Lecture 1

Shear Alfvén waves in tokamaks. Mode structure decomposition

Shear Alfvén continuous spectrum and frequency gaps

Fulvio Zonca

http://www.afs.enea.it/zonca

ENEA C.R. Frascati, C.P. 65 - 00044 - Frascati, Italy.

Institute for Fusion Theory and Simulation, Zhejiang University, Hangzhou 310027, P.R.C.

June 4.th, 2019

Kinetic theory and global dispersion relation
of Alfvén waves in tokamaks
June 4 – 20, 2019, IFTS – ZJU, Hangzhou

ITALIAN NATIONAL AGENCY
FOR NEW TECHNOLOGIES, ENERGY AND
SUSTAINABLE ECONOMIC DEVELOPMENT

浙江大學聚變理論與模擬中心

Institute for Fusion Theory and Simulation, Zhejiang University

Lecture Plan and Teaching Material

Abstract: These Lectures on the *Kinetic theory and global dispersion relation of Alfvén waves in tokamaks* are the continuation of the Spring 2018 Course on *Physics of Alfvén waves and energetic particles in fusion plasmas. Mode structures and global dispersion of Alfvén waves in tokamaks.* These Lectures will cover with pedagogical style the first part of *Reviews of Modern Physics* **88**, *015008 (2016).* Prerequisite of the course is a preliminary knowledge of general plasma physics and classical mechanics, and the Lecture Notes of the 2018 Intensive Course. General theoretical aspects are treated in Lectures $(1 \div 6)$, which are complemented by corresponding Q&A Sessions.

Main areas that will be explored are:

(I) Shear Alfvén waves in tokamaks. Mode structure decomposition

i) Shear Alfvén continuous spectrum and frequency gaps. (Lecture 1)

(II) Variational methods for Alfvén waves in tokamaks

ii) Variational formulation and variational principle.
Integral form of the dispersion relation: Ideal MHD. (Lecture 2)

iii) Variational formulation and variational principle.
Integral form of the dispersion relation: Kinetic theory. (Lecture 3)

(III) Kinetic description of Alfvén waves in tokamaks

iv) The linear gyrokinetic equation and the global dispersion relation of Alfvén waves in toroidal geometry. (Lecture 4)

v) Kinetic Alfvén waves and low-frequency Alfvén Eigenmodes in tokamaks. (Lecture 5)

vi) Toroidal Alfvén Eigenmodes and Energetic Particle Modes. (Lecture 6)

Lecture Notes: Available in electronic form. At the end of each lecture and during the Q&A time, a list of specific reading material is given explicitly. Should you have difficulty in finding literatures and papers, please contact me at fulvio.zonca@enea.it.

Exercises and Research Projects: In the lecture notes, Exercises (E) of various difficulty levels are suggested. These are meant to be important part of the lectures themselves. Possible topics for Research Projects (RP) are also indicated, which require significant more in depth analysis and work.

Q&A Sessions: Devoted to detailed discussions and analysis of technical aspects. They are meant to be complementary to general lectures and are a necessary addendum for full appreciation of the material presented in the course.

Time and Location: Lectures (9am-11am) will be held at IFTS on June 4-6-10-12-18-20, with corresponding Q&A Sessions (3pm-5pm) of the same days.

Fluid description of Alfvén waves: one dimensional plasmas

- To illustrate how plasma nonuniformities modify the spectral properties of SAW, let us consider the simple configuration of a 1D plasma slab confined in straight magnetic field [Chen 74, Goedbloed 84].

- The equilibrium quantities (plasma density, pressure and magnetic field) are assumed to vary only along the x direction ($\varrho_{m0} = \varrho_{m0}(x)$, $P_0 = P_0(x)$, $\mathbf{B}_0 = \mathbf{B}_0(x)$) and the equilibrium magnetic field is taken to be,

$$\mathbf{B}_0(x) = B_{0z}(x)\mathbf{e}_z \, .$$

- Here, we will adopt the well-known one-fluid ideal MHD description; which, we note, is valid for $|\omega/\omega_{ci}| \ll 1$, $|k_\perp \rho_i|^2 \ll 1$ and $m_e \ll m_i$.

E: demonstrate that one-fluid ideal MHD description is valid for $|\omega/\omega_{ci}| \ll 1$, $|k_\perp \rho_i|^2 \ll 1$ and $m_e \ll m_i$.

- Introducing the standard notation for the displacement vector $\delta \boldsymbol{u} \equiv \partial_t \delta \boldsymbol{\xi}$ and applying Faraday's and ideal Ohm's laws, such that

$$\delta \boldsymbol{B} = \boldsymbol{\nabla} \times (\delta \boldsymbol{\xi} \times \boldsymbol{B}_0) \ ,$$

it is straightforward to derive the following linearized force balance equation

$$\varrho_{m0} \frac{\partial^2}{\partial t^2} \delta \boldsymbol{\xi} = -\nabla \delta \tilde{p} + \frac{1}{4\pi} (\mathbf{B}_0 \cdot \nabla)^2 \delta \boldsymbol{\xi} - \frac{1}{4\pi} \mathbf{B}_0 (\mathbf{B}_0 \cdot \nabla)(\nabla \cdot \delta \boldsymbol{\xi}) \, .$$

- The total (kinetic plus magnetic) perturbed pressure

$$\delta \tilde{p} = \delta P + \frac{B_0 \delta B_\parallel}{4\pi}$$

corresponds to the eigenvector for the compressional Alfvén wave.

- Denoting by Γ the ratio of specific heats, from the equation of state, the perturbed plasma pressure can be obtained as

$$\delta P = -\delta\xi_x \frac{dP_0}{dx} - \Gamma P_0 (\nabla \cdot \delta\boldsymbol{\xi}) = \delta P_{\rm conv} + \delta P_{\rm comp} \, .$$

- Note that, by adopting a fluid description, we have suppressed important kinetic effects such as wave-particle interaction in the compressional component of δP, $\delta P_{\rm comp}$ (Lecture 4; cf. also discussions in Lecture 2 and Lecture 3).

- Use the following Fourier representation for the generic perturbed quantity f:

$$f(\mathbf{r},t) = \hat{f}(x) e^{-i(\omega t - k_y y - k_\parallel z)} \, .$$

- The force balance equation yields [Chen 74]

$$
\begin{aligned}
D_A \delta\xi_\parallel &= \frac{4\pi}{B_0^2} i k_\parallel \delta\tilde{p} + i k_\parallel \left(i k_\parallel \delta\xi_\parallel + i k_y \delta\xi_y + \frac{d}{dx}\delta\xi_x \right) , \\
D_A \delta\xi_y &= \frac{4\pi}{B_0^2} i k_y \delta\tilde{p} , \\
D_A \delta\xi_x &= \frac{4\pi}{B_0^2} \frac{d}{dx} \delta\tilde{p} ,
\end{aligned}
$$

- Here, we have denoted $v_A = (4\pi\varrho_{m0})^{-1/2} B_0$ the Alfvén speed, $k_\parallel = k_z$ and $\delta\xi_\parallel = \delta\xi_z$; while the local SAW dispersion function is

$$D_A \equiv (\omega^2/v_A^2) - k_\parallel^2 \, .$$

- Applying Faraday's and Ohm's laws to $\delta B_\parallel$ in the expression of $\delta\tilde{p}$, one can readily write $\delta\tilde{p}$ in terms of the three components of $\delta\boldsymbol{\xi}$,

$$\delta\tilde{p} = -i k_\parallel \Gamma P_0 \delta\xi_\parallel - \left(\Gamma P_0 + \frac{B_0^2}{4\pi} \right) \left(i k_y \delta\xi_y + \frac{d}{dx}\delta\xi_x \right) .$$

- Substituting $\delta\xi_\parallel$ and noting the equation for $\delta\xi_y$, one obtains

$$\delta\xi_y(x) = \frac{i\overline{\alpha}k_y}{\overline{\alpha}k_y^2 - D_A}\frac{d\delta\xi_x}{dx}\,,$$

where $\beta(x) \equiv 8\pi P_0(x)/B_0^2(x)$ and

$$\overline{\alpha}(x) \equiv 1 + \frac{\Gamma\beta\omega^2}{2\omega^2 - \Gamma\beta k_\parallel^2 v_A^2}\,.$$

E: Derive the above results step by step. What can you do next? What is the physical meaning of the equation that you are about to derive?

- Combining results, one finally arrives at the following wave equation in terms of $\delta\xi_x$ [Chen 74]:

$$\frac{d}{dx}\left(\frac{B_0^2 D_A\overline{\alpha}}{\overline{\alpha}k_y^2 - D_A}\frac{d\delta\xi_x}{dx}\right) - B_0^2 D_A\delta\xi_x = 0\,.$$

- This equation describes SAWs coupled with compressional Alfvén and ion-sound waves by equilibrium nonuniformities.

E: Show that this is indeed the case. Why can you say that the compressional Alfvén wave is described by this equation?

- We note that the obtained differential equation – and hence its solution ξ_x – is singular at the points where $B_0^2 D_A \overline{\alpha} = 0$.

E: Discuss what is the condition for a singular layer to appear in a differential equation. Apply this argument to the obtained differential equation.

The shear Alfvén continuous spectrum

- Singularity defines the following two continuous spectra. One is the SAW continuum given by $D_A = 0$; *i.e.*, by

$$\omega^2 = \omega_A^2(x) \equiv k_\parallel^2 v_A^2(x) \,.$$

The other is the ion-sound continuum given by $\overline{\alpha} = 0$; *i.e.*, with $v_S^2(x) = \Gamma P_0(x)/\varrho_{m0}(x)$ being the ion sound speed,

$$\omega^2 = \omega_S^2(x) \equiv \frac{v_S^2(x) k_\parallel^2}{1 + v_S^2(x)/v_A^2(x)} \,.$$

- Note that the spectra are continuous, since the wave frequency varies continuously in the non-uniformity direction x.

- To gain further insights into properties of SAW continuum, let us suppress the ion sound wave by assuming a cold plasma; that is $\Gamma\beta \to 0$ or $\overline{\alpha} \to 1$.

- Assume the plasma is confined by two perfectly conducting plates at $z = 0$ and $z = L$ [Chen 95].

- Our governing equation then describes shear and compressional Alfvén waves coupled by plasma non-uniformities; $\varrho_{m0}(x)$ in this case.

- In this limit, $\delta\xi_{\parallel} = 0$, and it is sufficient to consider the perpendicular components of the force balance equation

$$D_A \delta\boldsymbol{\xi}_{\perp} = \boldsymbol{\nabla}_{\perp}\left(\frac{\delta B_{\parallel}}{B_0}\right) .$$

- Here, $\delta\boldsymbol{\xi}_{\perp} = \delta\xi_y \boldsymbol{e}_y + \delta\xi_x \boldsymbol{e}_x$, $\boldsymbol{\nabla}_{\perp} = ik_y \boldsymbol{e}_y + \boldsymbol{e}_x \partial_x$ and meanwhile D_A is the local shear Alfvén operator

$$D_A \equiv \frac{\omega^2}{v_A^2} + \frac{\partial^2}{\partial z^2} .$$

- The compressional component of the magnetic field fluctuation is given by

$$\frac{\delta B_\parallel}{B_0} = -\boldsymbol{\nabla} \cdot \delta \boldsymbol{\xi}_\perp \, .$$

E: Justify the above expression using the results obtained so far.

- We now decompose all perturbed fields f in the complete orthonormal set of shear Alfvén eigenfunctions $\psi_{A\ell}(z|x)$

$$f(\mathbf{r}, t) \;\; = \;\; e^{-i(\omega t - k_y y)} \sum_{\ell=1}^{\infty} \hat{f}_\ell(x_0, x_1) \, \psi_{A\ell}(z|x_1) \, ,$$

where $\psi_{A\ell}(z|x)$, with boundary conditions $\psi_{A\ell} = 0$ at $z = 0$ and L, satisfies the differential equation ($\omega^2_{A\ell}(x)$ defines the local SAW frequency)

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2_{A\ell}(x)}{v^2_A(x)} \right] \psi_{A\ell}(z|x) \; = \; 0.$$

- Perturbed field decomposition takes into account the existence of continuous frequency spectra, with a corresponding singular eigenfunction behavior, introducing "fast" x_0 (singular) and "slow" x_1 (equilibrium) radial variables.

- Using the following definition of the internal product

$$\langle \psi_{A\ell} | \psi_{A\ell'} \rangle \equiv \int_0^L v_A^2 \psi_{A\ell} \psi_{A\ell'} dz = \delta_{\ell\ell'} ,$$

with $k_\ell = \ell\pi/L$ and $\omega_{A\ell}^2(x) = k_\ell^2 v_A^2(x)$, one has

$$\psi_{A\ell}(z|x) = (2/L v_A^2)^{1/2} \sin k_\ell z \, .$$

- We now exploit the existence of two spatial scales (x_0 and x_1) and solve the problem via asymptotic expansions of the fluctuating fields

$$\hat{f}_\ell = \hat{f}_\ell^{(0)} + \hat{f}_\ell^{(1)} + \dots \, ,$$

where $|\hat{f}_\ell^{(1)} / \hat{f}_\ell^{(0)}| \approx |\partial_{x_1} / \partial_{x_0}|$ etc.

- Noting that $|D_A| \ll k_y^2$ for a typical shear Alfvén wave (*i.e.*, $k_\parallel^2 \ll k_y^2$), the perpendicular force balance equation with the expression for $\delta B_\parallel$ may be combined and yield, at the lowest order,

$$\left[\frac{\partial}{\partial x_0} \epsilon_{A\ell} \frac{\partial}{\partial x_0} - k_y^2 \epsilon_{A\ell} \right] \delta \hat{\xi}_{x\ell}^{(0)} = 0 \,,$$

where $\epsilon_{A\ell} = \omega^2 - \omega_{A\ell}^2(x)$.

E: Derive the equation above step by step.

- Solutions become singular at the radial positions $x_{R\ell}$ where the "local dispersion relation", $\epsilon_{A\ell} = 0$ ($\omega^2 = \omega_{A\ell}^2(x_{R\ell})$), is satisfied; *i.e.*, at those positions where the shear Alfvén continuous spectrum is resonantly excited.

- A natural definition of the "fast" (singular) variable is $x_0 \equiv x - x_{R\ell}$, since $\epsilon_{A\ell} = \epsilon'_{A\ell}(x_1) x_0$.

- It is readily shown that solutions that may be written in the form

$$\delta\hat{\xi}^{(0)}_{x\ell} = \frac{C_\ell(x_1)}{\epsilon'_{A\ell}(x_1)} \ln(x_0) \; .$$

- Meanwhile, at the lowest order, $\nabla \cdot \delta\boldsymbol{\xi}^{(0)}_{\perp\ell} = 0$, from which we get

$$\delta\hat{\xi}^{(0)}_{y\ell} = \frac{i}{k_y}\partial_{x_0}\delta\hat{\xi}^{(0)}_{x\ell} = \frac{iC_\ell(x_1)}{k_y\epsilon'_{A\ell}(x_1)}\frac{1}{x_0} \; .$$

- Using this expression along with the y component of the perpendicular force balance, one can show that $\delta B_\parallel$ is given as

$$\delta B_\parallel = e^{-i(\omega t - k_y y)} \sum_{\ell=1}^{\infty} \delta\widehat{b_\parallel}_\ell(x_1)\,\psi_{A\ell}(z|x_1) \; .$$

- Here, the $\delta\widehat{b_{\|}}_{\ell}$ functions depend only on the slow variable and are directly related to the functions $C_\ell(x_1)$ introduced above:

$$\frac{\delta\widehat{b_{\|}}_{\ell}(x_1)}{B_0} = \frac{\epsilon'_{A\ell}(x_1)x_0}{\mathrm{i}k_y v_A^2(x_1)}\delta\hat{\xi}^{(0)}_{y\ell} = \frac{C_\ell(x_1)}{k_y^2 v_A^2(x_1)} \; .$$

- This result means that, in a non-uniform plasma, shear and compressional Alfvén waves are coupled together and their coupling is the origin of the singular solutions corresponding to the "local" shear Alfvén oscillations of the continuous spectrum $\omega^2 = \omega^2_{A\ell}(x)$.

- The group velocity of shear Alfvén waves is directed along the magnetic field lines (z-direction in the present case), whereas the compressional wave generally carries energy across the field itself.

- Thus, CAW "piles up" wave energy at the radial location where the shear Alfvén spectrum is resonantly excited, explaining the origin of "local singular oscillations" [Chen 95].

Resonant excitation and resonant absorption

- The concept of resonant absorption of shear Alfvén waves is introduced by [Chen 1974]. In fact, a finite amount of wave energy can be absorbed at the resonant layer, $x_{R\ell}$.

- The time-averaged energy absorption rate, $d\langle W\rangle/dt$, is given by the Poynting energy flux into the infinitely narrow layer at $x_{R\ell}$. Thus [Chen 74], with L_y is the system extension in the y-direction,

$$\frac{d\langle W\rangle}{dt} = -\frac{cL_y}{8\pi}\mathbb{R}\mathrm{e}\left\{\int_0^L dz\int_{-\infty}^{\infty}\nabla\cdot\left[\left(\frac{i\omega}{c}\delta\boldsymbol{\xi}\times\mathbf{B}_0\right)\times\delta\mathbf{B}^*\right]dx\right\}.$$

- Using the fluctuation decomposition and orthogonality relations, along with boundary conditions at $z=0$ and $z=L$,

$$\frac{d\langle W\rangle}{dt} = -\frac{B_0L_y}{8\pi}\mathbb{R}\mathrm{e}\left\{\sum_{\ell=1}^{\infty}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(-\frac{i\omega}{v_A^2}\delta\hat{\xi}_{x\ell}\delta\widehat{b_{\parallel}}_{\ell}^{*}\right)dx\right\}.$$

- The power absorption has contribution only at the resonant surface $x_{R\ell}$.

- Using the causality constraint $\mathbb{I}\mathrm{m}\,\omega \to 0^{(+)}$, it is possible to demonstrate that $\delta\hat{\xi}^{(0)}_{x\ell}$ (cf. p.16) has a jump given by

$$\Delta\delta\hat{\xi}^{(0)}_{x\ell} = -i\pi\,\mathrm{sgn}\left(\frac{\omega}{\epsilon'_{A\ell}(x_{R\ell})}\right)\frac{C_\ell(x_{R\ell})}{\epsilon'_{A\ell}(x_{R\ell})}\,.$$

- Thus, the power absorption becomes [Chen 95]

$$\begin{aligned}\frac{d\langle W\rangle}{dt} &= \frac{L_y}{8}k_y^2\sum_{\ell=1}^{\infty}\frac{|\omega_{A\ell}(x_{R\ell})|}{|\epsilon'_{A\ell}(x_{R\ell})|}\left|\delta\widehat{b_\parallel}_\ell(x_{R\ell})\right|^2 \\ &= \frac{L_yL}{8k_y^2}B_0^2\sum_{\ell=1}^{\infty}|\omega'_{A\ell}(x_{R\ell})|\frac{\ell^2}{L^3v_A^2(x_{R\ell})}\left|\Delta\delta\hat{\xi}^{(0)}_{x\ell}\right|^2\,.\end{aligned}$$

- This demonstrates resonant energy absorption at a rate $\propto \omega'_{A\ell}$. In turn, the resonant energy absorption of a given initial perturbation takes place on time scales $\approx(\omega'_{A\ell}\Delta x)^{-1}$, with Δx the perturbation *radial* extent.

Initial value approach

- The discussions on the singular solutions and resonant absorption are pertinent upon steady state excitations.
- It is also instructive to consider the wave perturbations from the initial-value approach.
- We are interested in the $|\omega_{\mathcal{A}\ell} t| \gg 1$ time asymptotic behaviors. Assuming, to be justified a posteriori, $|\partial_x| \gg |k_y|$, the governing equation (p. 15) with ω replaced by $i\partial_t$ then becomes

$$\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial t^2} + \omega^2_{\mathcal{A}\ell}(x)\right]\frac{\partial}{\partial x}\delta\hat{\xi}_{x\ell}(x,t) \;=\; 0\,.$$

- This equation can be straightforwardly integrated once and it yields

$$\frac{\partial}{\partial x}\delta\hat{\xi}_{x\ell}(x,t) \;=\; \hat{C}_\ell(x)e^{\pm i\omega_{\mathcal{A}\ell}(x)t}\,,$$

□ Here, $\hat{C}_\ell(x)$ is a function depending on equilibrium non-uniformities. Now, note that, as $\omega_{\mathcal{A}\ell}t \to \infty$ (the assumption $|\partial_x| \gg |k_y|$ is justified time asymptotically),

$$\partial_x \cong \pm i\omega'_{\mathcal{A}\ell}(x)t$$

and, thus,

$$\delta\hat{\xi}_{x\ell}(x,t) = \mp i \frac{\hat{C}_\ell(x)}{\omega'_{\mathcal{A}\ell}(x)t} e^{\pm i\omega_{\mathcal{A}\ell}(x)t} .$$

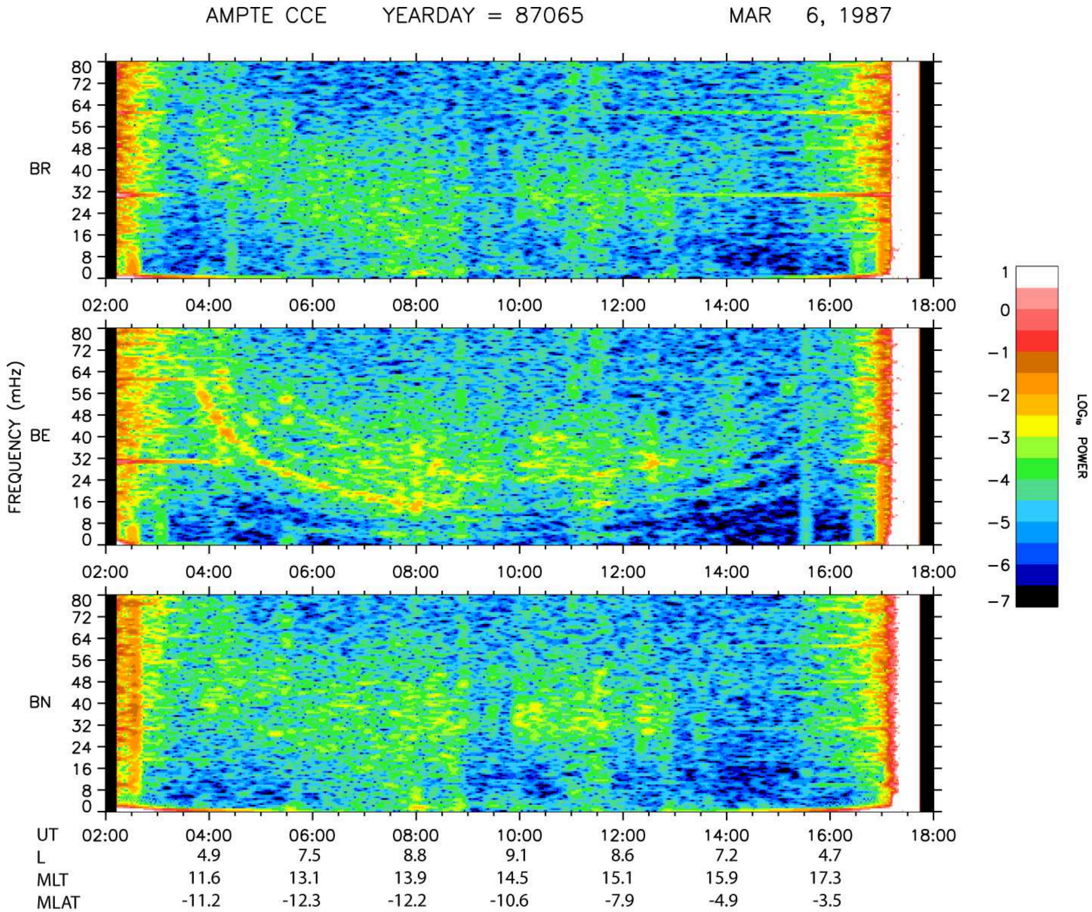
□ From the condition $\boldsymbol{\nabla} \cdot \delta\hat{\boldsymbol{\xi}}_{\perp\ell} = 0$, one derives $\delta\hat{\xi}_{y\ell} = (i/k_y)\partial_x\delta\hat{\xi}_{x\ell}$; i.e.,

$$\delta\hat{\xi}_{y\ell}(x,t) = \frac{i}{k_y}\hat{C}_\ell(x)e^{\pm i\omega_{\mathcal{A}\ell}(x)t} .$$

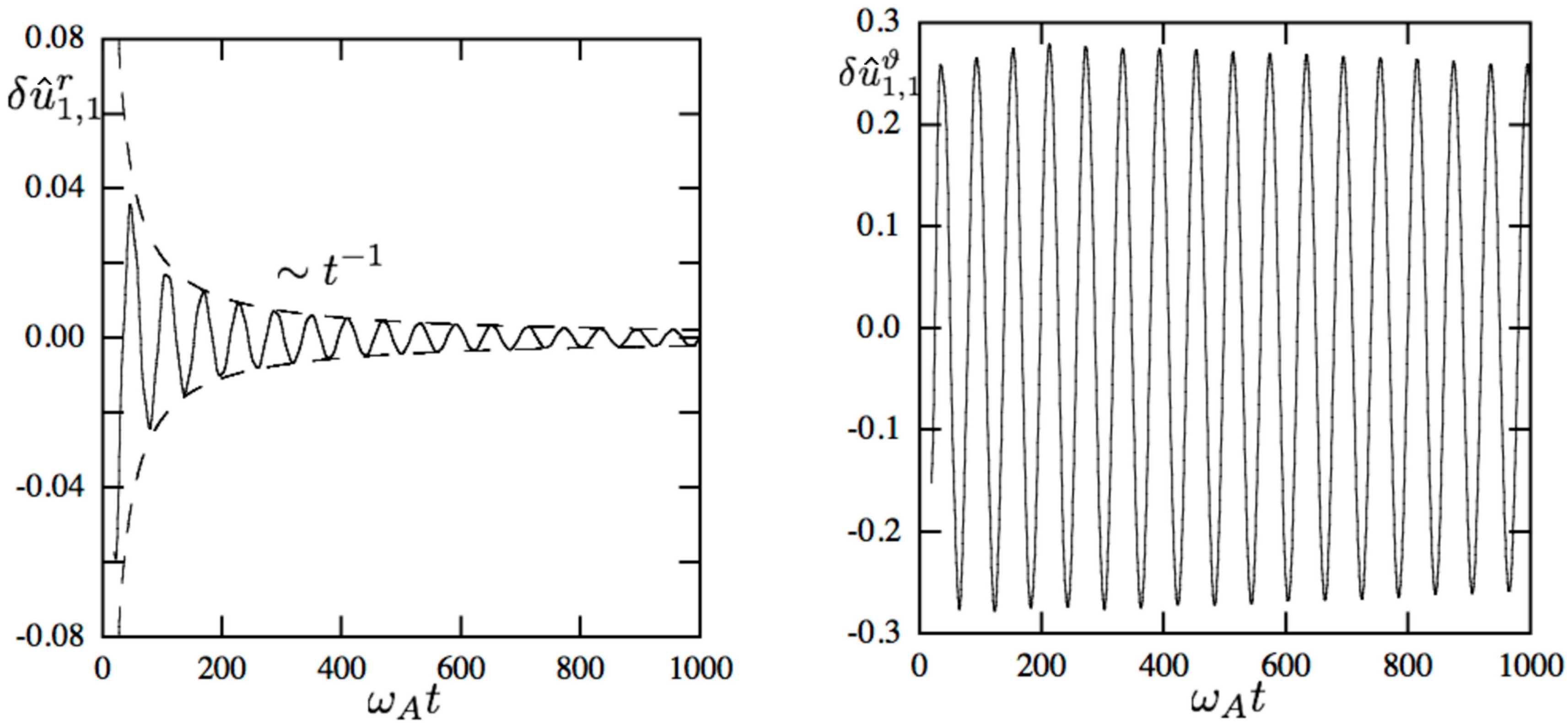
□ Insight in the dynamics associated with the resonant excitation of the shear Alfvén continuum and resonant wave absorption:

- the $\delta\hat{\xi}_{x\ell}$ component exhibits the characteristic $(1/t)$ decay via phase mixing of the continuous spectrum
- $\delta\hat{\xi}_{y\ell}$ shows undamped oscillations at frequencies corresponding to the shear Alfvén continuum

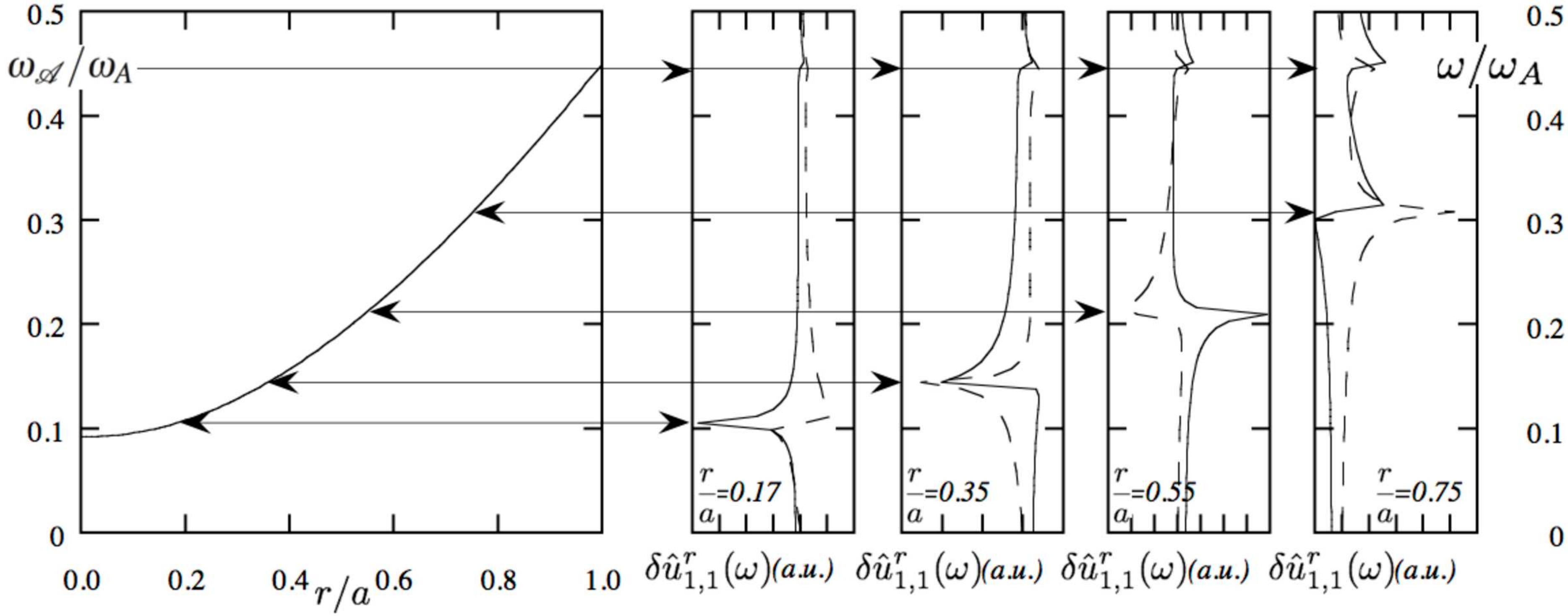
- These properties are nicely demonstrated by the satellite observations of magnetic field fluctuations in the Earth's magnetosphere [Engebretson 87]. B_R, B_E and B_N correspond to, respectively, $\delta B_x(\delta\hat{\xi}_x), \delta B_y(\delta\hat{\xi}_y), \delta B_\parallel$.



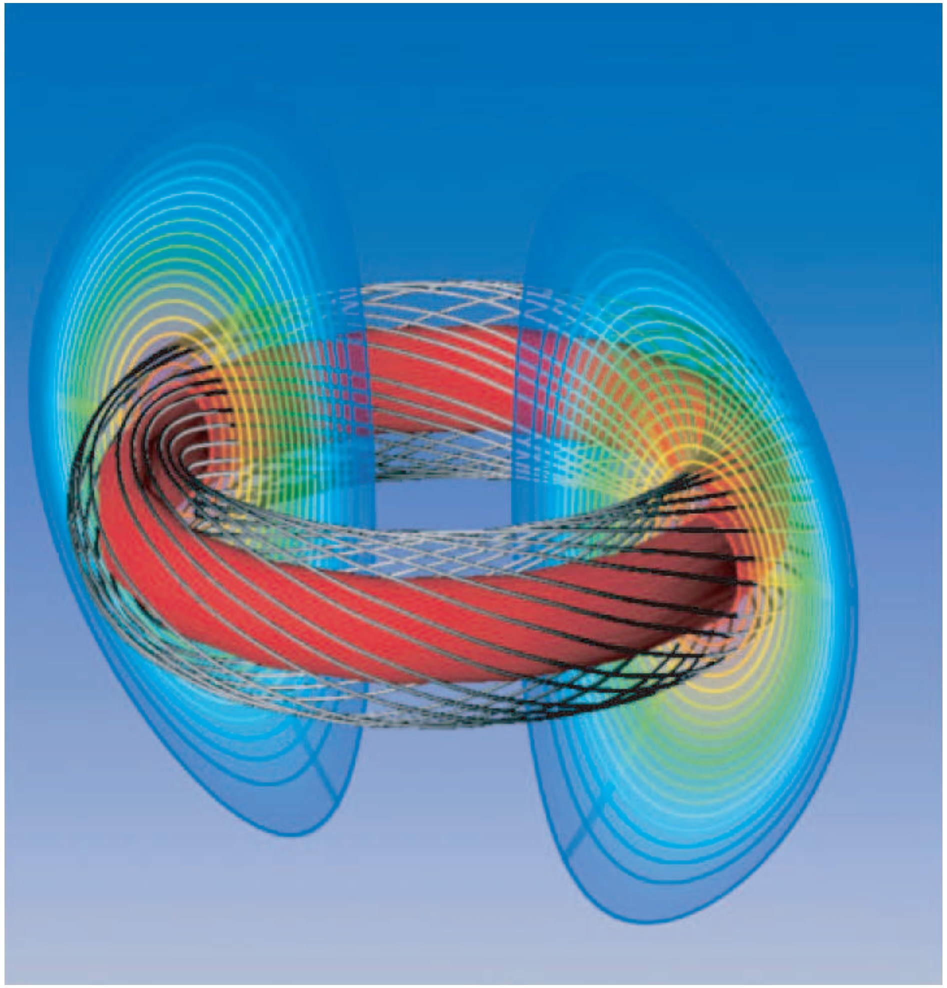
- The same behaviorshave also been demonstrated by ideal MHD initial value numerical simulations of Alfvén wave dynamics in a cylindrical plasma [Vlad 1999].
- Introducing (r, ϑ, z) as coordinate system in a cylindrical plasma of periodic length $2\pi R_0$ and radius a, $\delta \boldsymbol{u}(r, \vartheta, z, t) \equiv \partial_t \delta \boldsymbol{\xi} = e^{i(nz/R_0 - m\vartheta)} \delta \hat{\boldsymbol{u}}_{m,n}(r, t)$.



- The existence of "local singular oscillations" of the SAW continuous spectrum is further demonstrated by the Fourier frequency spectrum of $\delta \hat{u}^r_{1,1}(r,t)$ at several radial locations, reflecting the spatial structure of the continuum frequency $\omega_{\mathcal{A}}(r/a)/\omega_A$ [Vlad 1999].



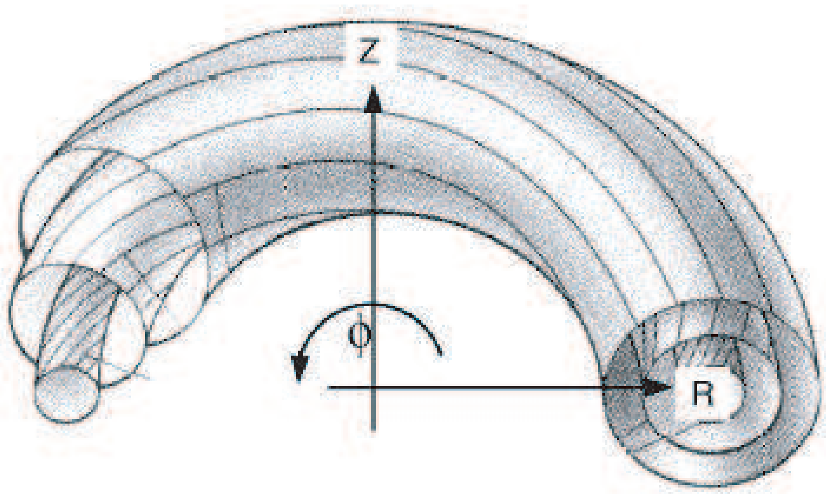
The importance of toroidal geometry

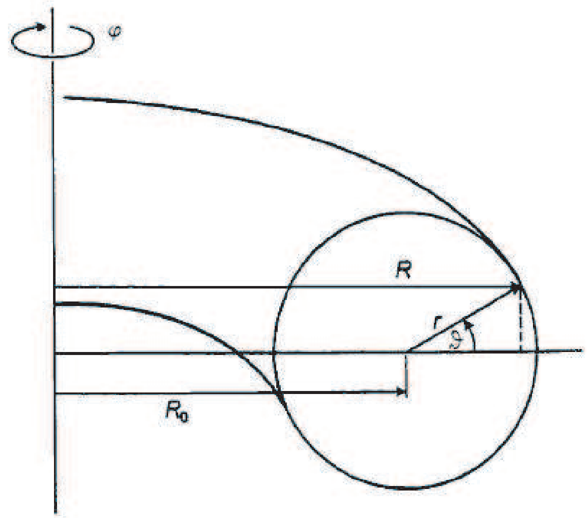


Toroidal geometry must be accounted for, since it crucially modifies

- Particle motions and transport processes
- Linear and nonlinear mode structures
- Plasma response: effective inertia, dispersiveness, etc.

Tokamak: 2D axisymmetric equilibrium





Jacobian $\mathcal{J} = (\nabla\psi \times \nabla\theta \cdot \nabla\phi)^{-1}$

$\mathbf{B} \cdot \nabla\psi = 0 \Rightarrow \mathbf{B} = B_1 \nabla\psi \times \nabla\theta + B_2 \nabla\phi \times \nabla\psi$

$\nabla \cdot \mathbf{B} = 0 \Rightarrow \partial_\theta B_2 = 0 \Rightarrow B_2 = B_2(\psi)$

Choose $\psi = \Phi_p / 2\pi \Rightarrow B_2 = 1$;
$\Phi_p = \int \mathcal{J}\mathbf{B} \cdot \nabla\theta d\psi d\phi = 2\pi \int B_2(\psi) d\psi$

$\mathbf{B} = F(\psi, \theta)\nabla\phi + \nabla\phi \times \nabla\psi$

$F = F(\psi)$ only if there are no toroidal forces on the system!

$$(4\pi/c)\mathbf{J} = \nabla F \times \nabla\phi + R^2\nabla\phi\nabla\cdot\left(R^{-2}\nabla\psi\right)$$

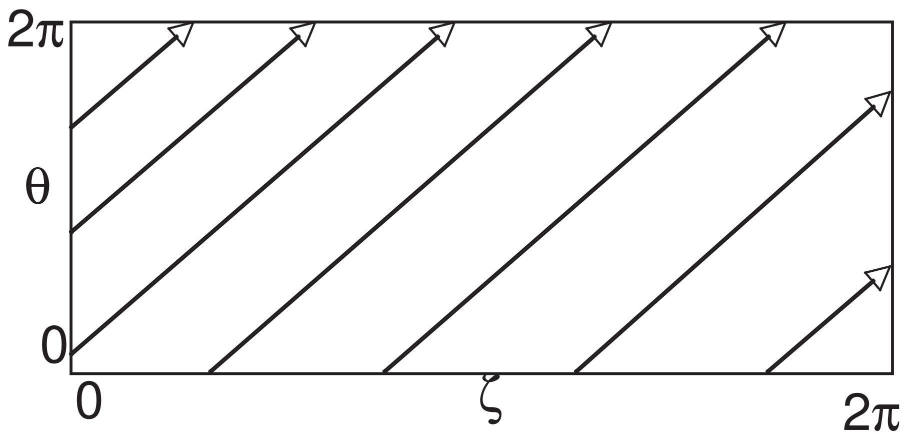
$$(4\pi/c)\mathbf{J}\times\mathbf{B} = -\nabla\psi\nabla\cdot\left(R^{-2}\nabla\psi\right) - R^{-2}F\nabla F + \mathcal{J}^{-1}\nabla\phi\partial_\theta F$$

$F = F(\psi)$ only if there are no toroidal forces on the system!

E: What is the effect of rotation? And of anisotropic pressure tensor (still diagonal)?

Straight Field line flux coordinates

2D with Generic $\zeta = \phi - \nu(\theta,\psi)$



$\vdots$

Choose $\nu(\theta,\psi)$ such that
$q = \mathbf{B}\cdot\nabla\zeta/\mathbf{B}\cdot\nabla\theta = q(\psi)$
E: Derive the expression of $\nu(\theta,\psi)$.

Clebsch representation for $\mathbf{B}$

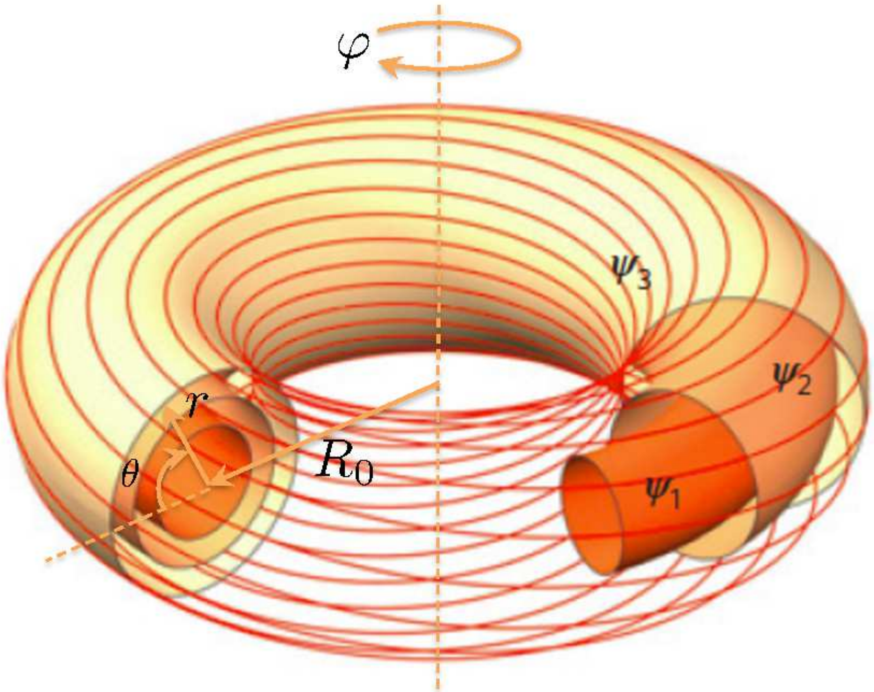
$$\mathbf{B} = \nabla\alpha(\psi,\theta,\zeta)\times\nabla\psi = \nabla\left(\zeta - q\theta\right)\times\nabla\psi$$

E: Show that magnetic field lines are intersections of surfaces $\alpha(\psi,\theta,\zeta) = \alpha_0$ and $\psi = \psi_0$.

Further freedom of choosing θ is used to select $\mathcal{J} = \left(\nabla\psi\times\nabla\theta\cdot\nabla\zeta\right)^{-1}$

Examples are:
$\mathcal{J} = \mathcal{J}_H(\psi)$: Hamada coordinates
$\mathcal{J} = \hat{\mathcal{J}}_B(\psi)/B^2$: Boozer coordinates

Shear Alfvén waves in tokamaks



- SAW continuous spectrum in one dimensional non-uniform plasma (this lecture)

$$\omega^2 = k_\parallel^2 v_A^2 = \omega_A^2(\psi) \ .$$

- In higher dimensionality systems, such as nearly two-dimensional (2D) or three-dimensional (3D) toroidal devices, the main additional complication is due to the modulations of v_A along $\mathbf{B_0}$.

- This causes the loss of translational symmetry for shear Alfvén waves traveling along $\mathbf{B_0}$ and sampling regions of periodically varying v_A.

- Similarly to electron wave packets traveling in a one-dimensional periodic lattice of period L, shear Alfvén waves in toroidal systems are characterized by gaps in their continuous spectrum, corresponding to the formation of standing waves at the Bragg reflection condition; *i.e.*,

$$2L = \ell\lambda \; , \qquad \lambda \equiv \frac{2\pi}{k} \; , \qquad \ell \in \mathbb{N} \; ,$$

E: Justify why the analogy of SAW with electron wave packets traveling in a one-dimensional periodic lattice is meaningful. Comment on the role of SAW group velocity, which is directed along $\mathbf{B_0}$.

- In toroidal systems, $L = 2\pi L_0$ is represented by the connection length; *i.e.*, the length of a magnetic field line connecting two distinct points on the same magnetic surface where the shear Alfvén wave frequency (or another equilibrium physical quantity of interest) is the same.

- Considering that $k \leftrightarrow k_\parallel$ in this analogy, the Bragg reflection condition becomes

$$k_\parallel = \frac{\ell}{2L_0} \; , \qquad \omega^2 = \frac{\ell^2 v_A^2}{4L_0^2} \; , \qquad \ell \in \mathbb{N} \; ,$$

with v_A representing now some "typical" value of the Alfvén speed on the reference magnetic surface.

E: Discuss what is the "typical" value of the Alfvén speed to be used in the equation above.

- Examples:
  - $\ell = 1$: Toroidal Alfvén Eigenmode (TAE) gap [Pogutse 78, D'Ippolito 80, Kieras 82]
  - $\ell = 2$: Ellipticity induced Alfvén Eigenmode (EAE) gap [Betti 91]
  - $\ell = 3$: Non-circularity induced Alfvén Eigenmode (NAE) gap [Betti 91]

- A low-frequency ($\ell = 0$) gap also exists because of finite plasma compressibility [Chu92, Turnbull 93] at $\omega \simeq \beta_i^{1/2} v_A/R_0 \ll v_A/R_0$.

Mode structure decomposition

- In axisymmetric tokamaks, a generic fluctuation field $f(r,\theta,\zeta)$, assuming implicitly $\exp(-i\omega t)$ time dependence, can be expressed as

$$f(r,\theta,\zeta) = e^{in\zeta} \sum_{m\in\mathbb{Z}} e^{-im\theta} f_n(m,r) \; .$$

- Here, in linear physics, only a single-n mode is considered. Note that for low-frequency ($|\omega| \ll |\Omega_i|$) modes of interest here, finite $k_\parallel = |\boldsymbol{b}_0 \cdot \boldsymbol{\nabla}|$ generally plays crucial stabilizing roles; such that instabilities tend to minimize $k_\parallel$.

- Noting the mode decomposition above, we have

$$K_\parallel = |k_\parallel q R_0| \simeq |nq - m| \; .$$

- Thus, $f_n(m,r)$ tends to be localized about its corresponding mode-rational surface located at $r = r_m$ where $nq(r_m) = m$. In other words, $f_n(m,r)$ generally should have a strong dependence on $(nq - m)$.

- This suggests to explicitly express $f_n(m,r)$ as $f_n(r, nq-m)$, where the first r dependence corresponds to other "weak" dependence on r:

$$f(r,\theta,\zeta) = e^{in\zeta} \sum_{m\in\mathbb{Z}} e^{-im\theta} f_n(r, nq-m) \ .$$

- Introducing $\hat{f}_n(r,\vartheta)$ as the Fourier conjugate of $f_n(r, nq-m)$; *i.e.*,

$$f_n(r, nq-m) = \int_{-\infty}^{\infty} e^{i(m-nq)\vartheta} \hat{f}_n(r,\vartheta) d\vartheta \ ,$$

the mode structure decomposition becomes

$$f(r,\theta,\zeta) = e^{in\zeta} \int_{-\infty}^{\infty} e^{-inq\vartheta} \hat{f}_n(r,\vartheta) \sum_{m\in\mathbb{Z}} e^{im(\vartheta-\theta)} d\vartheta \ .$$

- Applying the Poisson summation formula

$$\sum_{m\in\mathbb{Z}} e^{im(\vartheta-\theta)} = 2\pi \sum_{\ell\in\mathbb{Z}} \delta(\vartheta-\theta+2\pi\ell) \;,$$

we then have [Lu 12]

$$f(r,\theta,\zeta) = 2\pi \sum_{\ell\in\mathbb{Z}} e^{in\zeta - inq(\theta-2\pi\ell)} \hat{f}_n(r,\theta-2\pi\ell) \;.$$

- Mode structure decompositions (MSD) introduced above are valid for general toroidal mode number, n [Lu 12], and introduce and define the projection operator $\mathcal{P}_{Bn}(r,\vartheta)\;:\;f(r,\theta,\zeta)\mapsto\hat{f}_n(r,\vartheta)$, such that MSD reads

$$f(r,\theta,\zeta) = e^{in\zeta} \sum_{m\in\mathbb{Z}} e^{-im\theta} \int e^{i(m-nq)\vartheta} \mathcal{P}_{Bn}(r,\vartheta)\,[f]\,d\vartheta \;\;.$$

- $\mathcal{P}_{Bn}(r,\vartheta)$ properties allow interpreting ϑ as an extended poloidal angle
  - Multiplication by a periodic function $p(\theta)$ in (r,θ) space corresponds to multiplication by a periodic function $p(\vartheta)$ in (r,ϑ) space
  - in Clebsch coordinates (r,θ,ξ) with $\xi \equiv \zeta - q\theta$, $\boldsymbol{b}\cdot\boldsymbol{\nabla} = (\mathcal{J}B_0)^{-1}\partial_\theta \mapsto (\mathcal{J}B_0)^{-1}\partial_\vartheta$
  - Finally,

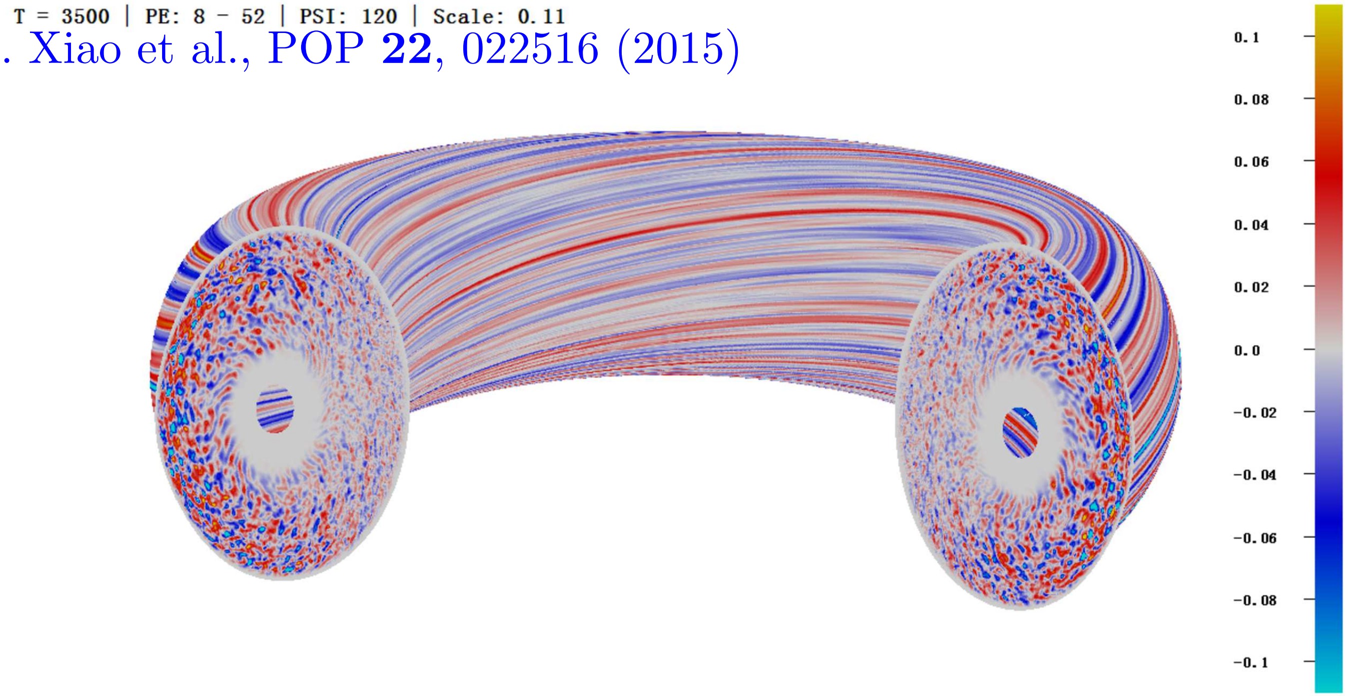
$$\begin{aligned}\boldsymbol{\nabla}_\perp \quad &\mapsto \quad \boldsymbol{\nabla} r\left(-inq'\vartheta + \frac{\partial}{\partial r}\right) + in\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\theta\left(\frac{\partial}{\partial\vartheta} - inq\right) - \frac{\boldsymbol{b}}{\mathcal{J}B_0}\frac{\partial}{\partial\vartheta}\\ &\mapsto \quad \boldsymbol{\nabla} r\frac{\partial}{\partial r} + \boldsymbol{\nabla}\xi\frac{\partial}{\partial\xi} + \boldsymbol{\nabla}\theta\frac{\partial}{\partial\vartheta} - \frac{\boldsymbol{b}}{\mathcal{J}B_0}\frac{\partial}{\partial\vartheta} \quad .\end{aligned}$$

E: Demonstrate all the properties of the projection operator $\mathcal{P}_{Bn}(r,\vartheta)$.

- The MSD approach is valid for general toroidal mode number, n [Lu 12]. For high toroidal mode numbers, $|n| \gg 1$, this approach reduces to the well-known "ballooning-mode" representation.

- Representation is valid for general mode structures in toroidal systems, from macroscopic to microscopic.

Courtesy of Y. Xiao et al., POP **22**, 022516 (2015)



RP: Exploit this formalism, showing it can recover usual representation and is readily extended to micro-scale (turbulence) and meso-scale fluctuations with its embedded field-aligned flux coordinates. Construct the corresponding algorithms for an elementary problem of your choice.

Ballooning mode representation

- In the $n \gg 1$ limit, one may define $\Delta_0 = 1/(nq')$, $r_1 \sim \mathcal{O}(a)$ and $|\Delta_0 r_1|^{1/2}$ as, respectively, the microscopic, macroscopic, and mesoscopic radial scale lengths.
- In the MSD representation, $f_n(r, nq - m)$ can then be expressed as

$$f_n(r, nq - m) = A_n(r) f_{0n}(r_1, nq - m) \; ,$$

  with $A_n(r)$ representing the meso-scale radial envelope.
- Noting that the functions $f_{0n}(r_1; nq - m)$ are nearly invariant under radial translations by multiples of Δ_0, a generic fluctuation $f(r, \theta, \zeta)$ can then be expressed in the following “ballooning-mode” representation

$$
\begin{aligned}
f(r,\theta,\zeta) &= \sum_{m\in\mathbb{Z}} e^{in\zeta - im\theta} f_{m,n}(r) = \sum_{m\in\mathbb{Z}} A_n(r) e^{in\zeta - im\theta} f_{0n}(r_1; nq-m) \\
&= \sum_{m\in\mathbb{Z}} A_n(r) e^{in\zeta - im\theta} \int e^{i(m-nq)\vartheta} \hat{f}_{0n}(r_1,\vartheta) d\vartheta \\
&= \sum_{m\in\mathbb{Z}} A_n(r) e^{in\zeta - im\theta} \int e^{i(m-nq)\vartheta} \mathcal{P}_{Bn}(r,\vartheta) \left[f_{0n} \right] d\vartheta \quad .
\end{aligned}
$$

- Due to the meso-scale nature of $A_n(r)$, it is possible to use the eikonal Ansatz [Zonca 93]

$$
A_n(r) \sim \exp i \int^r nq'\theta_k(r_1) dr_1 \quad ,
$$

introduced originally by [Dewar 81, 82] in theoretical analyses of short wavelength ballooning modes.

- In this way, the mapping of the perpendicular gradient can be written as

$$\boldsymbol{\nabla}_\perp \mapsto ik_\vartheta \boldsymbol{\nabla} r\,(s\vartheta - s\theta_k) + in\boldsymbol{\nabla}\zeta + ik_\vartheta r\boldsymbol{\nabla}\theta\ ,$$

with $k_\vartheta \equiv -(nq/r)$ and the magnetic shear defined as

$$s = s(r) = \frac{rq'(r)}{q(r)}\ .$$

E: Derive the mapping perpendicular gradient step by step.

- To proceed further analytically, in Lecture 2 we will adopt the so-called (s, α) model equilibrium for a local description of tokamaks with shifted circular magnetic surfaces [CHT 78]. Here, α is the dimensionless "ballooning" pressure gradient parameter

$$\alpha = -R_0 q^2 \frac{d\beta}{dr}\ .$$

References and reading material

L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

A. Hasegawa and L. Chen, Phys. Rev. Lett. **32**, 454 (1974).

L. Chen and A. Hasegawa, Phys. Fluids **17**, 1399 (1974).

L. Chen and A. Hasegawa, J. Geophys. Res. **79**, 1024 (1974).

J. P. Goedbloed, Physica **12D**, 107 (1984).

L. Chen and F. Zonca, Phys. Scr. **T60**, 81 (1995).

M. J. Engebretson, L. J. Zanetti, T. A. Potemra, W. Baumjohann, H. Lühr and M. H. Acuna, J. Geophys. Res. **92**, 10053 (1987).

G. Vlad, F. Zonca and S. Briguglio, Riv. Nuovo Cimento **22**, 1 (1999).

O. P. Pogutse and E. I. Yurchenko, Nucl. Fusion **18**, 1629 (1978).

D. A. D'Ippolito and J. P. Goedbloed, Plasma Phys. **22**, 1091 (1980).

C.E. Kieras and J.A. Tataronis, J. Plasma Phys. **28**, 395 (1982).

R. Betti and J. P. Freidberg, Phys. Fluids B **3**, 1865 (1991).

M. S. Chu, et al., Phys. Fluids B **4**, 3713 (1992).

A. D. Turnbull, et al., Phys. Fluids B **5**, 2546 (1993).

Z. X. Lu, F. Zonca and A. Cardinali, Phys. Plasmas **19**, 042104 (2012).

F. Zonca and L. Chen, Phys. Fluids B **5**, 3668 (1993).

R. L. Dewar and J. Manickam and R. C. Grimm and M. S. Chance, Nucl. Fusion **21**, 493 (1981).

R. L. Dewar and J. Manickam and R. C. Grimm and M. S. Chance, Nucl. Fusion **22**, 307 (1982).

A. D. Turnbull, et al., Phys. Fluids B **5**, 2546 (1993).

J. W. Connor, R. J. Hastie and J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).