Lecture 5

Wave propagation in slowly varying, weakly non-uniform media: The wave kinetic equation. Bound states and Schrödinger equation.

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The wave kinetic equation

- Summary from Lecture 4: Letting $A(r,t) = |A(r,t)| \exp i\varphi(r,t)$, the transport of the envelope intensity $|A|^2$ is

$$\frac{\partial}{\partial t}\left(\frac{\partial D_R^0}{\partial \omega}|A|^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D_R^0}{\partial k_r}|A|^2\right) = -2\left[D_A^1 - \mathbb{I}\mathrm{m}\left(\frac{\hat{\boldsymbol{e}}^+\cdot\boldsymbol{F}}{Ae^{iS}}\right)\right]|A|^2\,,$$

and the equation for evolution of the envelope phase:

$$\begin{aligned}\frac{\partial D_R^0}{\partial \omega}\frac{\partial \varphi}{\partial t} - \frac{\partial D_R^0}{\partial k_r}\frac{\partial \varphi}{\partial r} = & \left[D_R^1 - \mathbb{R}\mathrm{e}\left(\frac{\hat{\boldsymbol{e}}^+\cdot\boldsymbol{F}}{Ae^{iS}}\right)\right] + \frac{i}{2}\left(\hat{\boldsymbol{e}}^+\cdot\frac{d}{dt}\hat{\boldsymbol{e}} - \frac{d}{dt}\hat{\boldsymbol{e}}^+\cdot\hat{\boldsymbol{e}}\right)\frac{\partial D_R^0}{\partial \omega}\\ & -\frac{i}{2}\left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial \omega}\cdot\boldsymbol{D}_R^0\cdot\frac{\partial \hat{\boldsymbol{e}}}{\partial t} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial t}\cdot\boldsymbol{D}_R^0\cdot\frac{\partial \hat{\boldsymbol{e}}}{\partial \omega}\right)\\ & +\frac{i}{2}\left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot\boldsymbol{D}_R^0\cdot\frac{\partial \hat{\boldsymbol{e}}}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial r}\cdot\boldsymbol{D}_R^0\cdot\frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\,.\end{aligned}$$

- The intensity evolution equation in the absence of dissipation and external forcing, is in the form of a continuity equation that suggests the definition of a wave action density on the phase space (r, k_r) (cf. Lecture 4):

$$J(r,k_r,t) = \left.\frac{\partial D_R^0(r,t,k_r,\omega)}{\partial \omega}\right|_{\omega=\Omega(k_r,r,t)} |A|^2(r,t) \ .$$

- Introducing the (r, k_r) phase space and the concept of a wave packet evolving on it is natural, since the propagation follows the characteristics, consistent with the local dispersion relation (cf. Lecture 4, p. 26).

- Following McDonald [McDonald 88], the evolution equation for $J(r, k_r, t)$ can be obtained directly from the phase space representation of the wave equation, elegantly constructed from the spectral tensors (or tensor Wigner functions) through the Weyl transform.

- Here, we rather sketch the derivation originally provided by [Bernstein 77], which is less formal and more readily understood based on its physical meaning.

Bernstein's derivation of the wave kinetic equation

- Let us consider the value $\tilde{J}(r, k_{r0}, t)$ of the wave action density as a function of (r, t) corresponding to a given initial $r = r_0$ and $k_r = k_{r0}$.

- To do so, the solution of the equations of motion (characteristics, cf. Lecture 4, p. 26), *i.e.*

$$r = r(r_0, k_{r0}, t) \qquad \text{and} \qquad k_r = k_r(r_0, k_{r0}, t) \; ,$$

must be inverted as

$$r_0 = r_0(r, k_r, t) \qquad \text{and} \qquad k_{r0} = k_{r0}(r, k_r, t) \; .$$

- From the last relation, we can write $k_r = k_r(r, k_{r0}, t)$. Thus, $\tilde{J}(r, k_{r0}, t) = J\left(r, k_r(r, k_{r0}, t), t\right)$ with $J(r, k_r, t)$ defined as on p.3.

- It is then possible to construct the wave action as

$$W(t) = \int \tilde{J}(r, k_{r0}, t) dr_0 dk_{r0} \ ,$$

where the integral is extended to a specified and fixed volume of the initial phase space (r_0, k_{r0}).

- Therefore,

$$\begin{aligned} \dot{W}(t) &= \int (\partial_t + \tilde{v}_g \partial_r) \tilde{J}(r, k_{r0}, t) dr_0 dk_{r0} \\ &= -\int \left\{ \frac{\partial}{\partial r} \tilde{v}_g + \frac{2\left[D_A^1 - \mathbb{I}\mathrm{m}\left(\frac{\hat{e}^+ \cdot \boldsymbol{F}}{A e^{iS}} \right) \right]}{\left. \frac{\partial D_R^0(r, t, k_r, \omega)}{\partial \omega} \right|_{\omega = \Omega(k_r, r, t)}} \right\}_{k_r = k_r(r, k_{r0}, t)} \tilde{J}(r, k_{r0}, t) dr_0 dk_{r0} \ . \end{aligned}$$

- Here, we have defined the wave packet group velocity

$$v_g(r,k_r,t) = -\left(\frac{\partial D_R^0(r,t,k_r,\omega)}{\partial k_r}\right) \Big/ \left(\frac{\partial D_R^0(r,t,k_r,\omega)}{\partial \omega}\right)\Bigg|_{\omega=\Omega(k_r,r,t)} ,$$

  $\tilde{v}_g = v_g(r,k_r(r,k_{r0},t),t)$ and we made use of equation for the transport of the envelope intensity on p. 2.

- Meanwhile, due to the Hamiltonian nature of the wave characteristics (cf. Lecture 4, p. 26), the Jacobian of the invertible coordinate transformation $(r_0,k_{r0}) \to (r,k_r)$ is unity.

- Since the boundary of the integration domain of in (r,k_r) space is moving, the modification must be taken into account.

- More specifically, we can write

$$
\begin{aligned}
\dot{W}(t) &= \int \left[\partial_t J(r,k_r,t) + \partial_r \left(v_g(r,k_r,t)J(r,k_r,t)\right) + \partial_{k_r}\left(\dot{k}_r J(r,k_r,t)\right)\right] dr dk_r \\
&= \int \left(\partial_t + v_g\partial_r + \dot{k}_r\partial_{k_r}\right) J(r,k_r,t) dr dk_r \\
&= \int \left(\partial_t + v_g\partial_r + \dot{k}_r\partial_{k_r}\right) \tilde{J}(r,k_{r0},t) dr_0 dk_{r0} \ ,
\end{aligned}
$$

  where we have considered explicitly the equations of the wave characteristics (cf. Lecture 4, p. 26).

- The integration domain in the equation above and that at p. 5 is arbitrary. Thus, the integrands must be the same; and we can derive the wave kinetic equation for the wave action density on the phase space (r,k_r), $J(r,k_r,t)$, defined above.

- We therefore have the wave kinetic equation in the form derived by [Bernstein 77]:

$$\left(\partial_t + v_g\partial_r + \dot{k}_r\partial_{k_r}\right) J(r,k_r,t) =$$

$$= -\left\{ \frac{\partial}{\partial r}\tilde{v}_g + \frac{2\left[D_A^1 - \mathbb{I}\mathrm{m}\left(\frac{\hat{\boldsymbol{e}}^+\cdot\boldsymbol{F}}{Ae^{iS}}\right)\right]}{\left.\frac{\partial D_R^0(r,t,k_r,\omega)}{\partial\omega}\right|_{\omega=\Omega(k_r,r,t)}} \right\} J(r,k_r,t)\ .$$

- Note that we have maintained the notation $\partial_r\tilde{v}_g$ on the r.h.s., since ∂_r must be taken holding k_{r0} and t fixed.

- This implies having solved the equations of motion for the wave characteristics (cf. Lecture 4, p. 26), and makes the solution of the wave kinetic equation less straightforward than it may seem at first glance.

The Schrödinger equation

- The wave amplitude evolution equation (Lecture 4) and the corresponding wave kinetic equation, derived above, are not sufficient for solving the global problem of formation of bound states by SAW and SSW wave packets propagating in toroidal fusion plasmas.

E: Explain with your own words why you cannot solve the problem of bound state formation with the theoretical analysis described so far. What is the missing element? Could you suggest a simple way around to solve the problem?

- In a slowly varying and weakly non-uniform plasma, there are conditions that favor the formation of bound states.
- This means that wave packets satisfying the amplitude evolution equation (Lecture 4 and p. 2 of this lecture) may evolve into eigenstates, when the plasma non-uniformity provides an effective potential well.

- As a consequence, meso-scales are formed (Lecture 6), intermediate between $|\omega(r,t)|^{-1}$ and $|k_r(r,t)|^{-1}$, characterizing the micro-temporal and -spatial modulation of the wave packet, respectively; and T and L macro-scales accounting for the spatiotemporal evolution of the background medium.

- In particular, let us address the case where a radial potential well arises due to plasma background profiles (slowly evolving in time).

- We still assume that the slow temporal evolution satisfies

$$|\omega(r,t)|T = |\partial_t S(r,t)|T \sim \epsilon^{-1} \gg 1 \; .$$

- Meanwhile, let's assume that the spatial meso-scale is ordered as

$$|\partial_r \ln \boldsymbol{A}| \sim |\partial_r \ln k_r| \sim |k_r|\varepsilon^{1/2} \; .$$

- This ordering is quite general, and is consistent with the amplitude evolution equation provided that

$$|k_r| \left| \frac{\partial D_R^0}{\partial k_r} \right| \sim \varepsilon^{1/2} \; .$$

- In other words, the radial potential well can form only when the wave packet radial group velocity is sufficiently small, yielding bound states and, eventually, eigenstates characterized by meso-scales.

- Clearly, the meso-scales are self-consistently determined by the non-uniformity of the background plasma profiles, which, in the orderings above, are measured by the control parameter $\varepsilon \sim |\omega(r,t)T|^{-1}$.

E: Comment about the above conclusion and the importance of having a vanishing radial group velocity for properly defined radial bound states to exist. Why is radial non-uniformity so crucial?

- Reconsider the radial envelope evolution equation without separating intensity and phase

$$\frac{\partial}{\partial t}\left(\frac{\partial D_R^0}{\partial \omega}A^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D_R^0}{\partial k_r}A^2\right) + 2D_A^1 A^2 - 2iD_R^1 A^2$$
$$= -2ie^{-iS}A\hat{\boldsymbol{e}}^+ \cdot \boldsymbol{F} - \left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)A^2$$
$$- \left(\hat{\boldsymbol{e}}^+ \cdot \frac{d}{dt}\hat{\boldsymbol{e}} - \frac{d}{dt}\hat{\boldsymbol{e}}^+ \cdot \hat{\boldsymbol{e}}\right)\frac{\partial D_R^0}{\partial \omega}A^2 + \left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial \omega}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial t} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial t}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial \omega}\right)A^2 .$$

- Near the bottom of a local parabolic radial potential well, we can estimate

$$\frac{\Delta r^2}{L^2} \sim \frac{1}{k_r^2 L^2 \varepsilon} \sim \varepsilon \qquad \Rightarrow |k_r(r,t)|L \sim \varepsilon^{-1}$$

- Here, Δr measures the radial distance from the bottom of the potential well.
- This estimate is obtained assuming (express ε as function of $|k_r L|$)

$$\Delta r \sim |\partial_r \ln \boldsymbol{A}|^{-1} \sim |k_r|^{-1}\varepsilon^{-1/2} \sim |k_r|^{-1}|k_r L|^{1/2} \qquad \Rightarrow \qquad |\Delta r| \sim |L/k_r|^{1/2} \, ,$$

which is the estimate of the corresponding meso-scale, as anticipated in Lecture 4.

- Meanwhile, for a local linear profile of the radial potential well,

$$\frac{|\Delta r|}{L} \sim \frac{1}{|k_r| L \varepsilon^{1/2}} \sim \varepsilon \qquad \Rightarrow |k_r(r,t)| L \sim \varepsilon^{-3/2} \qquad .$$

- The estimate of the meso-scale becomes (express ε as function of $|k_r L|$)

$$\Delta r \sim |\partial_r \ln \boldsymbol{A}|^{-1} \sim |k_r|^{-1}\varepsilon^{-1/2} \sim |k_r|^{-1}|k_r L|^{1/3} \qquad \Rightarrow \qquad |\Delta r| \sim |L/k_r^2|^{1/3} \, .$$

- These estimates of the meso-scales, generated by the propagating radial wave packets, are consistently described by the ordering (p. 10-11; repeated here for convenience)

$$|\partial_r \ln \boldsymbol{A}| \sim |\partial_r \ln k_r| \sim |k_r|\varepsilon^{1/2} \; , \qquad |k_r| \left| \frac{\partial D_R^0}{\partial k_r} \right| \sim \varepsilon^{1/2} \; ,$$

and importantly depend on the non-uniformity of the plasma background, as expected [Chen RMP 16].

- Given this ordering, the envelope evolution equation on p. 12 must be modified since, at $\mathcal{O}(\varepsilon)$, the second line of (Lecture 4, p. 24)

$$\begin{aligned} \boldsymbol{A}(r-s, t-\tau) &= \boldsymbol{A}(r,t) - s\partial_r \boldsymbol{A} - \tau \partial_t \boldsymbol{A} \\ &+ \left[(s^2/2)\partial_r^2 + s\tau\partial_r\partial_t + (\tau^2/2)\partial_t^2 \right] \boldsymbol{A} + \dots \; , \end{aligned}$$

cannot be dropped as in the derivation of Lecture 4.

- In particular, we must retain the term originating from $\sim (s^2/2)\partial_r^2 \boldsymbol{A}$, which yields a contribution

$$i\hat{\boldsymbol{e}}^{+}\frac{\partial^2 \boldsymbol{D}_R^0}{\partial k_r^2}\cdot\frac{\partial^2}{\partial r^2}\boldsymbol{A}\ ,$$

to be added on the l.h.s. of the amplitude evolution equation on p. 12.

- This term can be rewritten as

$$i\hat{\boldsymbol{e}}^{+}\frac{\partial^2 \boldsymbol{D}_R^0}{\partial k_r^2}\cdot\hat{\boldsymbol{e}}\frac{\partial^2}{\partial r^2}A\ ,$$

since the polarization vector $\hat{\boldsymbol{e}}$ varies on the scale L but not on the meso-scale. That is,

$$\partial_r^2\boldsymbol{A} = A\partial_r^2\hat{\boldsymbol{e}} + 2\partial_r A\partial_r\hat{\boldsymbol{e}} + \hat{\boldsymbol{e}}\partial_r^2 A \simeq \hat{\boldsymbol{e}}\partial_r^2 A\ .$$

- Furthermore, extending the relations involving the polarization vector (Lecture 4, p. 29),

$$\hat{\boldsymbol{e}}^+ \cdot \frac{\partial^2 \boldsymbol{D}_R^0}{\partial k_r^2} = -2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r} \cdot \frac{\partial \boldsymbol{D}_R^0}{\partial k_r} - \frac{\partial^2 \hat{\boldsymbol{e}}^+}{\partial k_r^2} \cdot \boldsymbol{D}_R^0 + \frac{\partial^2}{\partial k_r^2}\left(D_R^0 \hat{\boldsymbol{e}}^+\right)$$

$$= \hat{\boldsymbol{e}}^+ \frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\frac{\partial D_R^0}{\partial k_r} - 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r} \cdot \frac{\partial \boldsymbol{D}_R^0}{\partial k_r} - \frac{\partial^2 \hat{\boldsymbol{e}}^+}{\partial k_r^2} \cdot \boldsymbol{D}_R^0 ,$$

from which

$$\hat{\boldsymbol{e}}^+ \cdot \frac{\partial^2 \boldsymbol{D}_R^0}{\partial k_r^2} \cdot \hat{\boldsymbol{e}} = \frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r} \cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r} .$$

E: Demonstrate explicitly the results on pp. 15-16. Convince yourself that they contain all necessary terms for the modification of the envelope evolution equation at the required order.

- The envelope evolution equation finally becomes

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial D_R^0}{\partial \omega}A^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D_R^0}{\partial k_r}A^2\right) + 2D_A^1 A^2 - 2iD_R^1 A^2 \\
&\quad + iA\left(\frac{\partial^2 D_R^0}{\partial k_r^2} + 2\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0 \cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)\frac{\partial^2 A}{\partial r^2} = -2ie^{-iS}A\hat{\boldsymbol{e}}^+\cdot \boldsymbol{F} \\
&\quad - \left(\hat{\boldsymbol{e}}^+\cdot \frac{d}{dt}\hat{\boldsymbol{e}} - \frac{d}{dt}\hat{\boldsymbol{e}}^+\cdot \hat{\boldsymbol{e}}\right)\frac{\partial D_R^0}{\partial \omega}A^2 + \left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial \omega}\cdot \boldsymbol{D}_R^0\cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial t} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial t}\cdot \boldsymbol{D}_R^0\cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial \omega}\right)A^2 \\
&\quad - \left(\frac{\partial \hat{\boldsymbol{e}}^+}{\partial k_r}\cdot \boldsymbol{D}_R^0\cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial r} - \frac{\partial \hat{\boldsymbol{e}}^+}{\partial r}\cdot \boldsymbol{D}_R^0\cdot \frac{\partial \hat{\boldsymbol{e}}}{\partial k_r}\right)A^2 \,.
\end{aligned}
$$

- This equation generalizes that on p. 12 (Lecture 4, p. 29) and reduces to it, when the orderings of p. 10-11 reduce to those of (Lecture 4, p. 21).

E: Demonstrate that this is indeed the case and that the envelope evolution equation of Lecture 4 is recovered in the proper limit.

- This equation is in the form of a Schrödinger-like equation and can be used for analyses of mode structures, stability properties and nonlinear dynamics.
- With proper boundary conditions, *e.g.* vanishing amplitude at $r = 0$ and $r = a$, corresponding to regularity of the solution on the magnetic axis and an ideal conducting wall at the plasma surface, this problem provides a well posed eigenvalue problem for coupled SAW and SSW in toroidal plasmas.
- The general solution of this equation requires a numerical approach. Lecture 6 and Lecture 7 will deal with the analytical solutions of this equation applied to the description of Alfvén waves in tokamaks.

References and reading material

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