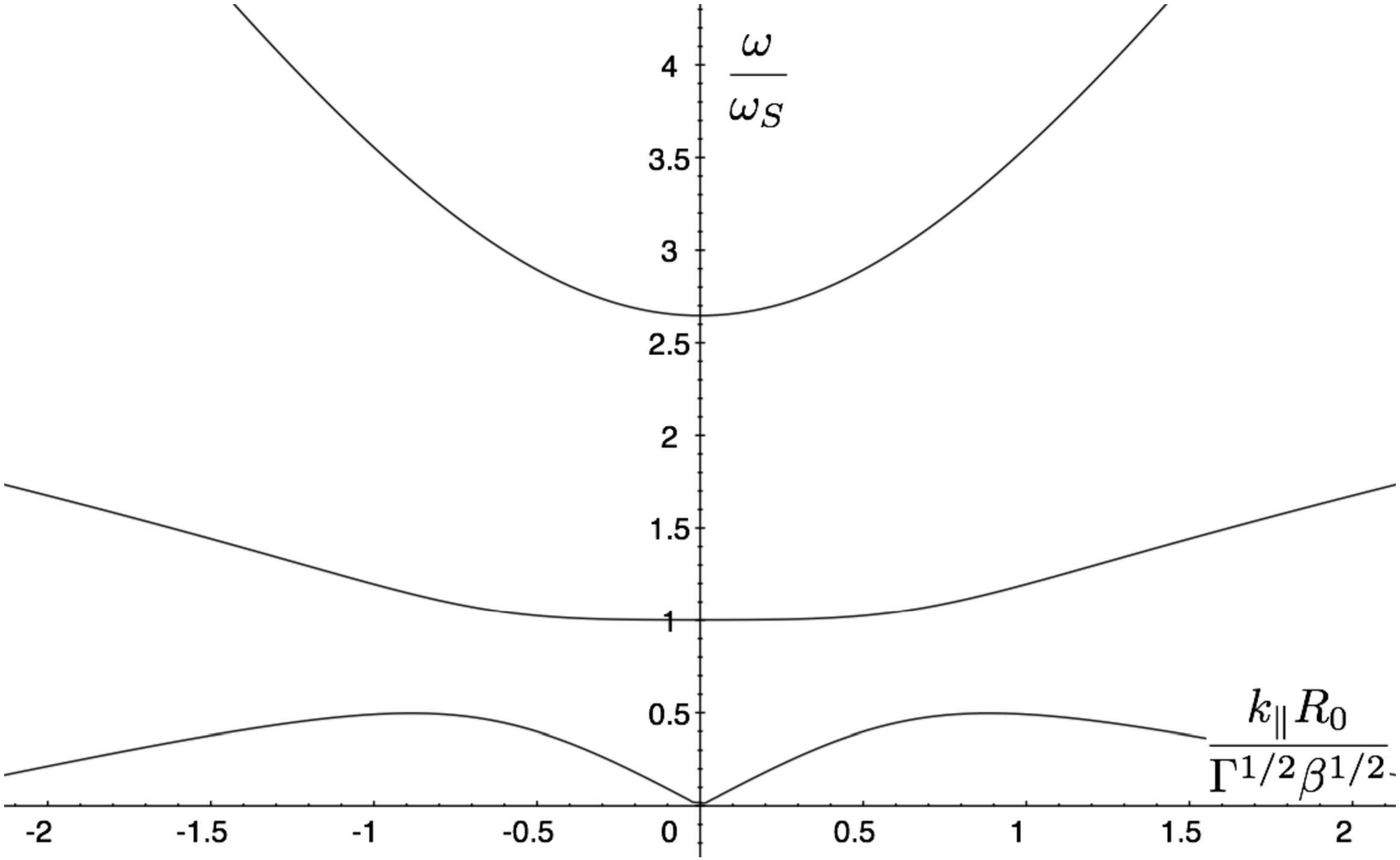
- **Low frequency MHD continuum**: Finally, let us consider the low frequency MHD limit, $|\omega^2| \ll \omega_S^2$. Thus, $\epsilon_S \simeq -\omega_S^2/\omega^2$ and

$$\Lambda^2 \simeq \frac{\omega^2}{\omega_A^2} - \frac{\Gamma\beta q^2}{\epsilon_S} \simeq \frac{\omega^2}{\omega_A^2}(1+2q^2) \ .$$

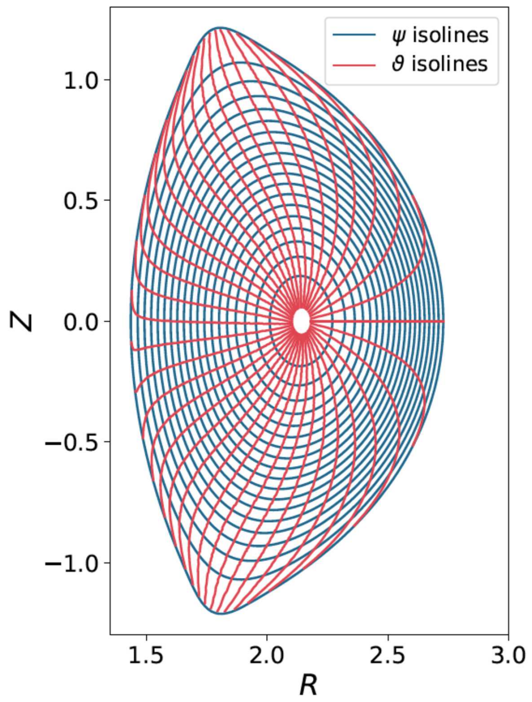
- Well-known inertia enhancement as originally derived by Glasser, Greene and Johnson.
- It is also connected to the ion flow within the considered magnetic flux surface under the incompressibility condition, as pointed out in the classic work by Pfirsch and Schlüter.
- The low-frequency MHD continuum is therefore given by

$$\omega^2 = \omega_A^2 \frac{k_\parallel^2 q^2 R_0^2}{1+2q^2} \ .$$

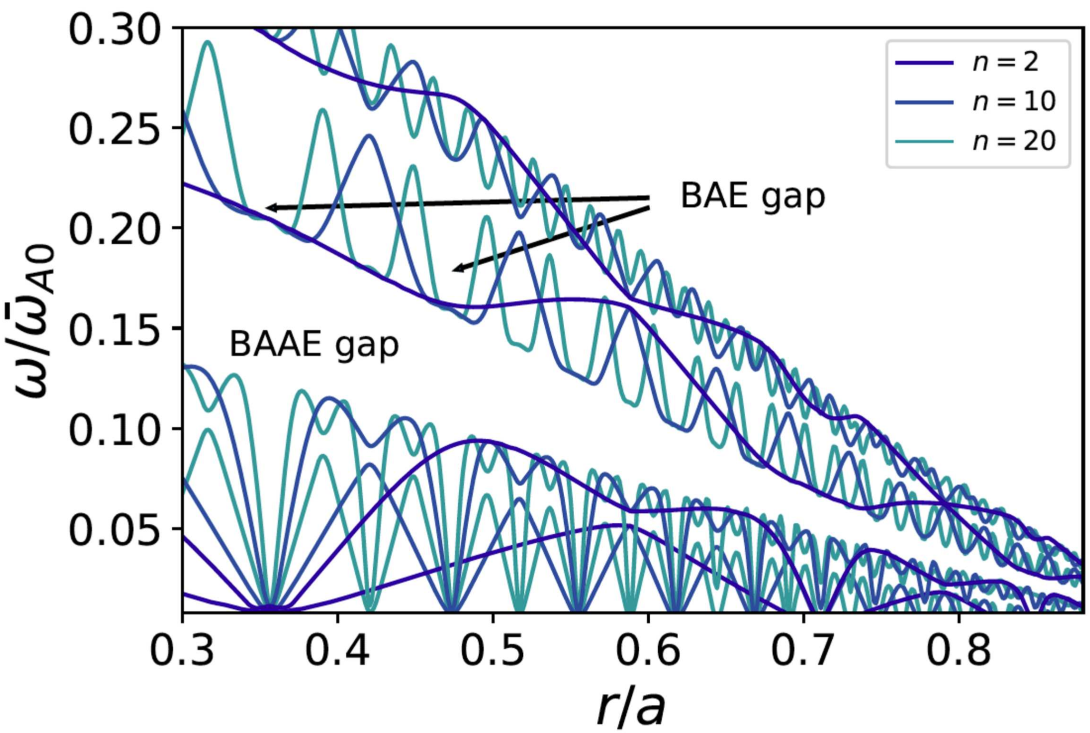
- Numerical solution of the low-frequency continuum: $\Lambda^2 = k_\parallel^2 q^2 R_0^2$



- Low-frequency continuum in DTT: $\Lambda^2 = k_\parallel^2 q^2 R_0^2$



DTT Boozer coordinates



Low frequency continuum in DTT

BAE/BAAE: kinetic effects in the layer

- Use the equations form pp. 20-21. For simplicity, write layer equations ($\hat{\kappa}_\perp \sim |s\vartheta| \to \infty$) neglecting FLR (FLR are treated by [Wang 2010].)
- QN condition for $\hat{\kappa}_\perp \to \infty$ (p. 20)

$$\left(1+\frac{1}{\tau}\right)\left(\delta\hat{\Phi}_n - \delta\hat{\Psi}_n\right) = \frac{T_i}{n_0 e}\hat{\kappa}_\perp \left\langle \delta\hat{K}_{in}\right\rangle_v \quad .$$

- GK vorticity equation for $\hat{\kappa}_\perp \to \infty$ (p. 21) with $g(\vartheta,\theta_k) \simeq s\vartheta \sin\vartheta$

$$\frac{\partial^2}{\partial\vartheta^2}\delta\hat{\Psi}_n + \frac{\omega\left(\omega-\omega_{*pi}\right)}{\omega_A^2}\delta\hat{\Phi}_n + \frac{4\pi R_0 q^2}{B_0}\frac{g(\vartheta,\theta_k)\omega}{c k_\vartheta \hat{\kappa}_\perp}\left\langle m_i\left(\mu B_0 + v_\parallel^2\right)\delta\hat{K}_{in}\right\rangle_v = 0 \quad .$$

- BAE are important since they can be excited by both fast ions (long wavelengths) as well by thermal ions (short wavelengths; AITG) [POP 1999; Chen RMP 2016].

- The particle distribution function $\delta\hat{K}_i$ is derived from the drift-kinetic equation (no FLR; $\omega_{tr} = v_\parallel/qR$)

$$[\omega_{tr}\partial_\vartheta - i(\omega - \omega_d)]_i \delta\hat{K}_i = i\left(\frac{e}{m}\right)_i QF_{0i}\left[\left(\delta\hat{\phi} - \delta\hat{\psi}\right) + \left(\frac{\omega_d}{\omega}\right)_i \delta\hat{\psi}\right]$$

- For BAE one typically has $\omega \approx \omega_{ti} \approx \omega_{*pi} \approx k_\parallel v_A$. Important effects are expected from kinetic interaction in the radial local region (kinetic layer) $k_\parallel qR_0 \approx \beta^{1/2}$.
- Typical paradigmatic case for application of MSD to drift Alfvén turbulence as well as MHD (no specification of mode number so far; just $k_\parallel qR_0 \approx \beta^{1/2}$).
- Derivation of layer equation via asymptotic expansion in $\beta^{1/2}$. Lowest order solution yields $\delta\hat{K}_i^{(0)} = -(e/m)_i(QF_{0i}/\omega)(\delta\hat{\phi}^{(0)} - \delta\hat{\psi}^{(0)})$, which yields $\delta\hat{\psi}^{(0)} = \delta\hat{\phi}^{(0)}$ when substituted back into the quasi-neutrality condition.

- At the next order, $\delta\hat{\phi}^{(1)} = e^{i\vartheta}\delta\hat{\phi}^{(1+)} + e^{-i\vartheta}\delta\hat{\phi}^{(1-)}$ with a corresponding $\delta\hat{K}_i^{(1)} = e^{i\vartheta}\delta\hat{K}_i^{(1+)} + e^{-i\vartheta}\delta\hat{K}_i^{(1-)}$. Let $\omega_{tr}^{(\pm)} = [1 \pm (nq - m)]\,(v_\parallel/qR_0)$. Then

$$i\left[\pm\omega_{tr}^{(\pm)} - \omega\right]\delta\hat{K}_i^{(1\pm)} = i\left(\frac{e}{m}\right)_i QF_{0i}^{(\mp)}\delta\hat{\phi}^{(1\pm)} \mp i\left(\frac{e}{m}\right)_i QF_{0i}\frac{v_\perp^2/2 + v_\parallel^2}{R_0\omega_{ci}\omega}\frac{k_r}{2i}\delta\hat{\phi}^{(0)}$$

- Note that the dominant effect comes from the geodesic curvature (large $k_r \propto \vartheta$), causing radial magnetic drifts. Sideband generation is evident in the wave-particle interaction as well as in the ω_* effect in $QF_{0i}^{(\mp)}$, which is computed at $m \mp 1$.

- Substituting back into the quasi-neutrality and letting $\omega_{ti}^{(\pm)} = (2T_i/m_i)^{1/2}\left[1 \pm (nq - m)\right]/qR_0$

$$\delta\hat{\phi}^{(1\pm)} = \mp i\frac{cT_i}{eB_0}\frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m\mp1}(\omega/\omega_{ti}^{(\pm)})}\frac{k_r}{\omega R_0}\delta\hat{\phi}^{(0)}$$

- Definitions: $Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-y^2}/(y-x)dy$ and subscripts m and $m \mp 1$ indicate poloidal mode numbers to compute

$$N(x) = \left(1 - \frac{\omega_{*ni}}{\omega}\right) \left[x + \left(1/2 + x^2\right) Z(x)\right] - \frac{\omega_{*Ti}}{\omega} \left[x \left(1/2 + x^2\right) + \left(1/4 + x^4\right) Z(x)\right] \; ,$$

$$D(x) = \left(\frac{1}{x}\right) \left(1 + \frac{1}{\tau}\right) + \left(1 - \frac{\omega_{*ni}}{\omega}\right) Z(x) - \frac{\omega_{*Ti}}{\omega} \left[x + \left(x^2 - 1/2\right) Z(x)\right] \; ,$$

- Corresponding solutions of the drift-kinetic equation are

$$\delta \hat{K}_i^{(1\pm)} = \mp i \frac{cT_i}{eB_0} \frac{e/m_i}{\omega \mp \omega_{tr}^{(\pm)}} \left[\frac{m_i}{2T_i} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) QF_{0i} - \frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m\mp1}(\omega/\omega_{ti}^{(\pm)})} QF_{0i}^{(\mp)} \right] \frac{k_r}{\omega R_0} \delta \hat{\phi}^{(0)}$$

- The large $|\vartheta|$ vorticity equation becomes finally (canonical form yielding the general fishbone like dispersion relation)

$$\left(\frac{\partial^2}{\partial\vartheta^2} + \Lambda^2\right)\delta\hat{\Psi}^{(0)} = 0$$

- Definitions: $\delta\Psi^{(0)} = \hat{\kappa}_\perp \delta\hat{\psi}^{(0)}$

$$\begin{aligned}
\Lambda^2 \;=\; & \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) + q^2\frac{\omega_{ti}^2}{2\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)\left((\omega/\omega_{ti}^{(+)})F(\omega/\omega_{ti}^{(+)}) + (\omega/\omega_{ti}^{(-)})F(\omega/\omega_{ti}^{(-)}\right)\right. \\
& -\frac{\omega_{*Ti}}{\omega}\left((\omega/\omega_{ti}^{(+)})G(\omega/\omega_{ti}^{(+)}) + (\omega/\omega_{ti}^{(-)})G(\omega/\omega_{ti}^{(-)})\right) \\
& -\left((\omega/\omega_{ti}^{(+)})N_m(\omega/\omega_{ti}^{(+)})\frac{N_{m-1}(\omega/\omega_{ti}^{(+)})}{D_{m-1}(\omega/\omega_{ti}^{(+)})} + (\omega/\omega_{ti}^{(-)})N_m(\omega/\omega_{ti}^{(-)})\frac{N_{m+1}(\omega/\omega_{ti}^{(-)})}{D_{m+1}(\omega/\omega_{ti}^{(-)})}\right)
\end{aligned}$$

$$\begin{aligned}
F(x) &= x\left(x^2+3/2\right) + \left(x^4+x^2+1/2\right)Z(x)\ , \\
G(x) &= x\left(x^4+x^2+2\right) + \left(x^6+x^4/2+x^2+3/4\right)Z(x)\ ,
\end{aligned}$$

- Remember $\Lambda^2 = k_\parallel^2 q^2 R_0^2$ represents the SAW continuum; i.e., frequency gap is at $\Lambda^2 < 0$
- To demonstrate that Λ^2 expression contains the physics of low-frequency (BAE) gap formation in the SAW continuum, take the fluid limit in which MHD is valid, $|\omega| \gg \omega_{ti}$
- Large argument expansion of the plasma Z function yields:

$$\Lambda^2 = \frac{1}{\omega_A^2}\left[\omega^2 - \left(\frac{7}{4} + \frac{T_e}{T_i}\right) q^2 \omega_{ti}^2\right] + i\sqrt{\pi} q^2 e^{-\omega^2/\omega_{ti}^2} \frac{\omega^2}{\omega_A^2} \left(\frac{\omega_{ti}}{\omega} - \frac{\omega_{*Ti}}{\omega_{ti}}\right) \left(\frac{\omega^2}{\omega_{ti}^2} + \frac{T_e}{T_i}\right)^2 .$$

- From solution of the low frequency MHD equations (p.24) we have that gap should occur at $\omega^2 < \Gamma(T_i + T_e)/(m_i R_0^2)$. Comparing results with kinetic theory:

$$\Gamma_e = 2 \qquad \text{electrons behave as adiabatic massless fluid in 2D}$$

$$\Gamma_i = 7/2???$$

Micro- and meso- scale excitation of low-frequency AE/EPM

- With the general expression of Λ ($k_\parallel = 0$). Reduces to fluid limit and incorporates Landau damping (esp. of Alfvén-acoustic branch)

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) + q^2\frac{\omega\omega_{ti}}{\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)F(\omega/\omega_{ti}) - \frac{\omega_{*Ti}}{\omega}G(\omega/\omega_{ti}) - \frac{N_m(\omega/\omega_{ti})}{2}\left(\frac{N_{m+1}(\omega/\omega_{ti})}{D_{m+1}(\omega/\omega_{ti})} + \frac{N_{m-1}(\omega/\omega_{ti})}{D_{m-1}(\omega/\omega_{ti})}\right)\right]$$

low frequency AE/EPM can be excited by both thermal ions (micro-scale) and energetic ions (meso-scales)

$$i\Lambda = \delta\bar{W}_f + \delta\bar{W}_k \qquad \text{(fast ions included)}$$

- Excitation of low-frequency AE by thermal ions is most easily seen using the simplified expression of Λ derived earlier: AITG excitation mechanism

$$\Lambda^2 = \frac{1}{\omega_A^2}\left[\omega^2 - \left(\frac{7}{4} + \frac{T_e}{T_i}\right) q^2\omega_{ti}^2\right] + i\sqrt{\pi} q^2 e^{-\omega^2/\omega_{ti}^2}\frac{\omega^2}{\omega_A^2}\left(\frac{\omega_{ti}}{\omega} - \frac{\omega_{*Ti}}{\omega_{ti}}\right)\left(\frac{\omega^2}{\omega_{ti}^2} + \frac{T_e}{T_i}\right)^2 .$$

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ accumulation point becomes unstable! The unstable continuum is not a concern
(E: compute how much the mode can grow before mode converting to KAW. Hint: compute how long takes for a wave-packet born at small ϑ-ballooning to reach large ϑ where KAW physics is important).

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ and equilibrium effects localize the AE, the Alfvénic ITG mode is excited (AITG) $\Rightarrow$ More details in Lecture 6

Low frequency Alfvén and acoustic fluctuations

- Near continuum accumulation points, equilibrium non-uniformity create the local potential well for bound states to exist [C&Z RMP 2016].
- Mode structure is obtained solving the governing equations in the ballooning space; that is for all values of ϑ and applying proper boundary conditions.
- Recall that SAW and SSW continuous spectra can be readily computed in the limit $\hat{\kappa}_\perp \to \infty$ ($|\vartheta| \to \infty$).
- Overall equation for $\hat{\phi}_s$, with b.c. $\hat{\phi}_s \sim e^{i\Lambda|\vartheta|}$ as $|\vartheta| \to \infty$ (here, $\theta_k = 0$)

$$\left[\frac{\partial^2}{\partial \vartheta^2} + \Lambda^2 + \frac{\alpha \cos\vartheta}{\hat{\kappa}_\perp^2} - \frac{(s - \alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^4} \right] \hat{\phi}_s = 0 \quad .$$

- Derive dispersion relation from asymptotic matching and/or variational approach.

- Asymptotic matching between external region solution ($\hat{\kappa}_\perp \gg 1$; $|\Lambda\vartheta| \sim 1$)

$$\left[\frac{\partial^2}{\partial \vartheta^2} + \Lambda^2 \right] \hat{\phi}_{sE} = 0 \; ;$$

and internal region solution ($\hat{\kappa}_\perp \sim 1$; $|\Lambda\vartheta| \ll 1$)

$$\left[\frac{\partial^2}{\partial \vartheta^2} + \frac{\alpha \cos \vartheta}{\hat{\kappa}_\perp^2} - \frac{(s - \alpha \cos \vartheta)^2}{\hat{\kappa}_\perp^4} \right] \hat{\phi}_{sI} = 0 \; .$$

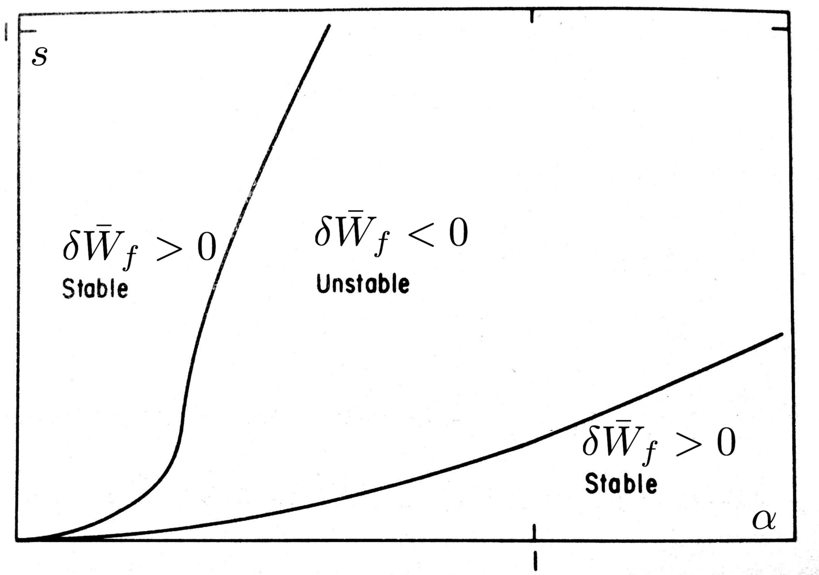
- External region solution, for $|\Lambda\vartheta| \ll 1$

$$\hat{\phi}_{sE} = e^{i\Lambda|\vartheta|} \sim 1 + i\Lambda|\vartheta| + \ldots \; .$$

- Internal region solution, for $\hat{\kappa}_\perp \gg 1$

$$\hat{\phi}_{sI} = \hat{\phi}_{sI}^{(1)} + \delta \bar{W}_f \hat{\phi}_{sI}^{(2)} \sim 1 + \delta \bar{W}_f |\vartheta| + \ldots \; .$$

- Here, $\hat{\phi}_{sI}^{(1)}$ and $\hat{\phi}_{sI}^{(2)}$ are the internal region solutions that satisfy, respectively $\hat{\phi}_{sI}^{(1)} = 1$ and $\hat{\phi}_{sI}^{(2)} = |\vartheta|$ boundary condition as $\hat{\kappa}_\perp \to \infty$.
- Meanwhile, $\delta\bar{W}_f(s,\alpha)$ is a function of $(s\alpha)$, which can be obtained from internal region solutions
  - odd modes: $\delta\bar{W}_f(s,\alpha) = -\hat{\phi}_{sI}^{(1)}(0)/\hat{\phi}_{sI}^{(2)}(0)$
  - even modes: $\delta\bar{W}_f(s,\alpha) = -\partial_\vartheta\hat{\phi}_{sI}^{(1)}(0)/\partial_\vartheta\hat{\phi}_{sI}^{(2)}(0)$



- $\delta\bar{W}_f(s,\alpha)$ is related to the MHD stability of high-n ballooning modes
- The dispersion relation by asymptotic matching is:

$$i\Lambda(\omega) = \delta\bar{W}_f(s,\alpha)$$

- Causality constraint requires $\mathbb{R}\mathrm{e}(i\Lambda) = \mathbb{R}\mathrm{e}\delta\bar{W}_f < 0$

- Causality: No low frequency gap modes (BAE/BAAE/MHD) exist in ideal MHD stable plasma.

RP: Write your own numerical code for solving the SAW vorticity equation in the internal region. Use the solutions $\hat{\phi}_{sI}^{(1)}$ and $\hat{\phi}_{sI}^{(2)}$ to calculate $\delta\bar{W}_f$ numerically. Draw your own ballooning mode marginal stability diagram.

- Given the mode dispersion relation, $i\Lambda(\omega) = \delta\bar{W}_f(s,\alpha)$, once the causality constraint is verified, the dispersion relation for Alfvén and Alfvén-acoustic modes can be obtained from the expression of the corresponding continuous spectra near the accumulation points by simple substitution

$$k_\parallel^2 q^2 R_0^2 \to -\delta\bar{W}_f^2 \; .$$

E: Use the expressions of continuous spectra near the accumulation point of BAE, BAAE and MHD fluctuations, respectively at p.24, p.25 and p. 26, to write the corresponding gap mode dispersion relation by the formal substitution $k_\parallel^2 q^2 R_0^2 \to -\delta\bar{W}_f^2$.

- Variational derivation: define Θ_1 in the overlapping region of internal and external solutions

$$\frac{1}{2}\int_{-\Theta_1}^{\Theta_1} \partial_\vartheta \left[\hat{\phi}_{sI}^* \partial_\vartheta \hat{\phi}_{sI} \right] d\vartheta = i\Lambda = \frac{1}{2}\int_{-\Theta_1}^{\Theta_1} \left[\left| \partial_\vartheta \hat{\phi}_{sI} \right|^2 + \left(-\frac{\alpha \cos\vartheta}{\hat{\kappa}_\perp^2} + \frac{(s - \alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^4} \right) \left| \hat{\phi}_{sI} \right|^2 \right] d\vartheta = \delta \bar{W}_f(s,\alpha) \; .$$

- This form illuminates the stabilizing/destabilizing role of various terms/contributions in the SAW vorticity equation and can be use for a simple estimate of $\delta\bar{W}_f$ by means of a trial function method.

E: Use the variational form of $\delta\bar{W}_f$ to give a simple analytic expression for small (s,α). Compare your analytic results with numerical simulation expressions obtained previously.

References and reading material

L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

Z. X. Lu, F. Zonca and A. Cardinali, Phys. Plasmas **19**, 042104 (2012).

F. Zonca, L. Chen, S. Briguglio, G. Fogaccia, G. Vlad and X. Wang, New J. Phys. **17**, 013052 (2015).

T. M. Antonsen and B. Lane, Phys. Fluids **23**, 1205 (1980).

P. J. Catto and W. M. Tang and D. E. Baldwin, Plasma Phys. **23**, 639 (1981).

E. A. Frieman and L. Chen, Phys. Fluids **25**, 502 (1982).

A. J. Brizard and T. S. Hahm, Rev. Mod. Phys. **79**, 421 (2007).

L. Chen and A. Hasegawa, J. Geophys. Res. **96**, 1503 (1991).

H. Qin, W. M. Tang and G. Rewoldt, Phys. Plasmas **5**, 1035 (1998).

G. Chew, F. Goldberger and F. Low, Proc. Roy. Soc. Ser. A **236**, 112 (1956).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072120 (2014).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072121 (2014).

L. Chen, R. B. White, and M. N. Rosenbluth, Phys. Rev. Lett. **52**, 1122 (1984).

A. Hasegawa and L. Chen, Phys. Rev. Lett. **35**, 370 (1975).

A. Hasegawa and L. Chen, Phys. Fluids **19**, 1924 (1976).

R. R. Mett and S. M. Mahajan, Phys. Fluids B **4**, 2885 (1992).

G. Vlad, F. Zonca and S. Briguglio, Riv. Nuovo Cimento **22**, 1 (1999).

X. Wang, F. Zonca and L. Chen, Plasma Phys. Control. Fusion **52**, 115005 (2010).

L. Chen and A. Hasegawa, J. Geophys. Res. **96**, 1503 (1991).

F. Zonca and L. Chen, Plasma Phys. Control. Fusion **48**, 537 (2006).

G. Y. Fu and C. Z. Cheng, Phys. Fluids B **4**, 3722 (1992).

H. L. Berk, B .N. Breizman and H. Ye, Phys. Lett. A **162**, 475 (1992).

S.-T. Tsai and L. Chen, Phys. Fluids B **5**, 3284 (1993).

L. Chen, Phys. Plasmas **1**, 1519 (1994).

A. Brizard, Phys. Plasmas **1**, 2460 (1994).