Lecture 7

Description of Alfvén waves in tokamaks: Time asymptotic solution and WKB dispersion relation.

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May 21.st, 2018

Physics of Alfvén waves and energetic particles in fusion plasmas
– Mode structures and global dispersion of Alfvén waves in tokamaks –
May 7 – 21, 2018, IFTS – ZJU, Hangzhou

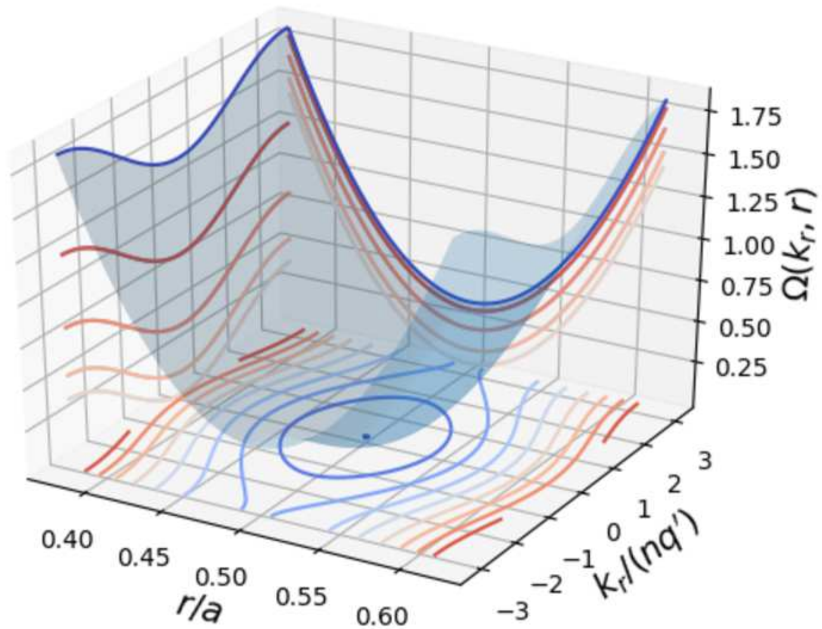
The WKB dispersion relation

- In order to understand the mechanism underlying the formation of an eigenmode, let us consider the time asymptotic WKB solution of the Schrödinger-like equation (Lecture 5 and Lecture 6) in the region between two WKB turning points located at r_{T1} and r_{T2} with $r_{T2} > r_{T1}$.
- From Lecture 6 [Z93, Lu 12], we can write the radial envelope in the form

$$A(r,t) \sim \frac{a_0 e^{-i\omega t+\gamma t}}{\sqrt{\partial D_R^0/\partial k_r}} \exp i \left(\int_{r_{T2}}^{r} K_r(r')dr' \right) + \frac{b_0 e^{-i\omega t+\gamma t}}{\sqrt{\partial D_R^0/\partial k_r}} \exp \left[-i \left(\int_{r_{T2}}^{r} K_r(r')dr' \right) \right] .$$

- Here, a_0 and b_0 are arbitrary constants and (Lecture 6, p. 16)

$$K_r = k_r + \Delta k_r - i\frac{D_A^1}{\partial D_R^0/\partial k_r} - i\gamma\frac{\partial D_R^0/\partial\omega}{\partial D_R^0/\partial k_r} \; .$$



- For the sake of clarity, we have also explicitly indicated the space and time dependences that belong to the eikonal $S(r, t)$, and we have reabsorbed the linear frequency shift $\Delta\omega$ into ω without loss of generality. In this way, ω becomes the eigenvalue of the problem, while γ is the eigenmode growth rate, which must also be determined self-consistently.

- Finally, we have explicitly separated waves propagating forward ($\propto a_0$) and backward ($\propto b_0$) in r, assuming that the equilibrium is up-down symmetric and, thus, that phase trajectories in the figure are symmetric for $k_r \to -k_r$.

- This assumption is by no means limiting the validity of the present analysis, which can be repeated in the general non-symmetric case.

E: With help of Lecture 6, convince yourself that this is the most general form of the WKB solution that you can derive.

E: Discuss what happens if up-down symmetry does not apply any longer. Why is this related with the symmetry in k_r and the propagation in the radial direction?

- Let r_{T2} be any of the turning points located at $r/a > 0.5$ in the figure above.

- Then, $k_r + \Delta k_r \simeq K_2(r_{T2} - r)^{1/2}$ for $r \to r_{T2}$, with K_2 an appropriate constant, and, with $\partial D_R^0/\partial k_r \propto (r_{T2} - r)^{1/2}$,

$$\int_{r_{T2}}^{r} (k_r + \Delta k_r)(r')dr' \simeq -\frac{2}{3} K_2 (r_{T2} - r)^{3/2} \, .$$

E: Repeat the calculation above. Can you say why the result should be negative?

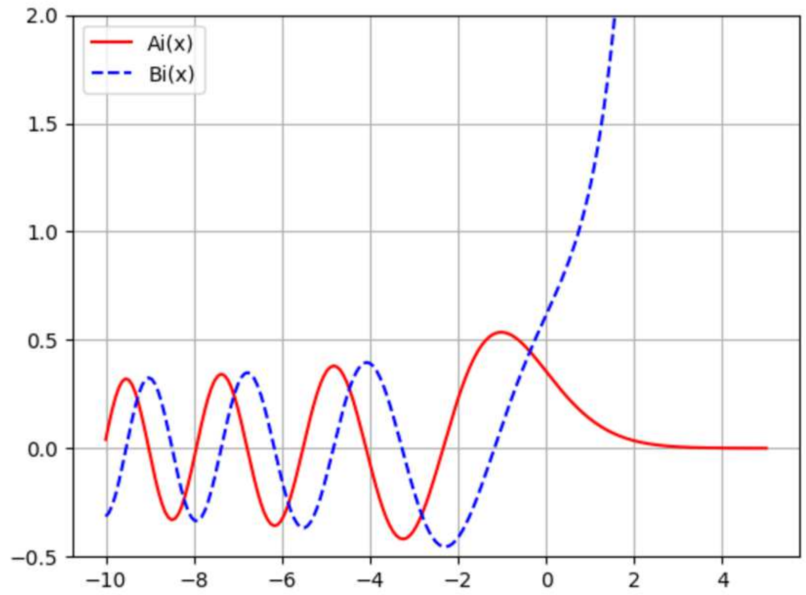
- Meanwhile, for $r \to r_{T2}$ the radial potential well can always be Taylor expanded and is, generally, linear. Thus, the Schrödinger-like equation for the eigenmode structure can be cast as

$$d^2|A|/dx^2 = x|A| \; ,$$

with $x = K_2^{2/3}(r - r_{T2})$.

E: Show that this is indeed the correct asymptotic form of the the Schrödinger-like equation. Can you recover the limiting form of the wave vector?

- This is the Airy equation and its linear independent solutions are given in terms of the Airy special functions, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, which are shown in the figure.



- The leading order behaviors of $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ for large $|x|$ are given by [Bender & 78]

$$
\begin{aligned}
\mathrm{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}x^{1/4}} \exp\left(-\frac{2}{3}x^{3/2}\right) \;, \qquad x \gg 1 \;, \\
&\sim \frac{1}{\sqrt{\pi}(-x)^{1/4}} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right) \;, \qquad x \ll -1 \;,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Bi}(x) &\sim \frac{1}{\sqrt{\pi}x^{1/4}} \exp\left(\frac{2}{3}x^{3/2}\right) \;, \qquad x \gg 1 \;, \\
&\sim \frac{1}{\sqrt{\pi}(-x)^{1/4}} \cos\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right) \;, \qquad x \ll -1 \;.
\end{aligned}
$$

E: Comment these asymptotic behaviors and discuss them making reference to the figure on p. 5.

- Letting by definition

$$b_0 \equiv R_2 a_0 e^{-i\pi/2} \; ,$$

with R_2 a complex number, and imposing asymptotic matching with Airy function behavior as $r \to r_{T2}$, one can show that the WKB solution for $r > r_{T2}$ is

$$A(r,t) \sim \frac{a_0 e^{-i\omega t + \gamma t}}{\sqrt{\partial D_R^0 / \partial k_r}} \left(\frac{1 + R_2}{2} \right) \exp \left[-i \left(\int_{r_{T2}}^{r} K_r(r')dr' \right) \right] + \frac{i a_0 e^{-i\omega t + \gamma t}}{\sqrt{\partial D_R^0 / \partial k_r}} (1 - R_2) \exp i \left(\int_{r_{T2}}^{r} K_r(r')dr' \right) \; .$$

- Here, we have adopted the convention that negative sign in the phase integral corresponds to the exponentially decaying, while positive to denotes the exponentially growing solution.

- Noting the definition of the a wave action current density, Γ_J, given in Lecture 6 (p. 11; reported here for convenience)

$$\Gamma_J = -\frac{\partial D_R^0}{\partial k_r}|A|^2 - \left(\frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\frac{\partial \varphi}{\partial r}|A|^2 \; ,$$

the physical interpretation of R_2 is that $|R_2|^2$ is the reflection coefficient of the radial potential well at $r = r_{T2}$, since it is the ratio of the reflected to the incoming wave action current density.

E: Demonstrate the physical interpretation of the reflection coefficient as the ratio of the reflected to the incoming wave action current density.

- When approaching $r = r_{T1}$, using the same method as above, $k_r + \Delta k_r \simeq K_1(r - r_{T1})^{1/2}$, with K_1 an appropriate constant, if $r_{T1}/a < 0.5$ and the phase trajectory is an oscillation.

- Meanwhile, $k_r + \Delta k_r \simeq \pi n q'(r_{T1}) - K_1 (r - r_{T1})^{1/2}$ if $r_{T1}/a > 0.5$ and the phase trajectory is a rotation. Thus,

$$\int_{r_{T1}}^{r} (k_r + \Delta k_r)(r') dr' \simeq \frac{2}{3} K_1 (r - r_{T1})^{3/2} , \qquad \text{(libration)}$$

$$\simeq \pi n q'(r_{T1})(r - r_{T1}) - \frac{2}{3} K_1 (r - r_{T1})^{3/2} , \qquad \text{(rotation)}$$

with $\partial D_R^0 / \partial k_r \propto (r - r_{T1})^{1/2}$.

E: Comment about the different qualitative features of phase space librations and rotations. Why is the wave phase spatial dependence so different? What does this represent?

E: In the case investigated here as an example, justify why phase space rotation is characterized by $k_r/nq' = \pi$ as the turning point at r_{T1} is approached.

E: What would be the difference of considering $k_r/nq' = -\pi$ as the turning point at r_{T1} is approached? Answer in the light of Lecture 6 (p. 6).

- Introducing the scaled radial variable $x = K_1^{2/3}(r_{T1} - r)$, and expressing the phase integrals using r_{T1} as lower integration point, it is possible to adopt the same approach based on asymptotic matching used above and show that the WKB solution for $r < r_{T1}$ is given by

$$
\begin{aligned}
A(r,t) \quad \sim \quad & \frac{b_1 e^{i(\pi \mp \pi)/4} e^{-i\omega t + \gamma t}}{\sqrt{\partial D_R^0 / \partial k_r}} \left(\frac{1 + R_1}{2} \right) \exp \left[-i \left(\int_{r_{T1}}^{r} K_r(r') dr' \right) \right] \\
& \pm \frac{i b_1 e^{i(\pi \mp \pi)/4} e^{-i\omega t + \gamma t}}{\sqrt{\partial D_R^0 / \partial k_r}} (1 - R_1) \exp i \left(\int_{r_{T1}}^{r} K_r(r') dr' \right) \; ,
\end{aligned}
$$

where the upper/lower signs refer to the case of libration/rotation and, again, we have adopted the convention that negative sign in the phase integral corresponds to the exponentially decaying solution away from r_{T1} and vice-versa.

- Meanwhile, R_1 has the same physical meaning as R_2; thus, $|R_1|^2$ is the reflection coefficient of the radial potential well at $r = r_{T1}$.

- In fact,

$$\begin{aligned} a_1 &= a_0 \exp\left(-i\Phi_R + \Phi_I\right) \equiv R_1 b_1 e^{\mp i\pi/2} \ , \\ b_1 &= R_2 a_0 \exp\left(i\Phi_R - \Phi_I - i\pi/2\right) \ ; \end{aligned}$$

and

$$\begin{aligned} \Phi_R &= \int_{r_{T1}}^{r_{T2}} \left(k_r + \Delta k_r\right)(r')dr' \ , \\ \Phi_I &= \left(\gamma - \bar{\gamma}_L\right)\tau_{|A|}/2 \ . \end{aligned}$$

E: Demonstrate these relations by expanding the WKB wave packet near the turning point at r_{T1}. Verify that R_1 is the proper definition of the corresponding reflection coefficient.

E: Derive the WKB solution for $r < r_{T1}$ given on p. 10. Hint: Use the asymptotic expressions for the Airy functions given on p.6.

- Letting $R_{1,2} \equiv |R_{1,2}|e^{i\varphi_{R_{1,2}}}$ and combining the conditions above, it is possible to derive the following global WKB dispersion relation [Z93, Chen RMP 16]

$$\oint (k_r + \Delta k)(r')dr' - \kappa_0 \pi + \varphi_{R_1} + \varphi_{R_2} = 2\ell\pi \ , \qquad \ell \in \mathbb{Z} \ ,$$

where the circulation line integral is taken along a complete libration/rotation, and $\kappa_0 = 1$ or $\kappa_0 = 0$ for librations or rotations, respectively.

- This equation is the analogue of the Bohr-Sommerfeld quantization condition for the Schrödinger-like equation. Meanwhile, the mode growth rate is given by

$$\gamma = \bar{\gamma}_L - \gamma_R \ .$$

- Here, $\tau_R \equiv \gamma_R^{-1}$ the characteristic lifetime, of the eigenmode/bound state (cf. Lecture 6)

$$\gamma_R \tau_{|A|} \equiv -\left(\ln|R_1| + \ln|R_2|\right) \ .$$

E: Explain the interpretation of the expression of τ_R as the lifetime of the bound state. Comment about the sign. Is it always a damping? Why?

- Equations on p. 12 illuminate the global character of the WKB dispersion relation. In fact, the rigorous calculation of reflection coefficients, $R_{1,2}$, implies the knowledge of the function $A(r,t)$ over the whole radial domain and imposing boundary conditions; *e.g.*, regularity of the solution on the magnetic axis and an ideal conducting wall at the plasma surface.
- Thus, global boundary conditions and equilibrium non-uniformity can affect frequency and growth rate of radially localized eigenmodes.

E: Give examples of how equilibrium non-uniformity can affect frequency and growth rate of radially localized eigenmodes.

E: Can you comment about the role of the SAW and SSW continuous spectrum?

- In general, for sufficiently localized modes, the radial potential well is assumed to be perfectly reflecting; that is, characterized by $R_1 = R_2 = 1$. Then, Eqs. on p. 12 reduce to the well-known form adopted in the literature.
- In the same case, it is possible to show that the WKB envelope becomes exponentially decaying away from the turning point positions.

E: Demonstrate that this is indeed the case.

Time asymptotic solution of the Schrödinger-like equation

- In order to illuminate the mechanism underlying the dynamic formation of an eigenmode as time asymptotic solution of a bound state, let us now consider a radially localized WKB wave packet, whose width is much less than and $r_{T2} - r_{T1}$, propagating back and forth between two WKB turning points located at r_{T1} and r_{T2} along libration/rotation phase trajectories shown in the figure on p. 3.
- When approaching r_{T2} from the left, the wave packet is reflected and phase shifted, according to $b_0 \equiv R_2 a_0 e^{-i\pi/2}$.
- In its further propagation toward r_{T1}, the wave packet is phase shifted by Φ_R, and amplified by $\exp(\bar{\gamma}_L \tau_{|A|}/2)$, consistent with the relations on p. 11.

- The reflection off r_{T1} is accompanied by a phase shift according to $a_1 \equiv R_1 b_1 e^{\mp i\pi/2}$ and, finally, the wave packet phase is shifted again by Φ_R and amplified by $\exp(\bar{\gamma}_L \tau_{|A|}/2)$ before returning to the original radial position.

- Thus, in one complete libration/rotation

$$A(r_0, t_0 + \tau_{|A|}) \to \mathcal{E}_0 A(r_0, t_0) \equiv \exp\left(i\Phi_{R0} + \gamma\tau_{|A|}\right) A(r_0, t_0) \ ,$$

with $\Phi_{R0} \equiv 2\Phi_R - \kappa_0 \pi + \varphi_{R_1} + \varphi_{R_2}$.

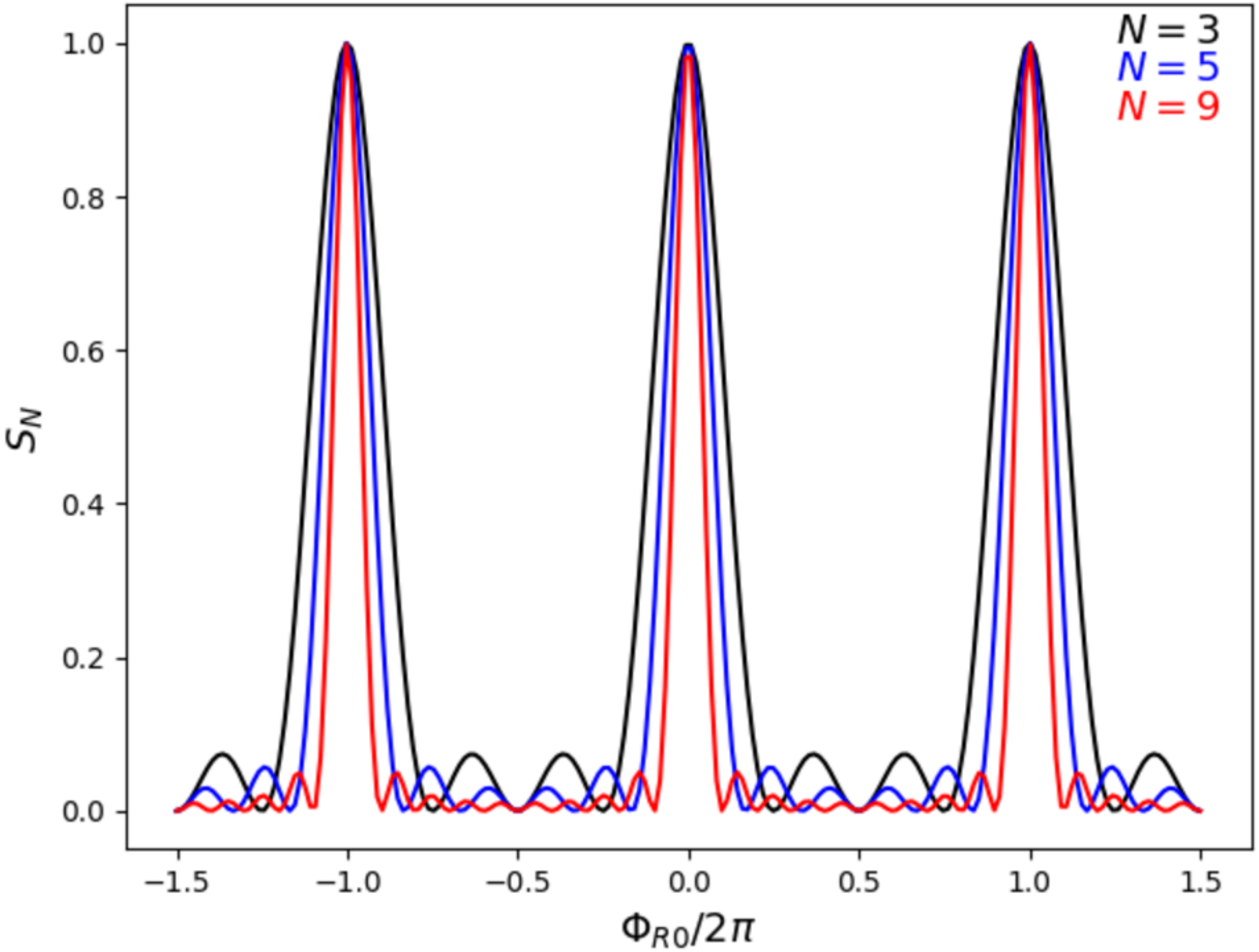
- If the same wave packet is launched at every passage by r_0, after N complete librations/rotations

$$\left|A(r_0, t_0 + N\tau_{|A|})\right|^2 \to \left|\sum_{p=0}^{N} \mathcal{E}_0^p\right|^2 |A(r_0, t_0)|^2 \ .$$

- Thus, $|A(r_0,t_0)|^2 \to (N+1)^2 S_N\,|A(r_0,t_0)|^2$, with [Z93, Lu 12]

$$S_N \equiv \frac{1}{(N+1)^2}\left|\sum_{p=0}^{N}\mathcal{E}_0^p\right|^2 = \frac{1}{(N+1)^2}\frac{\left|1-\mathcal{E}_0^{N+1}\right|^2}{\left|1-\mathcal{E}_0\right|^2}\;.$$

- This result suggests that the linear eigenmode can form from the time asymptotic solution of a bound state only if $|\gamma\tau_{|A|}| \ll 1$; that is if nonlinear effects are sufficiently weak.
- When this is not the case, the radial structures formed in the system may be completely different from the linear ones even if the parallel mode structures are weakly distorted by nonlinear effects.
- The following figure illustrates the dependence of the response function S_N on Φ_{R0} for different N values for $|\gamma\tau_{|A|}| \ll 1$.



The response function S_N dependence on Φ_{R0} for different N values.

- The interference patterns due to radial propagation demonstrate that the wave packet can have a finite amplitude and avoid phase mixing only if the global WKB dispersion relation is satisfied at the lowest order.
- The width of the frequency line is $\mathcal{O}(1/N)$, and the time scale for the formation of the radial eigenmode structure is $\sim \tau_{|A|}$ (cf. Lecture 6).

References and reading material

L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072120 (2014).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072121 (2014).

Z. X. Lu, F. Zonca and A. Cardinali, Phys. Plasmas **19**, 042104 (2012).

F. Zonca and L. Chen, Phys. Fluids B **5**, 3668 (1993).

F. Zonca, Plasma Phys. Control. Fusion **35**, B307 (1993).

F. Zonca, *Ph. D. Thesis*, Princeton University (Princeton, NJ, 1993).