Lecture 6

Description of Alfvén waves in tokamaks:
Global eigenvalue problem in toroidal plasmas.
The formation of meso-scales.

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Physics of Alfvén waves and energetic particles in fusion plasmas
– Mode structures and global dispersion of Alfvén waves in tokamaks –
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Global eigenvalue problem in toroidal plasmas

- Summarize main results from Lecture 3, Lecture 4, and Lecture 5.
- The eigenvalue problem for coupled SAW and SSW fluctuation in tokamak plasmas is given by the SAW vorticity and SSW equations (Lecture 4 pp. 11-12, 14) with proper boundary conditions; e.g., vanishing plasma displacement on a perfectly conducting wall assumed at the plasma boundary. Such a problem generally requires numerical approach.
- Further progress can be made assuming short wavelength (high mode number) modes, for which the mode structure decomposition (Lecture 3, p.24), reduces to the ballooning mode representation (Lecture 4, pp. 8-12).
- The great advantage is that wave equations become, thus, one dimensional, simplifying numerical analysis and often making the eigenvalue problem tractable even analytically.

- Proper boundary conditions in the ballooning space are outgoing waves as $|\vartheta| \to \infty$. In fact, $|\vartheta| \to \infty$ corresponds to vanishingly small radial scales, where no energy source can physically exist.

- The plasma response can be written introducing the radial envelope $A(r,t)$, the eikonal $S(r,t)$, the unit polarization vector $\hat{\boldsymbol{e}}(r,t)$ and the parallel mode structures, $y_1(\vartheta;r,t)$ and $y_2(\vartheta;r,t)$ satisfying dimensionless equations (Lecture 4, p.14) and normalization condition (Lecture 4, p.15).

- Meanwhile, the radial envelope evolution equation can be written in the Schrödinger-like form (Lecture 5)

$$\frac{\partial}{\partial t}\left(\frac{\partial D_R^0}{\partial \omega}A^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D_R^0}{\partial k_r}A^2\right) + 2D_A^1 A^2 - 2iD_R^1 A^2$$

$$+iA\left(\frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\frac{\partial^2 A}{\partial r^2} = -2ie^{-iS}A\hat{\boldsymbol{e}}^+ \cdot \boldsymbol{F} \ \ldots$$

$$
- \left(\hat{\mathbf{e}}^{+} \cdot \frac{d}{dt} \hat{\mathbf{e}} - \frac{d}{dt} \hat{\mathbf{e}}^{+} \cdot \hat{\mathbf{e}} \right) \frac{\partial D_R^0}{\partial \omega} A^2 + \left(\frac{\partial \hat{\mathbf{e}}^{+}}{\partial \omega} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial t} - \frac{\partial \hat{\mathbf{e}}^{+}}{\partial t} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial \omega} \right) A^2
$$

$$
- \left(\frac{\partial \hat{\mathbf{e}}^{+}}{\partial k_r} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial r} - \frac{\partial \hat{\mathbf{e}}^{+}}{\partial r} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\mathbf{e}}}{\partial k_r} \right) A^2 \; .
$$

- Here, $k_r = nq'\theta_k = \partial_r S(r,t)$ and $\omega = -\partial_t S(r,t)$ satisfy the local WKB dispersion relation

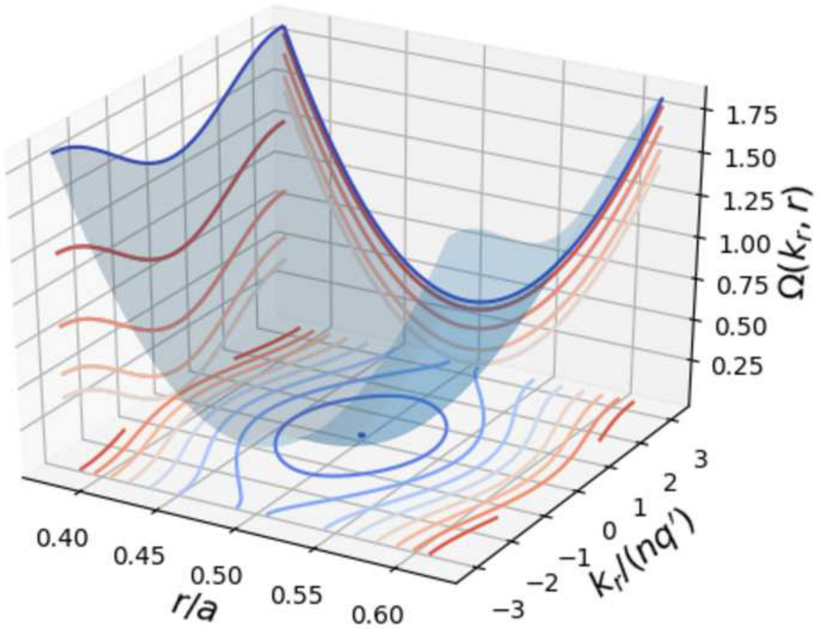
$$
D_R^0(r,t,k_r,\omega) \equiv \hat{\mathbf{e}}^{+} \cdot \boldsymbol{D}_R^0 \cdot \hat{\mathbf{e}} = 0 \; ,
$$

which can be solved as

$$
\omega = \Omega(k_r, r, t) \; .
$$

- In the present case, the dispersion matrix satisfies $D_R^0(r,t,k_r,\omega) = D_R^0(r,k_r,\omega)$, because the plasma equilibrium does not depend explicitly on time.

- Thus, the characteristics of wave packet propagation are obtained implicitly for $\omega = \text{const}$ and, thus, can be viewed as the contour plot of the local dispersion relation $\Omega(k_r, r, t) = \Omega(k_r, r)$.



- Qualitative three dimensional plot of $\Omega(k_r, r)$ in arbitrary units. Wave packet phase trajectories are visible as level curves projected on the (r, k_r) plane.

- Closed phase trajectories are called oscillations (or librations), while open trajectories are called rotations.

- The 2π periodicity of wave packet phase trajectories in $\theta_k = k_r/(nq')$ reflects the periodicity of the plasma equilibrium in θ.

- In an up-down symmetric plasma equilibrium, WKB turning points, $\partial D_R^0/\partial k_r = 0$ naturally appear at $\theta_k = p\pi$, $p \in \mathbb{Z}$.
- Because of the 2π periodicity in $\theta_k = k_r/(nq')$, one usually restricts the analysis to the first Brillouin zone, $-\pi \leq \theta_k = k_r/(nq') \leq \pi$.
- Note that the general case is readily recovered as

$$A(r,t) \to \sum_{h\in\mathbb{Z}} e^{i2\pi nqh} A(r,t) = \sum_{m\in\mathbb{Z}} \delta(nq-m) A(r,t) \ ,$$

where we have made use of the Poisson summation formula.
- This equation is a consequence of (connected with) the step involved in having expressed (Lecture 3, p.23; cf. also Lecture 4)

$$f(r,\theta,\zeta) = e^{in\zeta} \sum_{m\in\mathbb{Z}} e^{-im\theta} f_n(r, nq-m) \ ;$$

i.e., of having interpreted the envelope function defined for discrete values of its argument as a continuous function that interpolates it.

- In the notation of the Schrödinger - like equation (pp. 3-4), we maintain explicit time dependence for the sake of clarity and generality, since it can be used for a wide class of linear and nonlinear wave propagation problems in slowly varying and weakly non-uniform media with different forms of the dispersion matrix.

- For this reason, we have added a $\propto \boldsymbol{F}(r,t)$ term on the r.h.s., which can describe external forcing as well as nonlinear interactions [Chen RMP 16].

E: Read [Chen RMP 16] and explore the various possibilities for application of the Schrödinger - like equation, derived in Lecture 5, to topics that have broad interest in physics research beyond plasma physics and, e.g., share concepts and methods that are common in quantum mechanics and condensed matter physics.

- This equation with vanishing amplitude at $r = 0$ and $r = a$ boundary conditions, corresponding to regularity of the solution on the magnetic axis and an ideal conducting wall at the plasma surface, poses a well defined global eigenvalue problem in the time asymptotic limit (cf. Lecture 7).

- More precisely, the SAW vorticity and SSW equations (Lecture 4) with corresponding boundary conditions and the Schrödinger - like equation define a well-posed eigenvalue problem.

- The original 2D problem is, thus, reduced to two nested 1D problems. Note, however, that the present approach is not merely a WKB solution of the original equations. In fact, the local plasma response is solved as a full-wave problem along the field line, which provides the parallel mode structure, mode polarization and local plasma dispersive properties.

- This information is then used to generate the envelope evolution equation the Schrödinger - like form. In this respect, the solution strategy adopted here is a mixed full-wave WKB approach [Lu 12], and the global solution is a proper superposition of local solutions.

- The general solution of the Schrödinger - like equation generally requires a numerical approach. The characteristic spatiotemporal scales of the radial envelope $A(r,t)$ are manifestation of the structures that are formed due to the slowly varying weakly non-uniform plasma equilibrium.
- Next, we investigate the formation of meso-scales more in detail. The WKB dispersion relation will be discussed in Lecture 7.

The formation of meso-scales

- Adopting the same methodology of Lecture 4 and Lecture 5, it is instructive to let $A(r,t) = |A(r,t)| \exp i\varphi(r,t)$ and separate real and imaginary part in the Schrödinger - like equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial D_R^0}{\partial \omega}|A|^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D_R^0}{\partial k_r}|A|^2\right) &= -2\left[D_A^1 - \mathbb{Im}\left(\frac{\hat{\boldsymbol{e}}^+ \cdot \boldsymbol{F}}{Ae^{iS}}\right)\right]|A|^2 \\
&+ \left(\frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\frac{\partial}{\partial r}\left(\frac{\partial \varphi}{\partial r}|A|^2\right) ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial D_R^0}{\partial \omega}\frac{\partial \varphi}{\partial t} - \frac{\partial D_R^0}{\partial k_r}\frac{\partial \varphi}{\partial r} &= -\left(\frac{1}{2}\frac{\partial^2 D_R^0}{\partial k_r^2} + \frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\left(\frac{1}{|A|}\frac{\partial^2 |A|}{\partial r^2} - (\partial_r \varphi)^2\right) \\
&+ \left[D_R^1 - \mathbb{Re}\left(\frac{\hat{\boldsymbol{e}}^+ \cdot \boldsymbol{F}}{Ae^{iS}}\right)\right] + \frac{i}{2}\left(\hat{\boldsymbol{e}}^+ \cdot \frac{d}{dt}\hat{\boldsymbol{e}} - \frac{d}{dt}\hat{\boldsymbol{e}}^+ \cdot \hat{\boldsymbol{e}}\right)\frac{\partial D_R^0}{\partial \omega} \\
&- \frac{i}{2}\left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial \omega}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial t} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial t}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial \omega}\right) + \frac{i}{2}\left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right) .
\end{aligned}
$$

- These equations are the proper extension of those derived in Lecture 4 (pp. 29-30) and summarized in Lecture 5 (p. 2), when the wave packet radial group velocity becomes small, creating the optimal conditions for structure formation and eventually the occurrence of bound states.

- The first of these two equations can be viewed as the extended evolution equation for the wave action density, $J = (\partial_\omega D_R^0)|A|^2$ (Lecture 5, p. 3).

- In the absence of internal or external sources, the first equation is a continuity equation with a wave action current density, Γ_J, given by

$$\Gamma_J = -\frac{\partial D_R^0}{\partial k_r}|A|^2 - \left(\frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r} \right) \frac{\partial \varphi}{\partial r}|A|^2 \; ,$$

consistent with the fact that $\partial_r \varphi$ is the first order correction to the wave packet radial wave vector $k_r = \partial_r S$.

- Meanwhile, the second equation describes the evolution of the wave packet phase $\varphi(r,t)$ along the wave packet characteristics, qualitatively illustrated in the figure on p.5.

- Consistent with the present asymptotic approach, the wave packet spatial and temporal micro-scales are $|k_r|^{-1}$ and $|\omega|^{-1}$, while the macro-scales corresponding to the slowly evolving weakly non-uniform plasma are L and T.

- In addition to these, there are intermediate spatial and temporal meso-scales, which are due to the structures formed self-consistently because of the wave packet propagation.

- One fundamental temporal meso-scale is the time required for a complete oscillation/rotation of the wave packet between two WKB turning points, r_{T1} and r_{T2}, where $\partial_{k_r} D_R^0|_{r_{T1},r_{T2}} = 0$; *i.e.*,

$$\tau_{|A|} = 2 \int_{r_{T1}}^{r_{T2}} \left| \frac{\partial D_R^0/\partial \omega}{\partial D_R^0/\partial k_r} \right| dr \ .$$

- In Lecture 7, we will show that $\tau_{|A|}$ is the characteristic time for a bound state to form and for the plasma to respond at the linear eigenmode frequency.

- Another important time scale, τ_L, is defined by the exponential amplification of the wave packet during its propagation between r_{T1} and r_{T2}, that is

$$\tau_L \equiv \bar{\gamma}_L^{-1} = \tau_{|A|} \left(\oint \frac{D_A^1}{\partial D_R^0 / \partial k_r} dr \right)^{-1}$$
$$= \tau_{|A|} \left[2 \int_{r_{T1}}^{r_{T2}} \left(- \frac{D_A^1}{\partial D_R^0 / \partial \omega} \right) \left| \frac{\partial D_R^0 / \partial \omega}{\partial D_R^0 / \partial k_r} \right| dr \right]^{-1} .$$

where the circulation line integral connecting r_{T1} and r_{T2} is taken along the wave packet characteristics and, thus, the absolute value of the group velocity must be introduced in the second line.

- Note that

$$\gamma_L = - \frac{D_A^1}{\partial D_R^0 / \partial \omega}$$

is the local wave packet linear growth rate. Thus, the equation above defines a weighted average of the wave packet linear growth rate.

- This equation could be extended to include the effect of external and/or nonlinear sources through the $\propto \hat{\boldsymbol{e}}^{+} \cdot \boldsymbol{F}$ term. But this is beyond the scope of these lectures. More information on this point is given in [Chen RMP 2016].

- Another important time scale is the lifetime of the bound state, τ_R (cf. Lecture 7 for a detailed discussion), due to the possibility that the bound state may not be perfectly contained by the radial potential well and, thus, radiate energy away from the region where it is originally localized.

- **Now focus on the explicit solution of the equations above.**

- The phase φ evolution equation in the WKB limit is obtained neglecting the terms $\propto \partial_r^2 |A|$ and $\propto (\partial_r \varphi)^2$.

- Calling $R(r,t)$ this WKB limiting form of the right hand side for brevity, the solution can be written as

$$\varphi(r,t) = \varphi_0 \left(T^{-1}(T(r) - t + t_0) \right) - \int_{T^{-1}(T(r)-t+t_0)}^{r} \frac{R(r', t + T(r') - T(r))}{\partial D_R^0 / \partial k_r} dr' \, .$$

- Here, $\varphi_0(r)$ is the initial value of the phase at $t = t_0$ and, again, assuming a stationary plasma background, the function $T(r)$ and its inverse T^{-1} are defined by

$$t - t_0 = -\int_{r_0}^{r} \frac{\partial D_R^0\left(\omega, r', k_r(r', \omega)\right)/\partial \omega}{\partial D_R^0\left(\omega, r', k_r(r', \omega)\right)/\partial k_r} dr' \equiv T(r) - T(r_0) \ .$$

E: Demonstrate that the solution above coincides with integrating the corresponding linear PDE in the WKB limit along the characteristics.

- Now recall that the eikonal $S(r,t)$, in this case, is given by

$$S(r,t) = S(r_0) - \omega t + \int_{r_0}^{r} k_r(r') dr' \ ,$$

and note the explicit form of $R(r,t)$.

- Then, the solution for $\varphi(r,t)$ can be formally rewritten as

$$\varphi(r,t) = \varphi_0(r_0) - \Delta\omega(t-t_0) + \int_{r_0}^{r} \Delta k_r(r',\omega)dr' ,$$

where $\Delta\omega$ and $\Delta k_r(r,\omega)$ define, respectively, the linear frequency and radial wave vector shift due to the slowly evolving weakly nonuniform plasma background; and the corresponding meso spatiotemporal scales.

- Here, we have noted that, by definition,

$$r_0 = T^{-1}(T(r) - t + t_0) .$$

- Meanwhile, without the the $\propto \hat{\boldsymbol{e}}^{+} \cdot \boldsymbol{F}$ term, $R(r,t) = R(r)$ is independent of time and

$$\Delta k_r(r) = -\frac{R(r)}{\partial D_R^0(\omega, r, k_r)/\partial k_r} .$$

- Again, the effect of external and/or nonlinear sources is accounted for by the $\propto \hat{\boldsymbol{e}}^{+} \cdot \boldsymbol{F}$. Clearly, $|\Delta\omega|^{-1}$ and $|\Delta k_r|^{-1}$ are the corresponding temporal and spatial meso-scales defined by the phase φ evolution equation.

- The fundamental spatial meso-scales connected with the wave packet envelope, $|A|$, are due to the effective radial potential well that allows bound states to exist when the radial group velocity is sufficiently small (cf. Lecture 5).

- Near a local extremum of $\Omega(k_r, r)$, as in the figure on p. 5, it is possible to estimate
$$D_R^1 \sim \frac{(r-r_0)^2}{L^2} k_r^2 \frac{\partial^2 D_R^0}{\partial k_r^2} \; .$$

- This term must balance that $\propto \partial_r^2 |A|$ in the φ evolution equation, *i.e.*
$$\frac{1}{|A|} \frac{\partial^2 D_R^0}{\partial k_r^2} \frac{\partial^2 |A|}{\partial r^2} \sim \frac{\partial^2 D_R^0}{\partial k_r^2} \frac{1}{(r-r_0)^2} \; .$$

- Thus, the meso-scale is $|r - r_0| \sim |L/k_r|^{1/2}$, consistent with the ordering anticipated in Lecture 5. For consistency, we also have $|\Delta\omega/\omega| \sim |\Delta k_r/k_r|^2 \sim 1/|k_r L|$ and, as a consequence, the local linear mode frequency is a good estimate of the actual eigenmode frequency of a bound state near an extremum of the dispersion function.

- When the radial profile of the dispersion function is monotonic, the above estimate of the corresponding meso-scale is modified. In fact, we have

$$D_R^1 \sim \frac{(r - r_0)}{L} k_r^2 \frac{\partial^2 D_R^0}{\partial k_r^2} \; .$$

  Thus, the balance with the term $\propto \partial_r^2 |A|$ in the φ evolution equation gives the meso-scale $|r - r_0| \sim |L/k_r^2|^{1/3}$ consistent with the ordering anticipated in Lecture 5.

- Using these results, in Lecture 7 we will calculate the time asymptotic behavior of the wave packet and the WKB dispersion relation.

References and reading material

L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072120 (2014).

F. Zonca and L. Chen, Phys. Plasmas **21**, 072121 (2014).

Z. X. Lu, F. Zonca and A. Cardinali, Phys. Plasmas **19**, 042104 (2012).