Lecture 5

Kinetic energy principle and variational approach.

Kinetic Alfvén waves

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Kinetic Alfvén Waves

- As a simple example to illustrate the application of the linear gyrokinetic theory, let us analyze the linear wave properties of the kinetic Alfvén wave (KAW).
- KAW is a SAW modified by the finite ion Larmor radius effects.
- When approaching the continuous spectrum, Lecture 1 suggests that the radial wave-vector is

$$|k_x| \cong |\omega'_{\mathcal{A}\ell}(x)t|$$

  and, thus, $|k_x| \to \infty$ as $t \to \infty$; *i.e.*, the wave function becomes singular in the asymptotic time limit.
- This indicates the break down of the ideal MHD assumption; i.e., $|k_\perp \rho_i|^2 \ll 1$. Proper treatments of microscopic ρ_i-scale SAW are what require kinetic-theory analysis.

- The most relevant new (with respect to the ideal MHD model) dynamics that appear on short scales are associated with charge separation
  - finite parallel electric field fluctuations ($\delta E_\parallel$) due to, *e.g.*, finite ion Larmor radius, small but finite electron inertia and finite plasma resistivity
  - finite $\delta E_\parallel \Rightarrow$ additional effects are to be expected also from wave-particle interactions, which yield collisionless wave dissipation (Landau damping).
  - $\Rightarrow$ finite energy propagation across the resonant surfaces $x = x_{R\ell}$ and wave-function singularities are removed on short scales.

Kinetic Alfvén Waves: uniform plasma

- Consider an infinite uniform plasma immersed in a magnetic field $\boldsymbol{B}_0 = B_0 \mathbf{e}_z$.

- Furthermore, let the equilibrium distribution functions be isotropic Maxwellians; *i.e.*, for $j = e, i$,

$$\bar{F}_{0j} = n_{0j} F_{Mj} = \frac{n_{0j}}{(\sqrt{\pi} v_{tj})^3} \exp\left(-v^2/v_{tj}^2\right) \;,$$

  where $v_{tj}^2 = 2T_j/m_j$ and n_{0j} is the equilibrium particle density.

- Taking perturbations of the form $\sim \exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x})$, where $\boldsymbol{k} = k_\perp \mathbf{e}_x + k_\parallel \mathbf{e}_z$, we then have, from the linear GKE (Lecture 4),

$$\delta g_{kj} = -\left[\frac{e}{T} F_M J_0 \frac{\omega}{k_\parallel v_\parallel - \omega} \left(\delta\phi - \frac{v_\parallel}{c} \delta A_\parallel\right)\right]_j \;.$$

- By direct substitution into the quasineutrality condition (Lecture 4)

$$\sum_{j=e,i} \left(\frac{n_0 e^2}{T} \right)_j \delta\phi_k + \sum_{j=e,i} \left(\frac{n_0 e^2}{T} \right)_j \Gamma_{0kj} \left[\xi_{kj} Z_{kj} \delta\phi_k - (1 + \xi_{kj} Z_{kj}) \delta\psi_k \right] = 0 \ .$$

- Here, we have defined, for modes with finite $k_\parallel$, $\delta\psi_k = (\omega \delta A_\parallel / c k_\parallel)_k$, $\Gamma_{0kj} = \int J_0^2 F_{Mj} d\boldsymbol{v} = I_0(b_{kj}) \exp(-b_{kj})$, I_0 is the modified Bessel function, $b_{kj} = k_\perp^2 \rho_j^2 / 2 = k_\perp^2 (T_j / m_j) / \Omega_j^2$, $\xi_{kj} = \omega / |k_\parallel| v_{tj}$ and

$$Z_{kj} = Z(\xi_{kj}) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(y - \xi_{kj})} dy$$

is the plasma dispersion function.

- Similarly, the vorticity equation (Lecture 4) becomes

$$i\frac{c^2}{4\pi\omega_k}k_\parallel^2 k_\perp^2 \delta\psi_k - i\sum_{j=e,i}\left(\frac{n_0 e^2}{T}\right)_j (1-\Gamma_{0kj})\omega_k\delta\phi_k = 0 \; .$$

E: Derive the nonadiabatic particle response from the GKE. Then demonstrate the quasineutrality condition and the gyrokinetic vorticity equation.

- Taking further approximations, $b_{ke} = k_\perp^2\rho_e^2/2 \ll 1$, the gyrokinetic vorticity equation reduces to

$$\delta\phi_k = \frac{k_\parallel^2 v_A^2}{\omega_k^2}\frac{b_k}{(1-\Gamma_k)}\delta\psi_k \; ,$$

with $v_A^2 = B_0^2/(4\pi n_{0i} m_i)$ the Alfvén speed.

- Here, for simplicity of notation, $b_k = b_{ki}$ and $\Gamma_k = \Gamma_{0ki}$.

- Noting that $|\omega/k_\parallel| \sim v_A$ and, for a wide parameter range of interest, $1 \gg \beta \gg m_e/m_i$, we have $|\xi_{ki}| \gg 1 \gg |\xi_{ke}|$.

- Expanding the plasma dispersion function for small/large arguments where appropriate, the quasineutrality condition then becomes (with $\tau = T_e/T_i$; and assuming $e_i = |e|$)

$$\delta\psi_k = [1 + \tau(1 - \Gamma_k)]\, \delta\phi_k \equiv \sigma_k \delta\phi_k \; .$$

- Combining GK vorticity and QN gives the following KAW dispersion relation with wave particle resonances neglected

$$\omega_k^2 = k_\parallel^2 v_A^2 \frac{b_k \sigma_k}{(1 - \Gamma_k)} \; .$$

E: Discuss the properties of the KAW dispersion relation using this result.

- One important feature of KAW is that it possesses a significant component of parallel electric field.

- More specifically, noting that $\delta E_{\parallel k} = -ik_\parallel(\delta\phi_k - \delta\psi_k)$ and the QN condition, we have

$$\delta E_{\parallel k} = ik_\parallel \tau(1-\Gamma_k)\delta\phi_k \ .$$

- The existence of a finite $\delta E_{\parallel k}$ then leads to finite wave-particle resonance and, thereby, the important implications of charged particle heating as well as transport across $\boldsymbol{B}_0$.

E: Explain why finite KAW heating is connected with the existence of a finite $\delta E_{\parallel k}$. How about cross field transport?

Kinetic Alfvén Waves: 1D non-uniform plasmas

- Consider the one dimensional non-uniform plasma equilibrium introduced in Lecture 1.
- Make same approximations introduced on pp. 6-7 for deriving coupled GK vorticity and QN. Furthermore, for the sake of simplicity, we also assume $(k_x^2 + k_y^2)\rho_i^2 \equiv k_\perp^2 \rho_i^2 \ll 1$.
- It is then possible to show that the vorticity equation (p. 17 Lecture 1) becomes [Hasegawa and Chen 75, 76]

$$\left[\omega^2 \nabla_\perp^2 \rho_K^2 \nabla_\perp^2 + \nabla_\perp \cdot \epsilon_{A\ell} \nabla_\perp\right] \delta\hat{\xi}_{x\ell} = 0 \, ,$$

where

$$\rho_K^2 = \left[\frac{3}{4}(1 - i\delta_i) + \frac{T_e}{T_i}(1 - i\delta_e)\right] \rho_i^2 - i\frac{c^2\eta}{4\pi\omega} \, .$$

- Here, δ_i and δ_e indicate, respectively, ion and electron Landau damping contributions, whereas the term proportional to η is due to finite plasma resistivity.
- δ_i and δ_e are "simplified model" expressions that account the finite wave-particle resonant interactions due to the general plasma dispersion function response in the QN condition on p. 5.
- The plasma resistivity response is derived noting that a term $\eta\delta j_{\|k}$ should be added on the right hand side of Ohm's law (p. 8).

E: Derive KAW equation from QN condition and GK vorticity equation dropping magnetic curvature and diamagnetic terms in the latter, and adding resistive dissipation in the parallel Ohm's law.

E: Derive the small FLR limit of KAW dispersion relation noting parallel Ampère's law and QN condition.

- In the KAW equation, the singularity at $\epsilon_{A\ell} = 0$ is clearly removed by the term including the 4th order derivative.
- The 4th order derivative is also proportional to $\rho_K^2 \ll 1$, indicating the formation of a boundary layer around the SAW resonant surface.

E: Demonstrate that a boundary layer indeed appears at the SAW resonant surface $\epsilon_{A\ell} = 0$.

- The KAW equation,describes the mode-conversion of a long wavelength MHD mode (the SAW) to a short wavelength kinetic mode, the KAW [Hasegawa and Chen 75, 76].
- The WKB local dispersion relation of KAW's is

$$\omega^2 = \left(1 + k_\perp^2 \rho_K^2\right) \omega_{A\ell}^2 ,$$

and indicates that KAW's are propagating for $\epsilon_{A\ell} > 0$ and become cut-off for $\epsilon_{A\ell} < 0$.

- Assuming that $\omega_{\mathcal{A}\ell}(x) \simeq \omega_{\mathcal{A}\ell}(x_{R\ell})(1+\kappa\zeta)$ near the resonance absorption layer, with $\zeta = x - x_{R\ell}$ and $\kappa = (d/dx_{R\ell}) \ln \omega_{\mathcal{A}\ell}(x_{R\ell}) > 0$, one has [Hasegawa and Chen 76]

$$\left(\rho_K^2 \frac{d^2}{d\zeta^2} + \kappa\zeta \right) \delta\hat{\xi}_{x\ell} = 0 \quad .$$

- General solutions are written in terms of Airy functions and have the following form away from the SAW resonant absorption layer

$$\delta\hat{\xi}_{x\ell} = \frac{\delta\hat{\xi}_{x\ell 0}}{\kappa\zeta} \qquad \zeta < 0 \ , \qquad \text{SAW}$$

$$\begin{aligned} \delta\hat{\xi}_{x\ell} &= \frac{\delta\hat{\xi}_{x\ell 0}}{\kappa\zeta} - \frac{\pi^{1/2}\delta\hat{\xi}_{x\ell 0}}{(\kappa\rho_K)^{2/3}} \left(\frac{\rho_K^{2/3}}{\kappa^{1/3}\zeta} \right)^{1/4} \qquad \text{SAW} \oplus \text{KAW} \\ &\times \exp \left\{ i \left[\frac{2}{3} \left(\frac{\kappa^{1/3}\zeta}{\rho_K^{2/3}} \right)^{3/2} + \frac{\pi}{4} \right] \right\} \qquad \zeta > 0 \ . \end{aligned}$$

Bound states with Kinetic Alfvén Waves

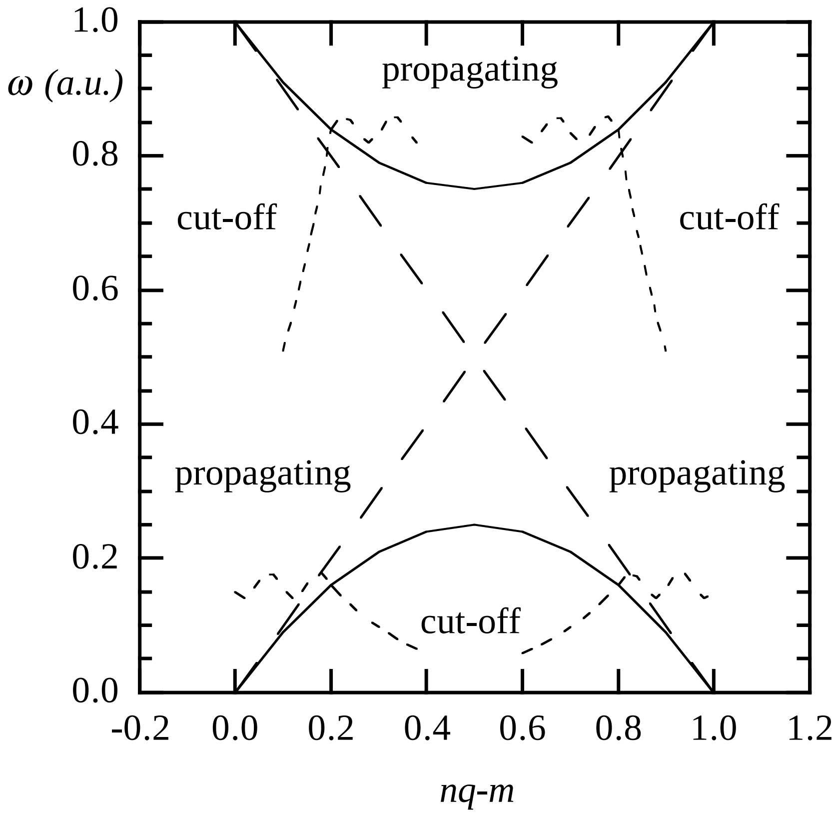
- Kinetic Alfvén Waves (KAW) are characterize by the following dispersion relation

$$\omega^2 = k_\parallel^2 v_A^2 \left(1 + k_\perp^2 \rho_i^2 (3/4 + T_e/T_i)\right)$$

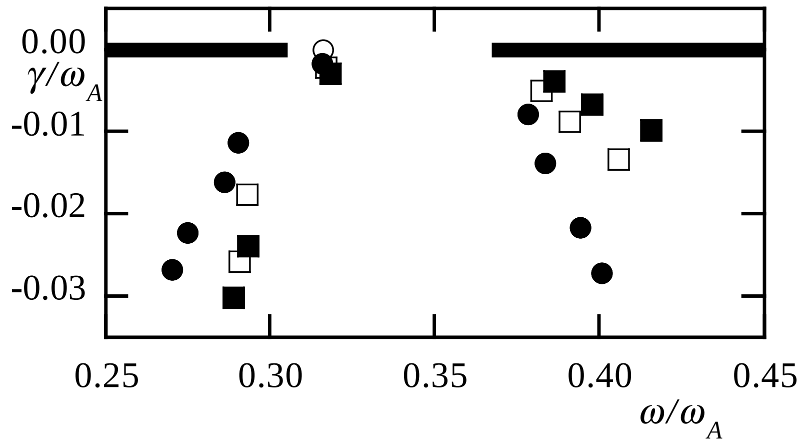
- KAW are propagating on the high-frequency side $(\omega_*/\omega, \omega_{ti}/\omega \rightarrow 0)$ and are generic to magnetized plasmas: no toroidal geometry needed
- At SAW continuum resonances, SAW can mode convert to KAW: Alfvén wave heating [Hasegawa and Chen 76]
- In toroidal system, KAW may generate standing waves near the TAE frequency gap. These waves are dubbed Kinetic TAE (KTAE); [Mett and Mahajan 1992].

Lower and Upper KTAE structures

[Vlad et al. 1999]



- Wave structures and strong (lower branch) damping vs. weak (upper branch) damping [Mett and Mahajan 1992]. (Lecture 6)



E: Demonstrate the formation of Kinetic BAE (KBAE) [X. Wang 10]. How about other modes? E.g. KBAAE?

Reduced drift Alfvén wave equations in tokamaks

- In burning plasmas of fusion interest, supra-thermal energetic particles are characterized by an energy density, which is comparable to that of the thermal plasma, so that $\beta_E \sim \beta$.
- However, due to the significantly higher energy $T_{0i}/T_{0E} = \mathcal{O}(10^{-2})$, the energetic particle density is typically low, $n_{0E}/n_{0i} \sim T_{0i}/T_{0E}$.
- Thus, it is generally possible to consider reactor relevant plasmas consisting of two components [Chen 84]:
  - a core or thermal plasma component, essentially providing an isotropic Maxwellian background made of electrons (e) and ions (i)
  - an energetic component (E), which is often anisotropic and non-Maxwellian
- General wavelength and frequency orderings for the case of DAW resonantly excited by energetic particles in space-plasmas was given by [Chen 91].

- Later specialized by to low-β laboratory plasmas [Z & C 2006], where

$$n_{0E}/n_{0i} \sim T_{0i}/T_{0E} = \mathcal{O}(10^{-2}) \lesssim \beta_i \sim \beta_E \lesssim \mathcal{O}(10^{-1}) \ .$$

- Meanwhile, most unstable energetic particle driven modes are characterized by $|k_\theta \rho_E| \lesssim 1$ [Fu 92, Berk 92, Tsai 93, Chen 94], where ρ_E is the characteristic energetic particle magnetic drift orbit width, corresponding to the relevant wave-particle resonance (Lecture 4).

- Finally, thermal electrons typically have $v_{te} \gg v_A$, corresponding to $\beta \gg m_e/m_i$, and, hence, can be approximated as a massless fluid.

- These orderings, in addition to the general gyrokinetic orderings (Lecture 4) and the low-β assumption, allow us to further simplify QN condition and GK vorticity equation for the description of SAW excited by energetic particles in burning plasmas.

- From the GKE (Lecture 4) the thermal electron response as a massless fluid is readily solved for and yields (e denotes the electron charge as positive number; neglect barely passing/trapped electrons)

$$\boldsymbol{b}\cdot\boldsymbol{\nabla}\delta g_e = -\left(\frac{e}{m_e c}\frac{\partial\delta A_\parallel}{\partial t}\frac{\partial\bar{F}_{0e}}{\partial\mathcal{E}} + \frac{\delta\boldsymbol{B}_\perp}{B_0}\cdot\boldsymbol{\nabla}\bar{F}_{0e}\right) \quad .$$

- For a Maxwellian electron core, the quasineutrality condition (Lecture 4), can be cast as parallel Ohm's law

$$\frac{n_{0e}e^2}{T_{0e}}\left[\boldsymbol{b}\cdot\boldsymbol{\nabla}\left(\delta\phi-\delta\psi\right)\right] = \boldsymbol{b}\cdot\boldsymbol{\nabla}\sum_{\neq e} e\left\langle\delta f\right\rangle_v + \frac{\delta\boldsymbol{B}_\perp}{B_0}\cdot\boldsymbol{\nabla}\sum_{\neq e} e\left\langle\bar{F}_0\right\rangle_v \quad ,$$

where $\sum_{\neq e}$ denotes summation on particle species except for core electrons and equilibrium charge neutrality has been used explicitly.

E: Demonstrate that the previous equation is just the extended Ohm's law

$$\delta \boldsymbol{E}_{\parallel} = -\boldsymbol{b} \cdot \frac{\boldsymbol{\nabla} \delta P_e}{n_{0e} e} - \frac{\delta \boldsymbol{B}_{\perp}}{B_0} \cdot \frac{\boldsymbol{\nabla} P_{e0}}{n_{0e} e} \quad ,$$

having assumed isothermal electron response.

- The new ordering allows ignoring the contribution of energetic particles to the plasma density (some exception may apply to present day experiments), while the wavelength ordering $|k_\theta \rho_E| \lesssim 1$ indicates that $\epsilon_\perp \ll 1$ for the core plasma component.
- Thus, the QN condition or its revised version above, at the lowest order reduces to the ideal MHD approximation $\delta E_{\parallel} = 0$ or $\delta\phi = \delta\psi$ [Chen 91].

- The gyrokinetic vorticity equation is also greatly simplified with this additional ordering

$$
\begin{aligned}
&B_0 \nabla_\parallel \left(\frac{\delta j_\parallel}{B_0} \right) - \frac{c^2}{4\pi} \boldsymbol{\nabla} \cdot \left\{ \left[\frac{1}{v_A^2} + \frac{3\pi}{B_0^2} \left(\frac{P_{0\perp i}}{\Omega_i^2} \right) \nabla_\perp^2 \right] \boldsymbol{\nabla}_\perp \frac{\partial}{\partial t} \delta\phi \right\} \\
&+ \frac{c^2}{4\pi} \boldsymbol{b} \times \boldsymbol{\nabla} \left[\frac{4\pi}{B_0^2} \left(\frac{P_{0\perp i}}{\Omega_i} \right) \right] \cdot \boldsymbol{\nabla} \nabla_\perp^2 \delta\phi + \delta \boldsymbol{B}_\perp \cdot \boldsymbol{\nabla} \left(\frac{j_{\parallel 0}}{B_0} \right) \\
&+ \frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \sum \left\langle m \left(\mu B_0 + v_\parallel^2 \right) J_0 \delta g \right\rangle_v = 0 \quad .
\end{aligned}
$$

- Here, we have used the definition $P_{0\perp} = \left\langle m \mu B_0 \bar{F}_0 \right\rangle_v$ and have adopted the long wavelength limit for thermal ions, which contribute to both finite Larmor radius correction to the plasma inertia (KAW) as well as to the diamagnetic response [Chen RMP 2016].

- Note that energetic ions typically do not contribute to the inertial/kinetic layer physics, due to their large orbits. However, in burning plasmas with high magnetic fields (small dimensionless orbits) energetic particles may actually enter, and the above equation must be suitably modified [Chen RMP 2016] (not in these Lectures).

- The long wavelength limit form of the GK vorticity equation above coincides with the gyrokinetic vorticity equation discussed by [Qin 1998] and, dropping KAW and diamagnetic terms, with the reduced form of the linear kinetic-MHD model by [Brizard 1994]. $\Rightarrow$ Connection with hybrid MHD-GK models [Chen RMP 2016].

- Next: use δg_e from p. 17 and exploit the long wavelength limit for thermal ions to explicitate the coupled QN and GK vorticity equation in the MSD formulation (Lecture 2). Then derive integral form for SAW dispersion relation in torus: the general fishbone like dispersion relation (GFLDR).

MSD form of reduced drift Alfvén wave equations

- It is useful to introduce the following decomposition:

$$\delta g \equiv \delta K + i\frac{e}{m} Q\bar{F}_0 \partial_t^{-1} \langle \delta\psi_g \rangle$$

with $\langle \delta A_{\parallel g} \rangle = -c\partial_t^{-1} \nabla_\parallel \langle \delta\psi_g \rangle$ and the operator $Q\bar{F}_0$ defined as [Chen 84]

$$iQ\bar{F}_0 = -\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} + \frac{\boldsymbol{b} \times \boldsymbol{\nabla} \bar{F}_0}{\Omega} \cdot \boldsymbol{\nabla} \quad .$$

- This subdivision of the particle response is particularly convenient in linear analyses [Chen 84, Chen 91], since $\delta K \to 0$ in the fluid ion ($\omega \gg \bar{\omega}_d, \omega_b$) and massless electron ($\omega_b \gg \omega, \bar{\omega}_d$) limits (Lecture 4).

- The GKE can be cast as

$$\left(\frac{\partial}{\partial t}+v_\parallel\nabla_\parallel+\boldsymbol{v}_d\cdot\boldsymbol{\nabla}_\perp\right)\delta K=i\frac{e}{m}\left[Q\bar{F}_0\left\langle\delta\phi_g-\delta\psi_g\right\rangle\right.$$
$$\left.-\boldsymbol{v}_d\cdot\boldsymbol{\nabla}_\perp Q\bar{F}_0\partial_t^{-1}\left\langle\delta\psi_g\right\rangle\right] \quad .$$

- Using the MSD representation (Lecture 2), the field equations are the QN condition for $\delta\phi\to\delta\hat{\Phi}=\hat{\kappa}_\perp\delta\hat{\phi}$, and the GK vorticity equation for $\delta\psi\to\delta\hat{\Psi}=\hat{\kappa}_\perp\delta\hat{\psi}$.

- QN condition (from Lecture 4 p.9 or this Lecture p.17), with $k_\perp^2=k_\vartheta^2\hat{\kappa}_\perp^2$,

$$\left(1+\frac{1}{\tau}\right)\left(\delta\hat{\Phi}_n-\delta\hat{\Psi}_n\right)+\left(1-\frac{\omega_{*pi}}{\omega}\right)k_\vartheta^2\rho_i^2\hat{\kappa}_\perp^2\delta\hat{\Psi}_n$$
$$=\frac{T_i}{n_0e}\hat{\kappa}_\perp\left\langle\left(1-k_\vartheta^2\frac{\mu B_0}{2\Omega^2}\hat{\kappa}_\perp^2\right)\delta\hat{K}_{in}\right\rangle_v \quad .$$

where $\omega_{*pi}=\omega_{*ni}+\omega_{*Ti}$ is the thermal ion diamagnetic frequency and

$$\omega_{*ni} = \left(\frac{T_0 c}{e n_0 B_0}\right)_i (\boldsymbol{b} \times \boldsymbol{\nabla} n_{0i}) \cdot \boldsymbol{k}_\perp \quad ,$$

$$\omega_{*Ti} = \left(\frac{c}{e B_0}\right)_i (\boldsymbol{b} \times \boldsymbol{\nabla} T_{0i}) \cdot \boldsymbol{k}_\perp \quad .$$

- Note that energetic particle density has been neglected, but this approximation may not be good in some of present day expt.
- The GK vorticity equation [Z & C POP14],
  with $g(\vartheta, \theta_k) = [s(\vartheta - \theta_k) - \alpha \sin\vartheta] \sin\vartheta + \cos\vartheta$

$$\left(\frac{\partial^2}{\partial\vartheta^2} + \Delta' \cos\vartheta\right) \delta\hat{\Psi}_n + \frac{\omega}{\omega_A^2} [1 + 4(r/R_0)\cos\vartheta] \left[\omega - \omega_{*pi} - \frac{3}{4} k_\vartheta^2 \rho_i^2 \hat{\kappa}_\perp^2 \left(\omega - \omega_{*pi} - \omega_{*Ti}\right)\right] \delta\hat{\Phi}_n$$

$$+ \frac{4\pi R_0 q^2}{B_0} \frac{g(\vartheta, \theta_k)\omega}{c k_\vartheta \hat{\kappa}_\perp} \left[\left\langle m_i \left(\mu B_0 + v_\parallel^2\right) \left(1 - k_\vartheta^2 \frac{\mu B_0}{2\Omega_i^2} \hat{\kappa}_\perp^2\right) \delta\hat{K}_{in} \right\rangle_v + \left\langle m_E \left(\mu B_0 + v_\parallel^2\right) J_0 \delta\hat{g}_{En} \right\rangle_v \right]$$

$$+ \left[\frac{\alpha\cos\vartheta}{\hat{\kappa}_\perp^2} - \frac{(s - \alpha\cos\vartheta)^2}{\hat{\kappa}_\perp^4} + \frac{(\alpha_c - \alpha) g(\vartheta, \theta_k)}{\hat{\kappa}_\perp^2}\right] \delta\hat{\Psi}_n = 0 \quad ,$$

Generalized FL dispersion relation: derivation

(More rigorous and detailed derivation by Chen and Zonca RMP 2016)

- Consider the simple case in which $|\omega_{di}| \ll |\omega| \ll \omega_A$ in the vorticity equation; no FLR effects

$$\mathbf{B}\cdot\nabla\left(\frac{k_\perp^2}{k_\vartheta^2 B^2}\mathbf{B}\cdot\nabla\delta\psi\right) + \frac{\omega(\omega-\omega_{*pi})}{v_A^2}\frac{k_\perp^2}{k_\vartheta^2}\delta\phi - \left\langle \sum_{s=i}\frac{4\pi e_s}{k_\vartheta^2 c^2}\omega\hat{\omega}_{ds}\delta K_s\right\rangle$$

$$+\frac{4\pi}{k_\vartheta^2 B^2}\mathbf{k}\times\mathbf{b}\cdot\tilde{\nabla}\left(\mathbf{P}_\perp + \mathbf{P}_\parallel\right)\boldsymbol{\Omega}_\kappa\delta\psi = \left\langle \frac{4\pi e_E}{k_\vartheta^2 c^2}J_0(k_\perp\rho_E)\omega\hat{\omega}_{dE}\delta K_E\right\rangle \;,$$

- This means that $\delta\psi$ and $\delta\phi$ must have a two-scale structure: $\theta_0 \sim 1$ and $\theta_1 \sim \omega_A/|\omega|$.

- Write asymptotic expansions in smallness parameter $|\omega|/\omega_A$; e.g., $\delta\psi = \delta\psi^{(0)} + \delta\psi^{(1)} + \ldots$, $\delta\phi = \delta\phi^{(0)} + \delta\phi^{(1)} + \ldots$

- From quasi-neutrality ...

$$\left\langle \sum_{s\neq E} \frac{\mathrm{e}_s^2}{m_s} \frac{\partial F_{0s}}{\partial \mathcal{E}} \right\rangle (\delta\phi - \delta\psi) + \sum_{s=i} \mathrm{e}_s \left\langle \delta K_s \right\rangle = 0 \quad ,$$

- ... and from gyrokinetic equation ...

$$\left[\omega_{tr}\partial_\theta - \mathrm{i}\left(\omega - \omega_d\right)\right]_s \delta K_s = \mathrm{i}\left(\frac{\mathrm{e}}{m}\right)_s QF_{0s}\left[J_0(k_\perp\rho_s)\left(\delta\phi - \delta\psi\right) + \left(\frac{\hat{\omega}_d}{\omega}\right)_s J_0(k_\perp\rho_s)\delta\psi\right]$$

- the lowest order solution is function of long scale only: $\delta\phi^{(0)}(\theta_1) = \delta\psi^{(0)}(\theta_1)$ and $\delta K_i^{(0)} \propto \left(\delta\phi^{(0)}(\theta_1) - \delta\psi^{(0)}(\theta_1)\right)$
- Look at kinetic (inertial) singular structures solving vorticity equation at large $|\theta_1|$: no contribution from fast ions (orbit averaging) and ballooning-interchange pressure term

- Large $|\theta_1|$ vorticity equation; definition $\omega_A = v_A/(qR_0)$; $\overline{(\ldots)} = (2\pi)^{-1}\oint(\ldots)d\theta_0$

$$\frac{\partial^2}{\partial\theta_1^2}\overline{(k_\perp/k_\vartheta)}\delta\phi^{(0)} + \frac{\omega(\omega-\omega_{*pi})}{\omega_A^2}\overline{(k_\perp/k_\vartheta)}\delta\phi^{(0)} - \overline{\left\langle \sum_{s=i}\frac{4\pi q^2R_0^2 e_s}{k_\vartheta k_\perp c^2}\omega\hat{\omega}_{ds}\delta K_s^{(1)}\right\rangle} = 0$$

- Introduce $\delta\Phi^{(0)} = \overline{(k_\perp/k_\vartheta)}\delta\phi^{(0)}$. Then, above equation can be cast into the form (note that $\delta K_i^{(1)} \propto \delta\Phi^{(0)}$) (E: show it!)

$$\partial_{\theta_1}^2\delta\Phi^{(0)} + \Lambda^2\delta\Phi^{(0)} = 0$$

$$\Lambda^2\delta\Phi^{(0)} = \frac{\omega(\omega-\omega_{*pi})}{\omega_A^2}\delta\Phi^{(0)} - \overline{\left\langle \sum_{s=i}\frac{4\pi q^2R_0^2 e_s}{k_\vartheta k_\perp c^2}\omega\hat{\omega}_{ds}\delta K_s^{(1)}\right\rangle}$$

E: Demonstrate this equation. How general is it? Could you do for arbitrary geometry or only for simple equilibria, e.g. the $(s-\alpha)$ model equilibrium?

- Causality constraints (outgoing wave boundary conditions): $\delta\Phi^{(0)} \sim \exp(i\Lambda|\theta_1|)$

- Multiply vorticity equation by $\delta\phi^{(0)*}$ and integrate in θ, constructing the quadratic form ($|\Theta_1| \gg 1$, $|\Lambda\Theta_1| \ll 1$):

$$\frac{1}{2}\int_{-\Theta_1}^{\Theta_1} d\theta \delta\Phi^{(0)*}\partial_\theta\delta\Phi^{(0)} = \mathrm{i}\Lambda = \frac{q^2R_0^2}{2}\int_{-\Theta_1}^{\Theta_1} d\theta\delta\phi^{(0)*}\left\langle \sum_{s=i} \frac{4\pi e_s}{k_\vartheta^2 c^2}\omega\hat{\omega}_{ds}\delta K_s \right\rangle$$

$$-\frac{q^2R_0^2}{2}\int_{-\Theta_1}^{\Theta_1} d\theta\delta\phi^{(0)*}\frac{4\pi}{k_\vartheta^2 B^2}\mathbf{k}\times\mathbf{b}\cdot\tilde{\nabla}\left(P_\perp + P_\parallel\right)\Omega_\kappa\delta\phi^{(0)}$$

$$+\frac{q^2R_0^2}{2}\int_{-\Theta_1}^{\Theta_1} d\theta\delta\phi^{(0)*}\left\langle \frac{4\pi e_E}{k_\vartheta^2 c^2} J_0(k_\perp\rho_E)\omega\hat{\omega}_{dE}\delta K_E \right\rangle$$

$$+\frac{1}{2}\int_{-\Theta_1}^{\Theta_1} d\theta\left(|\partial_\theta \frac{k_\perp}{k_\vartheta}\phi^{(0)}|^2 - \frac{k_\perp'' k_\perp}{k_\vartheta^2}|\phi^{(0)}|^2\right) \quad ,$$

- This is the generalized fishbone like dispersion relation: $\mathrm{i}\Lambda = \delta\bar{W}_f + \delta\bar{W}_k$

E: identify term by term the various contributions in the GFLDR.

E: repeat the derivation of the GFLDR. What terms are missing? Can you include kink drive ($\propto \boldsymbol{\nabla} J_{\parallel}$)?

RP: derive the GFLDR for arbitrary geometry (drop the large aspect ratio assumption). Then generalize the derivation to waves with finite but small $|k_{\parallel} q R_0| \ll 1$. Can you solve the problem generally also for finite $|k_{\parallel} q R_0|$ (e.g. TAE; Lecture 3)? [Hint: see Z & C POP14]

RP: Establish a link between $\delta \bar{W}_f$ and the well known expression of MHD $\delta \hat{W}$. Then, assuming that the mode structure is known, give a close expression for $\delta \hat{W}_k$. At this point, use a (local) gyrokinetic code for determining numerically the expression of Λ. Which dispersion relation can you investigate in this way? Can you recover known analytical results?
[Hint: see Z & C POP14; Chen RMP 2016]

Generalized FL dispersion relation: properties

- In general, demonstrate that the mode dispersion relation can be always written in the form of a *fishbone-like* dispersion relation [Chen 1984]

$$-i\Lambda + \delta\bar{W}_f + \delta\bar{W}_k = 0 \quad ,$$

where $\delta\bar{W}_f$ and $\delta\bar{W}_k$ play the role of fluid (core plasma) and kinetic (fast ion) contribution to the potential energy, while Λ represents a generalized inertia term.

- The generalized fishbone-like dispersion relation can be derived by asymptotic matching the regular (ideal MHD) mode structure with the general (known) form of the SA wave field in the singular (inertial) region, as the spatial location of the shear Alfvén resonance, $\omega^2 = k_\parallel^2 v_A^2$, is approached.

- Examples are : $\Lambda^2 = \omega(\omega - \omega_{*pi})/\omega_A^2$ for $|k_\parallel q R_0| \ll 1$ and $\Lambda^2 = (\omega_l^2 - \omega^2)/(\omega_u^2 - \omega^2)$ for $|k_\parallel q R_0| \approx 1/2$, with ω_l and ω_u the lower and upper accumulation points of the shear Alfvén continuous spectrum toroidal gap [Chen 94].

- $\delta\bar{W}_f$ is generally real, whereas $\delta\bar{W}_k$ is characterized by complex values, the real part accounting for non-resonant and the imaginary part for resonant wave particle interactions with energetic ions.

- The fishbone-like dispersion relation demonstrates the existence of two types of modes (note: $\Lambda^2 = k_\parallel^2 q^2 R_0^2$ is SAW continuum; see later):

  - a discrete gap mode, or Alfvén Eigenmode (AE), for $\mathbb{R}\mathrm{e}\Lambda^2 < 0$;
  - an Energetic Particle continuum Mode (EPM) for $\mathbb{R}\mathrm{e}\Lambda^2 > 0$.

- For EPM, the iΛ term represents continuum damping. Near marginal stability [Chen 84, Chen 94]

$$\mathbb{R}\mathrm{e}\delta\bar{W}_k(\omega_r) + \delta\bar{W}_f = 0 \quad \text{determines } \omega_r$$

$$\gamma/\omega_r = (-\omega_r \partial_{\omega_r} \mathbb{R}\mathrm{e}\delta\bar{W}_k)^{-1} (\mathbb{I}\mathrm{m}\delta\bar{W}_k - \Lambda) \quad \text{determines } \gamma/\omega_r$$

- For AE, the non-resonant fast ion response provides a real frequency shift, *i.e.* it removes the degeneracy with the continuum accumulation point, while the resonant wave-particle interaction gives the mode drive. Causality condition imposes

  - $\delta\bar{W}_f + \mathbb{R}\mathrm{e}\delta\bar{W}_k > 0$ when AE frequency is above the continuum accumulation point: inertia in excess w.r.t. field line bending

$$\Lambda^2 = \lambda_0^2(\omega_\ell - \omega) \; ; \qquad \omega > \omega_\ell \; \Rightarrow \; \Lambda \to -\mathrm{i}\sqrt{-\Lambda^2}$$

  - $\delta\bar{W}_f + \mathbb{R}\mathrm{e}\delta\bar{W}_k < 0$ when AE frequency is below the continuum accumulation point: inertia in lower than field line bending

$$\Lambda^2 = \lambda_0^2(\omega - \omega_u) \; ; \qquad \omega < \omega_u \; \Rightarrow \; \Lambda \to \mathrm{i}\sqrt{-\Lambda^2}$$

- For AE, $i\Lambda$ represents the shift of mode frequency from the accumulation point
- For both AE and EPM, the SAW accumulation point is the natural gateway through which modes are born at marginal stability
- For EPM, ω is set by the relevant energetic ion characteristic frequency and mode excitation requires the drive exceeding a threshold due to continuum damping. However, the non-resonant fast ion response is crucially important as well, since it provides the compression effect that is necessary for balancing the positive MHD potential energy of the wave.

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