Lecture 6

Energetic particle excitations of shear Alfvén instabilities in tokamaks

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Energetic particle excitations of shear Alfvén instabilities in tokamaks

- Kinetic effects and resonant wave-particle interactions enter with different spatiotemporal scales. Kinetic response $\delta K \propto \omega_d$ and pressure-curvature coupling, also $\propto \omega_d$, favor important role of geodesic curvature (radial drift, coupling through k_r). Thus:
  - inertial layer: mostly thermal ions/electrons (drive and damping) enhanced by $\propto \omega_d^2$ response
  - ideal region: mostly energetic particles (drive) as layer response is suppressed by finite magnetic drift orbit width effects
  - meso-scales: could be due to both thermal plasma and EP, *e.g.*, in fusion plasmas; they need careful analysis (not this lecture)

  $\Rightarrow$ EP excitation of BAE/BAAE in kinetic thermal ion gap [Chen 2007]

- Illustrate the possible role of EP in the layer by EP excitation of TAE/KTAE [Chen RMP 2016] assuming small EP drift orbits.

BAE/BAAE: kinetic effects in the layer

- Use the equations form Lecture 5. For simplicity, write layer equations ($\hat{\kappa}_\perp \sim |s\vartheta| \to \infty$) neglecting FLR (FLR are treated by [Wang 2010].)
- QN condition for $\hat{\kappa}_\perp \to \infty$ (p. 22 of Lecture 5)

$$\left(1+\frac{1}{\tau}\right)\left(\delta\hat{\Phi}_n - \delta\hat{\Psi}_n\right) = \frac{T_i}{n_0 e}\hat{\kappa}_\perp \left\langle \delta\hat{K}_{in}\right\rangle_v \quad .$$

- GK vorticity equation for $\hat{\kappa}_\perp \to \infty$ (p. 23 of Lecture 5) with $g(\vartheta,\theta_k) \simeq s\vartheta \sin\vartheta$

$$\frac{\partial^2}{\partial\vartheta^2}\delta\hat{\Psi}_n + \frac{\omega\left(\omega-\omega_{*pi}\right)}{\omega_A^2}\delta\hat{\Phi}_n + \frac{4\pi R_0 q^2}{B_0}\frac{g(\vartheta,\theta_k)\omega}{ck_\vartheta\hat{\kappa}_\perp}\left\langle m_i\left(\mu B_0 + v_\parallel^2\right)\delta\hat{K}_{in}\right\rangle_v = 0 \quad .$$

- BAE are important since they can be excited by both fast ions (long wave-lengths) as well by thermal ions (short wavelengths; AITG) [POP 1999; Chen RMP 2016].

- The particle distribution function $\delta\hat{K}_i$ is derived from the drift-kinetic equation (no FLR; $\omega_{tr} = v_\parallel/qR$)

$$[\omega_{tr}\partial_\vartheta \; - \; i \; (\omega - \omega_d)]_i \; \delta\hat{K}_i \; = \; i\left(\frac{e}{m}\right)_i QF_{0i}\left[\left(\delta\hat{\phi} - \delta\hat{\psi}\right) + \left(\frac{\omega_d}{\omega}\right)_i \delta\hat{\psi}\right]$$

- For BAE one typically has $\omega \approx \omega_{ti} \approx \omega_{*pi} \approx k_\parallel v_A$. Important effects are expected from kinetic interaction in the radial local region (kinetic layer) $k_\parallel qR_0 \approx \beta^{1/2}$.

- Typical paradigmatic case for application of MSD to drift Alfvén turbulence as well as MHD (no specification of mode number so far; just $k_\parallel qR_0 \approx \beta^{1/2}$).

- Derivation of layer equation via asymptotic expansion in $\beta^{1/2}$. Lowest order solution yields $\delta\hat{K}_i^{(0)} = -(e/m)_i(QF_{0i}/\omega)(\delta\hat{\phi}^{(0)} - \delta\hat{\psi}^{(0)})$, which yields $\delta\hat{\psi}^{(0)} = \delta\hat{\phi}^{(0)}$ when substituted back into the quasi-neutrality condition.

- At the next order, $\delta\hat{\phi}^{(1)} = e^{i\vartheta}\delta\hat{\phi}^{(1+)} + e^{-i\vartheta}\delta\hat{\phi}^{(1-)}$ with a corresponding $\delta\hat{K}_i^{(1)} = e^{i\vartheta}\delta\hat{K}_i^{(1+)} + e^{-i\vartheta}\delta\hat{K}_i^{(1-)}$. Let $\omega_{tr}^{(\pm)} = [1 \pm (nq - m)]\,(v_\parallel/qR_0)$. Then

$$i\left[\pm\omega_{tr}^{(\pm)} - \omega\right]\delta\hat{K}_i^{(1\pm)} = i\left(\frac{e}{m}\right)_i QF_{0i}^{(\mp)}\delta\hat{\phi}^{(1\pm)} \mp i\left(\frac{e}{m}\right)_i QF_{0i}\frac{v_\perp^2/2 + v_\parallel^2}{R_0\omega_{ci}\omega}\frac{k_r}{2i}\delta\hat{\phi}^{(0)}$$

- Note that the dominant effect comes from the geodesic curvature (large $k_r \propto \vartheta$), causing radial magnetic drifts. Sideband generation is evident in the wave-particle interaction as well as in the ω_* effect in $QF_{0i}^{(\mp)}$, which is computed at $m \mp 1$.

- Substituting back into the quasi-neutrality and letting $\omega_{ti}^{(\pm)} = (2T_i/m_i)^{1/2}\,[1 \pm (nq - m)]\,/qR_0$

$$\delta\hat{\phi}^{(1\pm)} = \mp i\frac{cT_i}{eB_0}\frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m\mp1}(\omega/\omega_{ti}^{(\pm)})}\frac{k_r}{\omega R_0}\delta\hat{\phi}^{(0)}$$

- Definitions: $Z(x) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-y^2}/(y-x)dy$ and subscripts m and $m \mp 1$ indicate poloidal mode numbers to compute

$$N(x) = \left(1 - \frac{\omega_{*ni}}{\omega}\right) \left[x + \left(1/2 + x^2\right) Z(x)\right] - \frac{\omega_{*Ti}}{\omega} \left[x \left(1/2 + x^2\right) + \left(1/4 + x^4\right) Z(x)\right] \; ,$$

$$D(x) = \left(\frac{1}{x}\right) \left(1 + \frac{1}{\tau}\right) + \left(1 - \frac{\omega_{*ni}}{\omega}\right) Z(x) - \frac{\omega_{*Ti}}{\omega} \left[x + \left(x^2 - 1/2\right) Z(x)\right] \; ,$$

- Corresponding solutions of the drift-kinetic equation are

$$\delta \hat{K}_i^{(1\pm)} = \mp i \frac{cT_i}{eB_0} \frac{e/m_i}{\omega \mp \omega_{tr}^{(\pm)}} \left[\frac{m_i}{2T_i} \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right) QF_{0i} - \frac{N_m(\omega/\omega_{ti}^{(\pm)})}{D_{m\mp1}(\omega/\omega_{ti}^{(\pm)})} QF_{0i}^{(\mp)} \right] \frac{k_r}{\omega R_0} \delta \hat{\phi}^{(0)}$$

- The large $|\vartheta|$ vorticity equation becomes finally (canonical form yielding the general fishbone like dispersion relation)

$$\left(\frac{\partial^2}{\partial\vartheta^2} + \Lambda^2\right)\delta\hat{\Psi}^{(0)} = 0$$

- Definitions: $\delta\Psi^{(0)} = \hat{\kappa}_\perp \delta\hat{\psi}^{(0)}$

$$\begin{aligned}
\Lambda^2 \;=\; & \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) + q^2\frac{\omega_{ti}^2}{2\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)\left((\omega/\omega_{ti}^{(+)})F(\omega/\omega_{ti}^{(+)}) + (\omega/\omega_{ti}^{(-)})F(\omega/\omega_{ti}^{(-)})\right)\right. \\
& -\frac{\omega_{*Ti}}{\omega}\left((\omega/\omega_{ti}^{(+)})G(\omega/\omega_{ti}^{(+)}) + (\omega/\omega_{ti}^{(-)})G(\omega/\omega_{ti}^{(-)})\right) \\
& -\left((\omega/\omega_{ti}^{(+)})N_m(\omega/\omega_{ti}^{(+)})\frac{N_{m-1}(\omega/\omega_{ti}^{(+)})}{D_{m-1}(\omega/\omega_{ti}^{(+)})} + (\omega/\omega_{ti}^{(-)})N_m(\omega/\omega_{ti}^{(-)})\frac{N_{m+1}(\omega/\omega_{ti}^{(-)})}{D_{m+1}(\omega/\omega_{ti}^{(-)})}\right)
\end{aligned}$$

$$\begin{aligned}
F(x) &= x\left(x^2+3/2\right) + \left(x^4+x^2+1/2\right)Z(x)\ , \\
G(x) &= x\left(x^4+x^2+2\right) + \left(x^6+x^4/2+x^2+3/4\right)Z(x)\ ,
\end{aligned}$$

- Reminder from Lecture 3: $\Lambda^2 = k_\|^2 q^2 R_0^2$ represents the SAW continuum; i.e., frequency gap is at $\Lambda^2 < 0$

- To demonstrate that Λ^2 expression contains the physics of low-frequency (BAE) gap formation in the SAW continuum, take the fluid limit in which MHD is valid, $|\omega| \gg \omega_{ti}$

- Large argument expansion of the plasma Z function yields:

$$\Lambda^2 = \frac{1}{\omega_A^2}\left[\omega^2 - \left(\frac{7}{4} + \frac{T_e}{T_i}\right) q^2 \omega_{ti}^2\right] + \mathrm{i}\sqrt{\pi} q^2 e^{-\omega^2/\omega_{ti}^2} \frac{\omega^2}{\omega_A^2}\left(\frac{\omega_{ti}}{\omega} - \frac{\omega_{*Ti}}{\omega_{ti}}\right)\left(\frac{\omega^2}{\omega_{ti}^2} + \frac{T_e}{T_i}\right)^2 .$$

- From solution of the low frequency MHD equations (Lecture 3) we have that gap should occur at $\omega^2 < \Gamma(T_i + T_e)/(m_i R_0^2)$. Comparing results with kinetic theory:

$$\Gamma_e = 2 \qquad \text{electrons behave as adiabatic massless fluid in 2D}$$

$$\Gamma_i = 7/2???$$

Micro- and meso- scale excitation of low-frequency AE/EPM

- With the general expression of Λ ($k_\parallel = 0$). Reduces to fluid limit (Lecture 3) and incorporates Landau damping (esp. of Alfvén-acoustic branch)

$$\Lambda^2 = \frac{\omega^2}{\omega_A^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) + q^2\frac{\omega\omega_{ti}}{\omega_A^2}\left[\left(1-\frac{\omega_{*ni}}{\omega}\right)F(\omega/\omega_{ti}) - \frac{\omega_{*Ti}}{\omega}G(\omega/\omega_{ti}) - \frac{N_m(\omega/\omega_{ti})}{2}\left(\frac{N_{m+1}(\omega/\omega_{ti})}{D_{m+1}(\omega/\omega_{ti})} + \frac{N_{m-1}(\omega/\omega_{ti})}{D_{m-1}(\omega/\omega_{ti})}\right)\right]$$

low frequency AE/EPM can be excited by both thermal ions (micro-scale) and energetic ions (meso-scales)

$$i\Lambda = \delta\bar{W}_f + \delta\bar{W}_k \qquad \text{(fast ions included)}$$

- Excitation of low-frequency AE by thermal ions is most easily seen using the simplified expression of Λ derived earlier: AITG excitation mechanism

$$\Lambda^2 = \frac{1}{\omega_A^2}\left[\omega^2 - \left(\frac{7}{4}+\frac{T_e}{T_i}\right) q^2\omega_{ti}^2\right] + \mathrm{i}\sqrt{\pi}q^2 e^{-\omega^2/\omega_{ti}^2}\frac{\omega^2}{\omega_A^2}\left(\frac{\omega_{ti}}{\omega}-\frac{\omega_{*Ti}}{\omega_{ti}}\right)\left(\frac{\omega^2}{\omega_{ti}^2}+\frac{T_e}{T_i}\right)^2 .$$

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ accumulation point becomes unstable! The unstable continuum is not a concern
  (E: compute how much the mode can grow before mode converting to KAW. Hint: compute how long takes for a wave-packet born at small ϑ-ballooning to reach large ϑ where KAW physics is important).

- When $\omega\omega_{*Ti} > \omega_{ti}^2$ and equilibrium effects localize the AE, the Alfvénic ITG mode is excited (AITG)

- Energetic particle contribution to potential energy from Lecture 5 (p. 27)

$$\delta \bar{W}_k = \left(\delta\hat{\Phi}_{0+}^{(0)\dagger}\delta\hat{\Phi}_{0+}^{(0)}\right)^{-1}\left(-\frac{1}{2}\right)\int_{-\infty}^{\infty}\delta\hat{\Phi}^{(0)\dagger}\left[\frac{4\pi R_0 q^2}{B_0}\frac{g(\vartheta,\theta_k)\omega}{ck_\vartheta\hat{\kappa}_\perp}\left\langle m_E\left(\mu B_0+v_\parallel^2\right)J_0\delta\hat{K}_E\right\rangle_v\right]d\vartheta$$

E: Derive this expression by using the form of $\delta\bar{W}_k$ of Lecture 5 and the expression of $\hat{\omega}_{dE}$ (*i.e.*, replacing $\nabla B_0 -$ with curvature-drift).

- This expression is ready for numerical implementation, once $\delta\hat{K}_E$ is provided. Complete analytical expressions are also available and given in [POP 2014; Chen RMP 2016].
- For simplicity, neglect finite orbit width effects, and assume $\omega_b \gg \bar{\omega}_d$.
- Since the thermal plasma has infinite conductivity, we can take $\delta\phi = \delta\psi$ at the lowest order.

- The GKE can be written as (p.22 Lecture 5)

$$\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\boldsymbol{v}_d\cdot\boldsymbol{\nabla}_\perp\right)\delta K=i\frac{e}{m}\left[Q\bar{F}_0\left\langle\delta\phi_g-\delta\psi_g\right\rangle\right.$$
$$\left.-\boldsymbol{v}_d\cdot\boldsymbol{\nabla}_\perp Q\bar{F}_0\partial_t^{-1}\left\langle\delta\psi_g\right\rangle\right]\quad .$$

- Applying present approximations and letting $-i\boldsymbol{v}_d\cdot\boldsymbol{\nabla}\equiv\omega_d\equiv\tilde{\omega}_d+\bar{\omega}_d$ (incorporating the toroidal mode number into the definition of $\bar{\omega}_d$, unlike in Lecture 4)

$$\left(\omega+iv_\parallel\nabla_\parallel-\tilde{\omega}_d-\bar{\omega}_d\right)\delta K=-\frac{e}{m}\frac{\omega_d}{\omega}Q\bar{F}_0\left\langle\delta\psi_g\right\rangle$$
$$\Rightarrow\qquad\left(\omega+iv_\parallel\nabla_\parallel-\bar{\omega}_d\right)\delta K\simeq-\frac{e}{m}\frac{\omega_d}{\omega}Q\bar{F}_0\delta\phi$$

- Here, we have noted that $\delta K=e^{-iQ_b}\delta K_b$, and $v_\parallel\nabla_\parallel Q_b-\tilde{\omega}_d=0$.

- The pull back operator e^{-iQ_b} from bounce-banana center to guiding center is nearly an identity for negligible orbit width and $\delta K \simeq \delta K_b$. Similarly $\langle \delta\psi_g \rangle \simeq \delta\psi \simeq \delta\phi$.

- Since $\bar{\omega}_d \ll \omega_b$, at the lowest order $\nabla_\parallel \delta K \simeq \partial_\vartheta \delta \hat{K} \simeq 0$. To avoid secularities in ϑ and introducing bounce averaging on unperturbed particle motions

$$\overline{[\ldots]} \equiv \left(\oint \frac{d\ell}{v_\parallel} \right)^{-1} \oint \frac{d\ell}{v_\parallel} [\ldots]$$

it is possible to obtain

$$\delta\hat{K} = \frac{e}{m} \frac{Q\bar{F}_0}{\omega} e^{-inq\vartheta} \left(\frac{\overline{e^{inq\vartheta} \omega_d \delta\hat{\phi}}}{\bar{\omega}_d - \omega} \right) \quad ,$$

which represents the integral particle response.

E: Discuss why this response of energetic particles is integral, even if we assumed negligible orbit width. What type of approximation could you introduce to further simplify the problem?

- In general, the integral particle response reflect the "memory" of the system in collisionless fusion plasmas

$$e^{-inq\vartheta}\overline{e^{inq\vartheta}\omega_d\delta\hat{\phi}} \equiv \hat{\hat{\omega}}_d\delta\hat{\phi}$$

- Here, note that $\hat{\omega}_d$ is a notation for recalling that we need to treat ∇B_0 and curvature drifts in the same fashion, for correctly accounting for equilibrium and perturbed parallel pressure balance (Lecture 2, Lecture 3 and Lecture 4). Meanwhile $\hat{\hat{\omega}}_d$ is defined by the above equation.
- Further approximation is generally possible either assuming deeply trapped particles and/or fixed mode structure; e.g., one single (m, n).

- Assuming $\bar{F}_{0E}$ to be symmetric in $v_\parallel$, and $\bar{\omega}_d = \Omega_d = -(\mu B_0 + v_\parallel^2)k_\vartheta/(\Omega R_0)$ for deeply trapped particles, with $\tau_b = 2\pi/\omega_b = 2\pi q R_0/[(r/R_0)\mathcal{E}]^{1/2}$

$$\delta\bar{W}_k = \frac{\pi^2}{|s|}\frac{e^2}{mc^2} q R_0 B_0 \sum_{v_\parallel/|v_\parallel|=\pm 1} \int d\mathcal{E} \int d\mu \left(\frac{\Omega_d}{k_\vartheta}\right)^2 \tau_b \frac{Q\bar{F}_0}{\bar{\omega}_d - \omega} \; .$$

- Consider EPs with an isotropic slowing down distribution function and injection energy E_F much larger than the critical energy E_c, so that for $E_F/m_E > \mathcal{E} > E_c/m_E$ the EP energy is predominantly transferred to thermal electrons by collisional friction (as for α-particles in fusion plasmas); *i.e.*,

$$\bar{F}_0 = \frac{3P_{0E}}{4\pi E_F} \frac{\mathrm{H}(E_F/m_E - \mathcal{E})}{(2\mathcal{E})^{3/2} + (2E_c/m_E)^{3/2}} \; ,$$

where H denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is $(3/2)P_{0E}$ for $E_F \gg E_c$.

- Substituting into the expression for $\delta\bar{W}_k$, we obtain

$$\delta\bar{W}_{nk} = \frac{3\pi(r/R_0)^{1/2}\alpha_E}{8\sqrt{2}|s|}\left[1+\frac{\omega}{\bar{\omega}_{dF}}\ln\left(\frac{\bar{\omega}_{dF}}{\omega}-1\right)+i\pi\frac{\omega}{\bar{\omega}_{dF}}\right] \quad ,$$

with $\bar{\omega}_{dF}=\bar{\omega}_d(\mathcal{E}=E_F/m_E)$ and having noted that $|\omega_{*E}|\gg|\omega|$ for SAW resonantly excited by EPs.

E: Derive this expression step by step filling in the missing algebra.

E: Use the simplified expression for Λ on p. 10 to show that BAE can be excited. Can you discuss the condition for exciting EPM?

RP: Use the same dispersion relation to show that there is a transition from EP driven modes at long wavelength (BAE) to thermal ion driven modes (AITG) for increasing mode number.

RP: Adopt the fluid expression of Λ derived in Lecture 3 and demonstrate the properties of the EP excitations of the low-frequency fluctuation spectrum. What can you conclude from the comparison of BAE vs BAAE excitation?

EP excitation of TAE/KTAE

- Illustrate the possible role of EP in the layer by EP excitation of TAE/KTAE [Chen RMP 2016] assuming small EP drift orbits.

- Relevant equations for TAE's (KAW's) as a vorticity equation. Derived assuming small Finite Larmor radius (FLR) correction to ideal Ohm's Law (p.3 this lecture and Lecture 5)

$$\left[\partial_\vartheta^2 + \Omega^2\left(1 + 2\epsilon_0\cos\theta\right) + \frac{\alpha\cos\vartheta}{\hat\kappa_\perp^2} - \frac{(s-\alpha\cos\vartheta)^2}{\hat\kappa_\perp^4} - \right.$$
$$\left. - \frac{s^4\rho_K^2}{\hat\kappa_\perp^2}(\vartheta-\theta_k)^4\right]\hat\Phi - \sum_j \frac{4\pi\omega e_j}{c^2}\frac{q^2R^2}{\hat\kappa_\perp k_\vartheta^2} < \omega_d\delta\hat K >_{vj} = 0 \ .$$

- Here, $\hat\Phi = \hat\kappa_\perp\delta\phi$, $\hat\kappa_\perp(\vartheta) = 1 + [s(\vartheta-\theta_k) - \alpha\sin\vartheta]^2$, $\alpha = -Rq^2\beta'$, $\beta = 8\pi P/B^2$, $\Omega^2 = \omega^2/\omega_A^2$, $\epsilon_0 = 2(r_o/R + \Delta')$, $k_\vartheta = -nq/r$, and Δ' is the derivative of the Shafranov shift.

$$\omega_{dj}(\vartheta) = \Omega_{dj}\left\{\cos\vartheta + [s(\vartheta-\theta_k) - \alpha\sin\vartheta]\sin\vartheta\right\}$$

$$\Omega_{dj} = -k_\vartheta \frac{cm_j}{e_j BR}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right) \text{ and } <\ldots>_v = \int d^3v(\ldots)$$

- Kinetic effects associated to finite ion Larmor orbits and electron collisions as well as resistivity are accounted for by Lecture 5)

$$\rho_K^2 = (k_\vartheta^2\rho_i^2/2)[3/4 + (T_e/T_i)(1-i\delta)] - ik_\vartheta^2 c^2\eta/16\pi\omega$$

$$\delta = \left(\frac{r}{R_0}\right)^{3/2}\sqrt{\frac{\nu_e}{\omega}}\left[\frac{1+\ln\left(8\frac{r}{R_0}\frac{\omega}{\nu_e}\right)}{(r/R_0)^{3/2}+(\nu_e/\omega)^{3/2}}\right] .$$

$$\nu_e = \frac{4\pi e^4 n_e \ln\Lambda}{m_e^{1/2}(2T_e)^{3/2}}$$

- The distribution function $\delta \hat{K}_j$ is derived from the gyrokinetic equation (p. 22 Lecture 5; with $\omega_t = v_\parallel / qR$ the transit frequency)

$$\left[\omega_t \partial_\vartheta \; - \; i \; (\omega - \omega_d)\right]_j \; \delta \hat{K}_j \; = \; i \left(\frac{e}{m}\right)_j \; Q F_{0j} \, \frac{1}{\hat{\kappa}_\perp} \; \left(\frac{\omega_d}{\omega}\right)_j \hat{\Phi} \;\; ,$$

E: Note that here fast ion finite Larmor radius is neglected. Is that consistent with the assumption of thermal FLR? Why? What is fast ion FLR producing?

- All characteristics of Alfvén waves are described:
  - $\epsilon_0 = 0$, $\rho_K^2 = 0$, $\Rightarrow$ ideal MHD continuum
  - $\epsilon_0 \neq 0$, $\rho_K^2 = 0$, $\Rightarrow$ TAE's
  - $\epsilon_0 = 0$, $\rho_K^2 \neq 0$, $\Rightarrow$ KAW's
  - $\epsilon_0 \neq 0$, $\rho_K^2 \neq 0$, $\Rightarrow$ KTAE's
  - Energetic particle effects $\propto < \omega_d \delta \hat{K} >_h$, $\Rightarrow$ EPM's (ϵ_0 not crucial; E: can you tel why?)

- Short scale mode structure from the large $\hat{\kappa}_\perp >> 1$ behavior

Eigenmode analysis

- Eigenmode analysis follows p. 30 Lecture 3 with $\hat{\Phi} \simeq A(\vartheta)\cos(\vartheta/2) + B(\vartheta)\sin(\vartheta/2)$.
- Results remain the same: coupled differential equations for $A(\vartheta)$ and $B(\vartheta)$, with the addition of FLR effects and energetic particles.
- FLR effects are $\propto \hat{\kappa}_\perp^2 \sim s^2\vartheta^2$ as $\hat{\kappa}_\perp \gg 1$: no coupling between $A(\vartheta)$ and $B(\vartheta)$.
- Energetic particle response is modulated by geodesic curvature coupling: $\omega_d \simeq \Omega_d s\vartheta \sin\vartheta$.

$$
\begin{aligned}
\left[\omega_t\partial_\vartheta - i\,(\omega-\omega_d)\right]_j \delta\hat{K}_j \;&=\; i\left(\frac{e}{m}\right)_j QF_{0j}\frac{s\vartheta}{|s\vartheta|}\left(\frac{\Omega_d}{\omega}\right)_j\left[\frac{A}{2}\left(\sin\frac{\vartheta}{2}+\sin\frac{3\vartheta}{2}\right)\right.\\
&\left. +\frac{B}{2}\left(\cos\frac{\vartheta}{2}-\cos\frac{3\vartheta}{2}\right)\right]
\end{aligned}
$$

- Solution is straightforward, using $|\omega_t| \gg |\omega_d|$ and $|A'/A|, |B'/B| \ll 1$.

$$
\begin{aligned}
\delta \hat{K}_j \quad = \quad & -\left(\frac{e}{m}\right)_j QF_{0j}\frac{s\vartheta}{|s\vartheta|}\left(\frac{\Omega_d}{\omega}\right)_j \left[\left(\frac{\omega B/2 - i\omega_t A/4}{\omega^2-\omega_t^2/4}\right)\cos\frac{\vartheta}{2}\right. \\
& +\left(\frac{\omega A/2 + i\omega_t B/4}{\omega^2-\omega_t^2/4}\right)\sin\frac{\vartheta}{2} - \left(\frac{\omega B/2 + 3i\omega_t A/4}{\omega^2-9\omega_t^2/4}\right)\cos\frac{3\vartheta}{2} \\
& \left. +\left(\frac{\omega A/2 - 3i\omega_t B/4}{\omega^2-9\omega_t^2/4}\right)\sin\frac{3\vartheta}{2}\right]
\end{aligned}
$$

- Note that, due to geodesic curvature coupling, the particle response has both $\sim \cos(\vartheta/2)$ and $\sim \sin(\vartheta/2)$ dependences, as well as $\sim \cos(3\vartheta/2)$ and $\sim \sin(3\vartheta/2)$. This is called sideband generation and is typical of toroidal geometry where magnetic drifts induce $k_\| \pm 1/(qR_0)$ behaviors in the parallel mode structure.

E: Derive the above equation step by step and discuss the physics implication of sideband generation.

- When substituted back into the vorticity equation, the particle response $\propto \omega_t$ vanishes if one assumes even particle distribution function in $v_\parallel$ space.

E: Convince yourself that this is indeed the case. Why should the particle distribution be even in $v_\parallel$ space? What happens if this is not the case?

- Compute now the coupling of kinetic particle compression via geodesic curvature in the vorticity equation, neglecting odd terms $\propto \omega_t$ and modulations that are not modifying the leading order AE eigenfunction

$$-\sum_j \frac{4\pi\omega e_j}{c^2}\frac{q^2R^2}{\hat{\kappa}_\perp k_\vartheta^2} < \omega_d \delta\hat{K} >_{vj} \;=\; \frac{\pi q^2}{B^2}\sum_j m_j \left\langle \left(\frac{v_\perp^2}{2}+v_\parallel^2\right)^2 QF_{0j} \right.$$
$$\left. \times \left(\frac{\omega}{\omega^2-\omega_t^2/4}+\frac{\omega}{\omega^2-9\omega_t^2/4}\right)\right\rangle_{vj} \left(A\cos\frac{\vartheta}{2}+B\sin\frac{\vartheta}{2}\right)+\ldots$$

E: Derive the above equation step by step. Is it valid for thermal particles as well, in addition to energetic particles? What does this response describe?

- Collecting terms and recalling Lecture 3, the relevant equations are:

$$
\begin{aligned}
-A' + \left(\Gamma_- - s^2\rho_K^2\vartheta^2\right) B &= 0 \\
B' + \left(\Gamma_+ - s^2\rho_K^2\vartheta^2\right) A &= 0 \ .
\end{aligned}
$$

$$
\Gamma_\pm = (\Omega^2 - 1/4 - \beta_1) \pm \epsilon_0\Omega^2
$$

$$
\begin{aligned}
\beta_1 = \omega \frac{\pi q^2}{B^2} \Bigg\langle \sum_j m_j \left(\frac{v_\perp^2}{2} + v_\parallel^2\right)^2 Q F_{0j} \Bigg[& \left(\frac{\omega_t^2}{4} - \omega^2\right)^{-1} \\
& + \left(\frac{9\omega_t^2}{4} - \omega^2\right)^{-1}\Bigg]\Bigg\rangle \ .
\end{aligned}
$$

- Two relevant limits:
  - Small (perturbative) kinetic effects $\Rightarrow$ TAE's
  - Kinetic effects $\approx$ Toroidal effects $\Rightarrow$ KTAE's

TAE Linear Stability

- Treat kinetic effects perturbatively
- Lowest order solution is (Lecture 3):

$$A(\vartheta) = \exp(-\Gamma|\vartheta|)$$
$$\Gamma = (-\Gamma_+\Gamma_-)^{1/2}$$

E: Show that this is the solution. Can you write the solution for $B(\vartheta)$?

- Condition for localized modes in ϑ space (nonsingular radial structures) is:

$$\Gamma^2 = \epsilon_0^2\Omega^4 - (\Omega^2 - 1/4)^2 > 0$$
$$|\Omega^2 - 1/4| < \epsilon_0\Omega^2 \quad \Rightarrow \text{FREQUENCY GAP}$$

- Expression for the *local* growth rate, including thermal electron and ion Landau damping:

$$\frac{\gamma}{\omega} = \frac{4\pi^2 q^3 R}{B^2} \left\langle \sum_j m_j \left(\frac{v_\perp^2}{2} + v_\parallel^2 \right)^2 Q F_{0j} \left[\delta(v_\parallel - v_A) + \frac{1}{3}\delta(v_\parallel - v_A/3) \right] \right\rangle + \frac{\mathrm{Im}(s^2 \rho_K^2)}{\Gamma^2} \; .$$

RADIATIVE DAMPING

$\approx k_\theta^2 \rho_i^2$

E: Use F_0 as Maxwellian distribution function to compute explicitly thermal electron and ion Landau damping. Comment on the importance of ion Landau damping at high-β.

- Energy flow to $|\vartheta| >> 1$ (small scale) $\Rightarrow$ radiative damping [Mett and Mahajan '92]
- Typical scale: $|\vartheta| \approx 1/\Gamma$. $\Gamma_{\pm} = (\Omega^2 - 1/4 - \beta_1) \pm \epsilon_0 \Omega^2 \Rightarrow$ perturbative treatment of energetic particles may overestimate R.D.
  E: Can you show why this is the case?
- Consistency condition:
  - $(\epsilon_0 s^2 \rho_K^2)/\Gamma^4 << 1$
  - In general: $|\gamma/\omega| << \epsilon_0 \omega_A$

  E: Comment both these conditions and explain what they mean
- It may break down because of *local* effects (large R.D.) or *global* effects $\Rightarrow$
  CONTINUUM
  DAMPING

Asymptotic matching with internal region

- Follow Lecture 3, pp. 31-32; and corresponding E: and RP: on p. 33.
- Internal region equation is

$$\left[\frac{\partial^2}{\partial\vartheta^2}+\frac{1}{4}+\frac{\alpha\cos\vartheta}{\hat{\kappa}_\perp}-\frac{(s-\alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^2}\right]\hat{\phi}_{sI}=0$$

- For $|s|,|\alpha|\ll 1$, the external region solution $\hat{\phi}_{sE}=A(\vartheta)\cos(\vartheta/2)+B(\vartheta)\sin(\vartheta/2)$ applies in the internal region as well
- For $|s|,|\alpha|\ll 1$, the mode is located very near the lower continuum accumulation point (verify a posteriori), so $|\Gamma_+|\ll -\Gamma_-\simeq\epsilon_0/2$

$$\begin{aligned}&\hat{\phi}_{sI}\simeq(-\Gamma_-)^{1/2}\cos\vartheta/2 && \text{internal region}\\ &\hat{\phi}_{sE}\simeq\left[(-\Gamma_-)^{1/2}\cos\vartheta/2\pm(\Gamma_+)^{1/2}\sin\vartheta/2\right]e^{\mp\Gamma\vartheta} && \text{external region}\end{aligned}$$

- Apply perturbation theory and method of variation of arbitrary constant to derive the asymptotic behavior of internal solution (E: fill in the algebra!)

$$\hat{\phi}_{sI} = (-\Gamma_-)^{1/2}\left\{1 - \int_0^{\vartheta}\left[\frac{(s-\alpha\cos\vartheta')^2}{\hat{\kappa}_\perp^2} - \frac{\alpha\cos\vartheta'}{\hat{\kappa}_\perp}\right]\sin\vartheta' d\vartheta'\right\}\cos(\vartheta/2)$$

$$+(-\Gamma_-)^{1/2}\left\{\int_0^{\vartheta}\left[\frac{(s-\alpha\cos\vartheta')^2}{\hat{\kappa}_\perp^2} - \frac{\alpha\cos\vartheta'}{\hat{\kappa}_\perp}\right](1+\cos\vartheta')d\vartheta'\right\}\sin(\vartheta/2)$$

- Matching the $\sin\vartheta/2$ terms one find the TAE dispersion relation as [Cheng, Chen, Chance 1985]

$$\frac{(\Gamma_+)^{1/2}}{(-\Gamma_-)^{1/2}} = \delta T_f = \frac{s\pi}{4}\left(1 - \alpha\frac{1+s}{s^2}\right)$$

- Real TAE frequency determined by same equation as in fluid theory (Lecture 3). E: Compute the real frequency shift due to kinetic particle compressions and radiative damping using perturbation theory.

KTAE Linear Stability

- Approach the lower continuum accumulation point:
  - $\Gamma_+ \simeq 0$ and $\Gamma_- \simeq -\epsilon_0/2$
- Relevant equations:

$$A'' + \Gamma_- \left(\Gamma_+ - s^2 \rho_K^2 \vartheta^2\right) A = 0$$

- Dispersion relation:

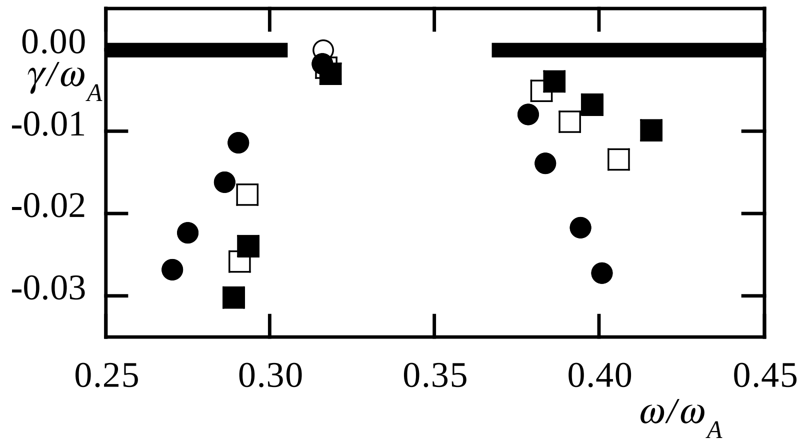
$$\Gamma_+ = -(2N+1) \left(\frac{s^2 \rho_K^2}{\Gamma_-}\right)^{1/2}$$

- At the upper continuum $\Gamma_+ \simeq \epsilon_0/2$ and $\Gamma_- \simeq 0$:

$$\Gamma_- = (2N+1)\left(\frac{s^2\rho_K^2}{\Gamma_+}\right)^{1/2}$$

- Radial width $\Delta r_{KTAE} \approx (k_\theta s)^{-1}(\epsilon_0 s^2 \rho_K^2)^{1/4}$. $\Delta r_{KTAE} \ll \Delta r_{TAE} \approx (k_\theta s)^{-1}\Gamma \Rightarrow$ $\Rightarrow \epsilon_o s^2\rho_K^2/\Gamma^4 << 1$.

E: Show that the derived KTAE dispersion relations explain this figure (shown before)
E: Show that for $\rho_i^2 = 0, \eta \neq 0$ lower and upper KTAE are on stable lines at 45^o departing from accumulation points [Cheng, Chen, Chance 1985]. How about $\rho_i^2 \neq 0, \eta = 0$? [Mett and Mahajan 1992]



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