Lecture 4

The linear gyrokinetic equation and the wave particle resonance condition in toroidal geometry

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Gyrokinetic orderings

- So far, we have investigated SAW spectral properties based on the simplified ideal MHD fluid description.
- This is, however, inadequate for reliable and accurate understanding of SAW dynamics in high-temperature collisionless fusion plasmas:
  - the "singular" behaviors associated with the ubiquitous SAW continuum is a manifestation that ideal MHD description ignores microscopic scales such as the finite ion Larmor radius
  - microscopic-scale dynamics $\Rightarrow$ SAW is modified into the so-called kinetic Alfvén wave (KAW) and the SAW continuum becomes a dense but regular discrete spectrum
  - crucial roles of wave-particle interactions: instability drive by energetic particles and damping/drive by thermal electrons/ions
  - crucial roles of plasma compression in the low-frequency SAW spectrum

- As the SAW frequency is generally much less than the charged particle's cyclotron frequencies, the cyclotron motion can be suitably averaged out, and the kinetic theory can, therefore, be based on a reduced dynamic description.

- Gyrokinetic theory, and the corresponding orderings for temporal and spatial scales:

$$|\rho_i/L_B| \sim \epsilon_B \; , \qquad |\rho_i/L_p| \sim \epsilon_F \; ;$$

$$|\omega/\Omega_i| \sim \epsilon_\omega \; , \qquad |\boldsymbol{k}_\perp \rho_i| \sim \epsilon_\perp \; , \qquad \left|k_\parallel/\boldsymbol{k}_\perp\right| \sim \epsilon_\omega/\epsilon_\perp \; .$$

- Physically, the smallness of ϵ_B and ϵ_F are obviously based on the argument that the plasmas of interest here are well confined.

- That $\epsilon_\perp$ can be $\mathcal{O}(1)$, instead, is based on the argument that SAW instabilities are typically excited via plasma inhomogeneities (*i.e.*, similar to the well-known drift waves) and, thus, the growth rates tend to increase with $\boldsymbol{k}_\perp$ and are limited by the orbital averaging due to the finite ion Larmor radii.
- Finally, the $|k_\parallel/\boldsymbol{k}_\perp|$ ordering enters because:
  - thermal ion Landau damping tends to limit the maximum excitable $|k_\parallel|$ to $|k_\parallel| \sim |\omega|/v_{ti}$
  - finite $k_\parallel$ tends to stabilize SAW instabilities via magnetic field-line bending

The linear gyrokinetic equation

- Introduce the notion of "guiding-center" of the particle trajectory in the equilibrium magnetic field $\boldsymbol{B}_0$.
- Particles guiding-center orbit can be described by the energy per unit mass, $\mathcal{E} = v^2/2$, and the magnetic moment adiabatic invariant, μ, which, at the leading order, can be expressed as $\mu = v_\perp^2/(2B_0) + \ldots$.
- Particle response can be obtained from the guiding-center response by the "pull-back" transformation from guiding-center to particle coordinates, $e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}$, with $\boldsymbol{\rho} \equiv \Omega^{-1}\boldsymbol{b} \times \boldsymbol{v}$ denoting the particle position in the guiding-center frame and $\Omega = eB_0/(mc)$ the cyclotron frequency.
- Separation of adiabatic response of the particle distribution function (all terms that are not acted upon by $e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}$) from the non-adiabatic response of the guiding-center distribution.

- Within this framework, one can derive the following expression for the linear perturbed distribution function, δf

$$\delta f = e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}} \left[\delta g - \frac{e}{m} \frac{1}{B_0} \frac{\partial \bar{F}_0}{\partial \mu} \langle \delta L_g \rangle \right] + \frac{e}{m} \left[\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \delta \phi + \frac{1}{B_0} \frac{\partial \bar{F}_0}{\partial \mu} \delta L \right] \quad .$$

Here, $\bar{F}_0$ is the equilibrium guiding-center particle distribution function,

$$\delta L_g = \delta \phi_g - \frac{v_\parallel}{c} \delta A_{\parallel g} = e^{\boldsymbol{\rho}\cdot\boldsymbol{\nabla}} \delta L = e^{\boldsymbol{\rho}\cdot\boldsymbol{\nabla}} \left(\delta \phi - \frac{v_\parallel}{c} \delta A_\parallel \right) \quad ,$$

and $\langle \cdots \rangle$ denotes gyrophase averaging.

- Note here that, consistent with the $\beta \ll 1$ and suppression of compressional Alfvén wave (CAW) approximations, we have neglected the $\delta B_\parallel$ term in δL.

- The non-adiabatic response δg is obtained from the solution of the linear gyrokinetic equation [Antonsen 80, Catto 81, Frieman & Chen 82]

$$\left(\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp\right) \delta g = - \left(\frac{e}{m} \frac{\partial}{\partial t} \langle \delta L_g \rangle \frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\nabla} \langle \delta L_g \rangle \cdot \boldsymbol{\nabla} \bar{F}_0 \right) \; .$$

- Here, the magnetic drift velocity $\boldsymbol{v}_d$ is

$$\boldsymbol{v}_d = \frac{\boldsymbol{b}}{\Omega} \times \left(\mu \boldsymbol{\nabla} B_0 + \boldsymbol{\kappa} v_\parallel^2 \right) \simeq \frac{\left(\mu B_0 + v_\parallel^2 \right)}{\Omega} \boldsymbol{b} \times \boldsymbol{\kappa} \;\; ,$$

where $\boldsymbol{\nabla} B_0 \simeq \boldsymbol{\kappa} B_0$ in the low-β limit.

- As in the ideal MHD analysis, this is consistent with well-known cancellations in the linear vorticity equation, arising from plasma equilibrium condition and the perpendicular pressure balance

$$\boldsymbol{\nabla}_\perp \left(B_0 \delta B_\parallel + 4\pi \delta P_\perp \right) \simeq 0 \;\; ,$$

when anisotropic pressure response is considered [Chen 91].

- In the long wavelength limit the last term on the right hand side of the GKE equation has to be slightly modified to accurately account for the perturbed particle motion in the perpendicular fluctuating magnetic field [Chen RMP 16]

$$\begin{aligned} \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle\delta L_g\right\rangle &\rightarrow \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle\delta L_g\right\rangle+\frac{v_\parallel}{B_0}\boldsymbol{\kappa}\left\langle\delta A_{\parallel g}\right\rangle \\ &= \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle\delta\phi_g\right\rangle+v_\parallel\frac{\left\langle\delta\boldsymbol{B}_{\perp g}\right\rangle}{B_0} \quad , \end{aligned}$$

  with $\left\langle\delta\boldsymbol{B}_{\perp g}\right\rangle=\boldsymbol{\nabla}\times\boldsymbol{b}\left\langle\delta A_{\parallel g}\right\rangle$.

- The field equations, meanwhile, are the quasineutrality condition and the gyrokinetic vorticity equation, which are derived below.

Quasi-neutrality condition

- Due to the gyrokinetic orderings, $k^2\lambda_D^2 \sim \lambda_D^2/\rho_i^2 \sim \Omega_i^2/\omega_{pi}^2 \ll 1$, with λ_D the Debye length and ω_{pi} the ion plasma frequency. Thus, Poisson's equation becomes approximately the quasineutrality condition

$$\sum e \left\langle \delta f \right\rangle_v = 0 \quad ,$$

where $\sum$ implicitly indicates summation on all particle species and $\langle \ldots \rangle_v$ denotes integration in velocity space.

- This equation can be readily rewritten as

$$\sum \left\langle \frac{e^2}{m} \frac{\partial \bar{F}_0}{\partial \mathcal{E}} \right\rangle_v \delta\phi + \sum \left\langle e J_0(\lambda) \delta g \right\rangle_v$$
$$+ \boldsymbol{\nabla} \cdot \sum \left\langle \frac{e^2}{m} \frac{2\mu}{\Omega^2} \frac{\partial \bar{F}_0}{\partial \mu} \left(\frac{J_0^2(\lambda) - 1}{\lambda^2} \right) \right\rangle_v \boldsymbol{\nabla}_\perp \delta\phi = 0 \quad .$$

- Here, we have assumed that most of the equilibrium current is carried by electrons and that relevant wavelengths are much longer than the electron Larmor radius.

- Furthermore, $\langle e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}} \rangle = J_0(\lambda)$, $J_0(\lambda)$ is the Bessel function and $\lambda^2 = 2\mu B_0 k_\perp^2/\Omega^2$.

E: Demonstrate that the approximations needed for deriving the quasi-neutrality conditions are justified.

- Here, we have followed [Qin 98] to properly account for the correct recovery of the long wavelength limit $\lambda^2 = \epsilon_\perp^2 \to 0$.

- For the finite λ case, spatial scale separation between fluctuations and equilibrium profiles applies and, therefore, $J_0(\lambda)$ can be made commute with equilibrium quantities at the leading order. This greatly simplifies formal analyses, although, in general, equilibrium nonuniformities can be rigorously accounted for at all orders [Brizard 07].

- As usual, the gyrophase average involves the introduction of Bessel functions as integral operators, e.g.

$$\langle \delta L_g \rangle = J_0(\lambda)\delta L = J_0(\lambda)\left(\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel\right) \quad .$$

- The definition of $J_0(\lambda)$ acting on a generic function $g(\boldsymbol{r}) = \int \hat{g}(\boldsymbol{k})\exp(i\boldsymbol{k}\cdot\boldsymbol{r})d\boldsymbol{k}$ is

$$J_0(\lambda)g(\boldsymbol{r}) \equiv \int e^{i\boldsymbol{k}\cdot\boldsymbol{r}} J_0(\lambda)\hat{g}(\boldsymbol{k})d\boldsymbol{k} \quad .$$

- From this definition and from the presence of velocity space integrals involving δg, one notes that governing equations are integro-differential in nature, reflecting the nonlocal plasma response due to finite orbit excursions on scale lengths that may be of the same order of the mode wavelength and, in some cases, of the characteristic equilibrium scales, especially when supra-thermal particles are involved.

Gyrokinetic vorticity equation

- The gyrokinetic vorticity equation is derived from a velocity space integral of the GKE, acted upon by $e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}$ and $\sum e$.

- Looking at various contributions separately, for the inertia term one finds

$$\sum\left\langle eJ_0\frac{\partial}{\partial t}\delta g\right\rangle_v+\sum\left\langle\frac{e^2}{m}J_0\left[\frac{\partial\bar{F}_0}{\partial\mathcal{E}}J_0\frac{\partial}{\partial t}\delta\phi\right]\right\rangle_v=$$

$$-\boldsymbol{\nabla}\cdot\sum\left\langle\frac{e^2}{m}\frac{2\mu}{\Omega^2}\left(B_0\frac{\partial\bar{F}_0}{\partial\mathcal{E}}+\frac{\partial\bar{F}_0}{\partial\mu}\right)\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v\boldsymbol{\nabla}_\perp\frac{\partial}{\partial t}\delta\phi\ ,$$

where we have omitted the λ dependence of J_0 for brevity of notation and noted the quasi-neutrality condition.

E: Fill in the missing details to derive the inertia response.

- Meanwhile, for the gyrokinetic diamagnetic response, we have

$$\sum e \left\langle J_0 \left[\frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\nabla} \left(J_0 \delta\phi \right) \cdot \boldsymbol{\nabla} \bar{F}_0 \right] \right\rangle_v = - \sum ec\boldsymbol{b} \times \boldsymbol{\nabla} \left\langle \frac{2\mu}{\Omega^2} \bar{F}_0 \left(\frac{J_0^2 - 1}{\lambda^2} \right) \right\rangle_v \cdot \boldsymbol{\nabla} \nabla_\perp^2 \delta\phi \quad .$$

- Note that here, as for the quasi-neutrality condition, we have followed [Qin 98] to properly account for the correct recovery of the long wavelength limit $\lambda^2 = \epsilon_\perp^2 \to 0$.

E: Fill in the missing details to derive the diamagnetic response. Discuss where you have to invoke equilibrium charge neutrality condition.

- Considering now the gyrokinetic kink term, we have

$$\sum e \left\langle J_0 \left[\frac{v_\parallel}{B_0} \langle \delta \boldsymbol{B}_{\perp g} \rangle \cdot \boldsymbol{\nabla} \bar{F}_0 \right] \right\rangle_v = \delta \boldsymbol{B}_\perp \cdot \boldsymbol{\nabla} \left(\frac{j_{\parallel 0}}{B_0} \right) \quad ,$$

  so that the gyrokinetic kink term reduces to the ideal MHD expression.

E: Fill in the missing details to derive the gyrokinetic kink response. What would you need to change in this expression, if energetic particles are contributing importantly to the equilibrium current? Why?

- Here, again, the total parallel current is assumed to be carried by electrons. This means that this expression must be modified when, *e.g.*, a significant fraction of the equilibrium current is carried by energetic ions.

- The gyrokinetic line bending term can be written as [Chen 91]

$$\sum e \left\langle J_0 \left[v_\parallel \nabla_\parallel \delta g \right] \right\rangle_v = B_0 \nabla_\parallel \left(\sigma_k \frac{\delta j_\parallel}{B_0} \right) \simeq B_0 \nabla_\parallel \left(\frac{\delta j_\parallel}{B_0} \right) \quad ,$$

where $\nabla_\parallel \equiv \boldsymbol{b} \cdot \boldsymbol{\nabla}$, the gyrokinetic firehose stability term

$$\sigma_k = 1 + \frac{4\pi}{c^2} \sum \left\langle \frac{e^2}{m} \frac{v_\parallel^2}{B_0} \frac{\partial \bar{F}_0}{\partial \mu} \left(\frac{1 - J_0^2}{k_\perp^2} \right) \right\rangle_v \simeq 1 \quad ,$$

for low-β plasmas and we have consistently dropped terms $\propto \boldsymbol{b} \cdot \boldsymbol{\nabla} J_0$.

E: Fill in the missing details to derive the gyrokinetic line bending term. What are the conditions that you need to assume to approximate the gyrokinetic firehose stability term?

- Finally, the ballooning-interchange contribution

$$\sum e \langle J_0 [\boldsymbol{v}_d \cdot \boldsymbol{\nabla} \delta g] \rangle_v \simeq \frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \sum \left\langle m \left(\mu B_0 + v_\parallel^2 \right) J_0 \delta g \right\rangle_v \quad ,$$

is the natural gyrokinetic extension of the ideal MHD response obtained from the Chew–Goldberger–Low pressure tensor [Chew 56].

- Combining terms, the linear GK vorticity equation can be cast as

$$B_0 \nabla_\parallel \left(\frac{\delta j_\parallel}{B_0} \right) - \boldsymbol{\nabla} \cdot \sum \left\langle \frac{e^2}{m} \frac{2\mu}{\Omega^2} \left(B_0 \frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{\partial \bar{F}_0}{\partial \mu} \right) \left(\frac{J_0^2 - 1}{\lambda^2} \right) \right\rangle_v \boldsymbol{\nabla}_\perp \frac{\partial}{\partial t} \delta\phi$$
$$- \sum ec\boldsymbol{b} \times \boldsymbol{\nabla} \left\langle \frac{2\mu}{\Omega^2} \bar{F}_0 \left(\frac{J_0^2 - 1}{\lambda^2} \right) \right\rangle_v \cdot \boldsymbol{\nabla} \nabla_\perp^2 \delta\phi + \delta \boldsymbol{B}_\perp \cdot \boldsymbol{\nabla} \left(\frac{j_{\parallel 0}}{B_0} \right)$$
$$+ \frac{c}{B_0} \boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} \sum \left\langle m \left(\mu B_0 + v_\parallel^2 \right) J_0 \delta g \right\rangle_v = 0 \quad .$$

- Here, $\delta j_\parallel$ is related to $\delta \boldsymbol{A}$ via the parallel component of the low-frequency Ampère's law; *i.e.*, adopting the $\boldsymbol{\nabla} \cdot \delta \boldsymbol{A} = 0$ Coulomb gauge,

$$\begin{aligned} \delta j_\parallel &= \frac{c}{4\pi} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \delta \boldsymbol{A}) \\ &= \frac{c}{4\pi} \left\{ \left[-\nabla^2 + \kappa^2 + (\boldsymbol{\nabla b}) : (\boldsymbol{\nabla b}) \right] \delta A_\parallel \right. \\ &\quad + (\boldsymbol{\nabla} \times \boldsymbol{b})_\parallel \delta B_\parallel + (\boldsymbol{\nabla b}) : (\boldsymbol{\nabla} \delta \boldsymbol{A}_\perp) + \boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla b}) \cdot \delta \boldsymbol{A}_\perp] \\ &\quad \left. + (\boldsymbol{\kappa} \cdot \boldsymbol{\nabla b}) \cdot \delta \boldsymbol{A}_\perp + (\boldsymbol{b} \cdot \boldsymbol{\nabla} \delta \boldsymbol{A}_\perp) \cdot \boldsymbol{\kappa} \right\} \quad . \end{aligned}$$

- Applying the linear gyrokinetic ordering and neglecting $\delta B_\parallel$ and $\delta \boldsymbol{A}_\perp$ in the $\beta \ll 1$ limit, $\delta j_\parallel$ reduces to (with $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \boldsymbol{b} \nabla_\parallel$)

$$\delta j_\parallel \simeq -\frac{c}{4\pi} \nabla_\perp^2 \delta A_\parallel \ .$$

E: Demonstrate the identities for $\delta j_\parallel$.

Equilibrium particle motion in toroidal geometry

- More information following Lecture 1 of Spring 2012 Lecture Notes.
- Note $\boldsymbol{b}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\psi = -(\mathcal{J}B_0)^{-1}F(\psi)$. Then, given $B_\parallel^* = B_0 + (c/e)\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\bar{p}_\parallel$ and $\Omega = eB_0/(mc)$,

$$\begin{aligned} \dot{\psi} &= -\frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{\partial G}{\partial\theta} \;, \\ \dot{\theta} &= \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{\partial G}{\partial\psi} \;. \end{aligned}$$

$$G = \psi - F(\psi)v_\parallel/\Omega \simeq -(c/e)P_\phi$$

which is connected with the lowest order expression (in the drift parameter expansion) for the toroidal canonical angular momentum.

- Consider also

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\psi = B_0 - \frac{q(\psi)F(\psi)}{\mathcal{J}B_0} \quad ,$$

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\theta = \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F}{\mathcal{J}B_0}\frac{\partial \nu(\psi,\theta)}{\partial\psi} \quad .$$

In this way, one obtains

$$\dot{\zeta} = \frac{qv_\parallel}{\mathcal{J}B_\parallel^*} + \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{mc}{e}\left[\frac{\partial}{\partial\psi}\left(\mathcal{J}B_0 v_\parallel - q\frac{Fv_\parallel}{B_0}\right) + \frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{Fv_\parallel}{B_0}\frac{\partial \nu(\psi,\theta)}{\partial\psi}\right)\right] \quad ,$$

$$\begin{aligned} \dot{\xi} \quad &= \quad \dot{\zeta} - q\dot{\theta} - q_\psi \theta \dot{\psi} = \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{mc}{e}\left\{\frac{\partial}{\partial\psi}\left(\mathcal{J}B_0 v_\parallel\right)\right. \\ &\left. + \frac{\partial}{\partial\theta}\left[\mathcal{J}v_\parallel \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{Fv_\parallel}{B_0}\left(\frac{\partial\nu(\psi,\theta)}{\partial\psi} + q_\psi\theta\right)\right]\right\} \quad . \end{aligned}$$

Integral invariants and action angle coordinates

- The quantities $(\bar{\mu}, \bar{\zeta})$ and (P_ϕ, ϕ) are action-angle coordinates for the unperturbed particle motion. For fully characterizing the equilibrium particle motion in action-angle coordinates, we are still missing (J, η), with

$$J = m \oint v_\parallel d\ell = m \oint \left| \dot{\boldsymbol{X}} \right|^2 dt = m \oint v_\parallel^2 dt$$

being the *second invariant* and η the respective conjugate canonical angle, defined as

$$\eta = \omega_b \int_0^\theta \frac{d\theta'}{\dot{\theta}'} \quad .$$

E: Justify the definition of η as the canonical angle conjugate to J and discuss the implications of interpreting $v_\parallel$ in the definition of J as parallel velocity along $\boldsymbol{b}$ or along the guiding center trajectory arc-length.

- With the definitions above, J can be computed once for all given the reference magnetic equilibrium, i.e.

$$J = J(H_0, \bar{\mu}, P_\phi) \quad \leftrightarrow \quad H_0 = H_0(\bar{\mu}, J, P_\phi) \quad .$$

- Meanwhile, from the definition of J and ω_b

$$\frac{\partial J}{\partial H_0} = \frac{2\pi}{\omega_b} \quad \leftrightarrow \quad \omega_b = 2\pi \frac{\partial H_0}{\partial J} \quad .$$

E: Demonstrate this last statement. Hint: refer to the solution of the take-home exam of the Spring 11 Lectures.

Note: The distinction between μ and $\bar{\mu}$ (guiding center and gyrocenter magnetic moment) is unneeded in the absence of fluctuations. We maintain it here for consistency of notation.

Resonance conditions and equilibrium particle motion in toroidal geometry

- Particle motions in toroidal geometry depend on constants of motion and reflect the equilibrium magnetic configuration. Meanwhile, resonance conditions also depend on geometry and breaking of constants of motion by wave-particle resonances reflects fluctuation induced transport.

- Consider a generic fluctuation $f(r, \theta, \xi \equiv \zeta - q(r)\theta)$ in Clebsch coordinates (r, θ, ξ) [Z.X. Lu etal 12], leaving time dependences implicit. We can generally write

$$f(r, \theta, \xi) = \sum_n e^{in\xi} F_n(r, \theta) \quad ,$$

- While periodicity in ξ is maintained, periodicity in θ is substituted by $F_n(r,\theta+2\pi) = e^{2\pi i n q} F_n(r,\theta)$. In (r,ϑ) mapping space, the transform corresponding to using Clebsch coordinates is obtained by the periodization operator

$$\begin{aligned} F_n(r,\theta) &= 2\pi \sum_\ell e^{2\pi i \ell n q} \hat{F}_n(r,\theta-2\pi\ell) \\ &= \sum_m e^{i(nq-m)\theta} \int e^{i(m-nq)\vartheta} \hat{F}_n(r,\vartheta) d\vartheta \ . \end{aligned}$$

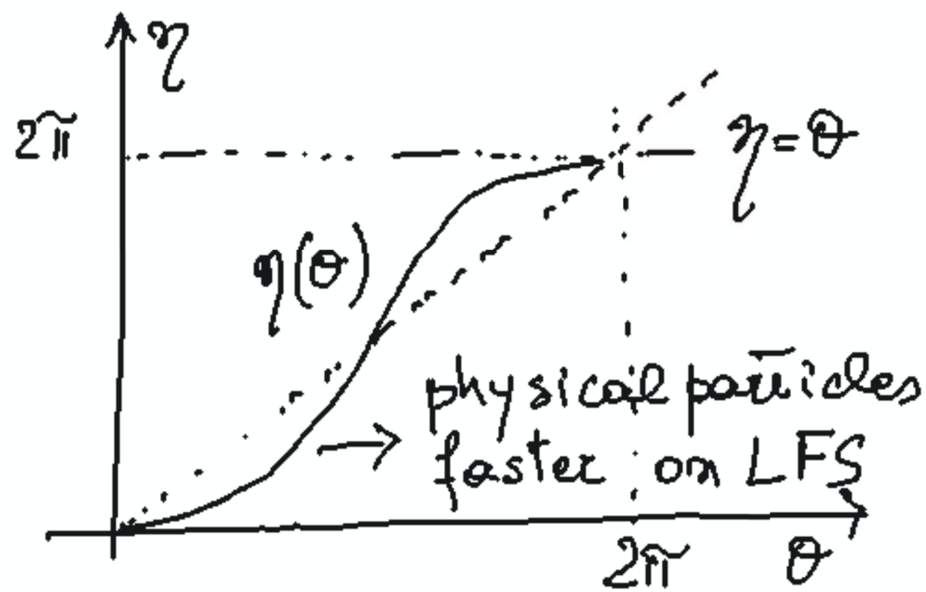
The governing equations are generally expressed either for $F_n(r,\theta)$ or for $\hat{F}_n(r,\vartheta)$, as discussed by [Z.X. Lu etal 12]. All together,

$$\begin{aligned} f(r,\theta,\xi) &= \sum_{m,n} e^{in\xi} e^{i(nq-m)\theta} F_{m,n}(r) \ , \\ F_{m,n}(r) &= \int e^{i(m-nq)\vartheta} \hat{F}_n(r,\vartheta) d\vartheta \ . \end{aligned}$$

- Recall η is the canonical angle conjugate to J (second invariant), obtained from integration along the particle orbit as

$$J = m \oint v_{\|} d\ell \quad , \qquad \eta = \omega_b \int_0^{\theta} \frac{d\theta'}{\dot{\theta}'} \quad , \qquad \omega_b = \frac{2\pi}{\oint d\theta / \dot{\theta}} .$$

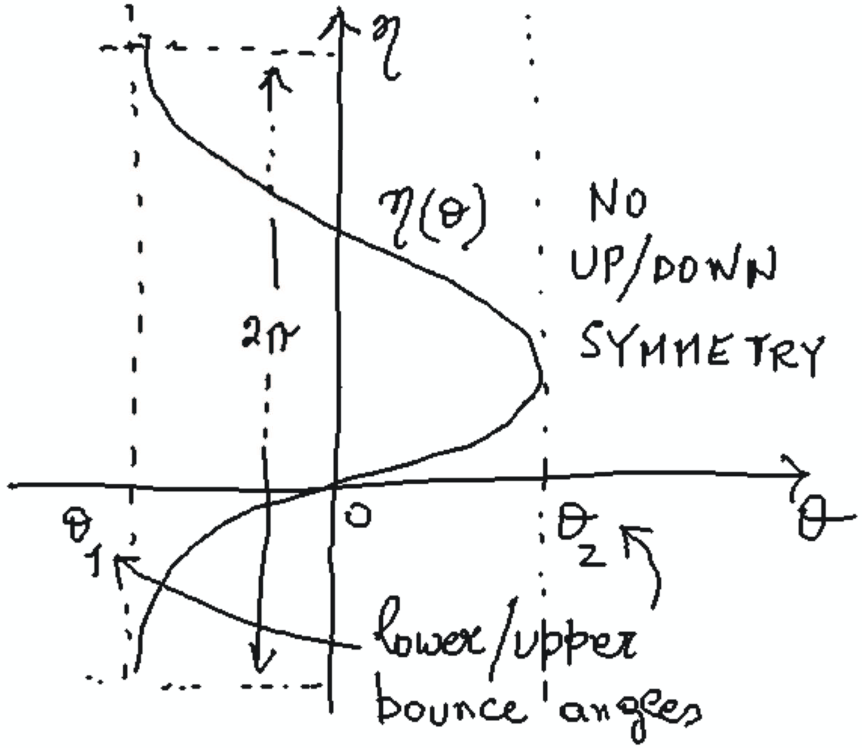
- For circulating particles, recalling that ω_b is the transit/bounce frequency,



$$\theta = \eta + \Theta_C(\eta) = \omega_b \tau + \Theta_C(\eta) \quad ,$$

where $\Theta_C(\eta)$ is a periodic functions of η, which depends also on H_0, μ, P_ϕ or, equivalently, of μ, J, P_ϕ; and τ is a time-like parameter describing the particle motion along the particle trajectory.

- For magnetically trapped particles



$$\theta = \Theta_T(\eta) \quad ,$$

where $\Theta_T(\eta)$ is a periodic function of η, which depends also on H_0, μ, P_ϕ or, equivalently, of μ, J, P_ϕ.

- From the periodicity of the radial particle motion in the equilibrium fields, we also have

$$r = \bar{r} + \rho(\eta) \quad ,$$

with $\rho(\eta)$ a periodic function of η, different for trapped and circulating particles, which depends also on H_0, μ, P_ϕ or, equivalently, of μ, J, P_ϕ.

E: From the equations of motions in the equilibrium magnetic field show that the particle trajectories in the (ψ, θ) plane are closed and discuss how this implies the results above for the radial particle motion.

- The connection of ξ with $\eta = \omega_b \tau$ is more subtle to obtain. Recall the definition

$$\xi = \zeta - q(\psi)\theta = \phi - \nu(\psi, \theta) - q(\psi)\theta$$

and that the particle orbit follows the magnetic field line and is accompanied by a periodic motion about it, of period $2\pi/\omega_b$, which is therefore a periodic function of η.

- Keeping this in mind we can conclude that, except for periodic dependences in η (indicated by the $\simeq$ sign and the $\ldots$)

$$\zeta \simeq \bar{\omega}_d \tau + \int q d\theta + \ldots \simeq \bar{\omega}_d \tau + \bar{q}\theta + \ldots$$

with $q(\bar{r} + \rho(\eta)) = \bar{q}(\bar{r}) + Q(\bar{r} + \rho(\eta))$ and $\oint Q d\theta = \oint (\dot{\theta}/\omega_b) Q d\eta = 0$.

- The representation for ξ is then obtained as

$$\xi = \bar{\omega}_d \tau + (\bar{q} - q)\,\theta + \Xi(\eta) \quad ,$$

with $\Xi(\eta)$ a periodic function of η, which depends also on H_0, μ, P_ϕ or, equivalently, of μ, J, P_ϕ.

- Given the above representation for ξ, the definition of the precession frequency is

$$\bar{\omega}_d = \frac{\omega_b}{2\pi} \oint \left(\dot{\zeta} - q\dot{\theta} \right) \frac{d\theta}{\dot{\theta}} \quad ,$$

E: Demonstrate the last statement. Hint: take the total time derivative of the ξ parametric expression and integrate on one full periodic orbit in η.

- Summarizing: introduce the periodic functions $\Theta(\eta)$, $\Xi(\eta)$ and $\rho(\eta)$, i.e. without distinction between trapped and circulating particles but keeping in mind that they are different for each class of particles, which depend also on H_0, μ, P_ϕ or, equivalently, of μ, J, P_ϕ

Circulating particles	Trapped particles
$\theta = \eta + \Theta(\eta) = \omega_b\tau + \Theta(\eta)$	$\theta = \Theta(\eta)$
$r = \bar{r} + \rho(\eta)$	$r = \bar{r} + \rho(\eta)$
$\xi = \bar{\omega}_d\tau + (\bar{q} - q)\,\theta + \Xi(\eta)$	$\xi = \bar{\omega}_d\tau + (\bar{q} - q)\,\theta + \Xi(\eta)$

with the definition

$$q(\bar{r} + \rho(\eta)) \equiv \bar{q}(\bar{r}) + Q(\bar{r} + \rho(\eta)) \qquad \oint Q d\theta = \oint (\dot{\theta}/\omega_b) Q d\eta = 0$$

- Here, τ is a time-like parameter tracking the position along the trajectory ($\eta \equiv \omega_b\tau$), but no time dependence has been assumed so far.

- With the above parametrization of particle trajectories in terms of action-angle variables and characteristic particle frequencies in the equilibrium magnetic configuration, we can systematically decompose the fluctuating field in action-angle coordinates, for given constants of motion.

- This procedure, which is not the usual practice, corresponds to decompose a mode structure in elements that are actually describing the wave-particle interactions, once the Fourier decomposition is known. This is somewhat tedious, but systematic, and has the advantage of:
  - identifying the resonance condition in general geometry and suggesting a practical way of computing it
  - describing wave-particle resonances and their effect on the particle motion in the most natural coordinates describing the unperturbed particle motion
  - lifting of f to the particle phase space

- Recall that (see p. 23)

$$f(r,\theta,\xi) = \sum_{m,n} e^{in\xi} e^{i(nq-m)\theta} F_{m,n}(r)$$

- Using the trajectory parametrization summarized on p. 28, we have

$$f(r,\theta,\xi) = \sum_{m,n,\ell} \lambda_{n,m} e^{i(n\bar{\omega}_d + \ell\omega_b)\tau} \mathcal{F}_{m,n,\ell}(\bar{r}) \quad ,$$

$$\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi) = \frac{1}{2\pi} \oint \exp\left\{ in\Xi(\eta) + i\left[n\bar{q}(\bar{r}) - m \right] \Theta(\eta) \right\} F_{m,n}(\bar{r} + \rho(\eta)) e^{-i\ell\eta} d\eta \quad .$$

- Note that, here, we have made explicit the dependences of $\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi)$ on μ, J, P_ϕ. Note also that no specific time dependence has been assumed here, so far, and that the time-like parameter τ parameterizes (r,θ,ξ) along the particle trajectory, as noted before.

- Note also that, because of the finite orbit width $\rho(\eta)$, the effective mode width – that of $\mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi)$ – can be significantly larger than that of the Fourier harmonic $F_{m,n}(r)$.

E: Explain qualitatively why the effective mode width can be larger than that of the Fourier components and discuss which physics this may impact. Explain what is the physical meaning of the dependence of the fluctuation strength on phase space variables such as μ, J, P_ϕ.

- The mode structure decomposition in action angle variables corresponds to a further Fourier decomposition of the amplitudes $F_{m,n}(r)$ in bounce harmonics ($\eta \equiv \omega_b\tau$), as effectively experienced by the particle moving on the phase space surface given by its constants of motion.

E: Can you recognize this further Fourier decomposition? Can you explain why it is induced by the periodic functions $\Theta(\eta)$, $\Xi(\eta)$ and $\rho(\eta)$?

- The quantity $\lambda_{n,m}$ is $\lambda_{n,m} = 1$ for trapped particles, while, for circulating particles, it is given by

$$\lambda_{n,m} = \exp\left[i\left(n\bar{q}(\bar{r}) - m\right)\omega_b \tau\right] \quad .$$

E: Derive this last equation step by step.

- The resonance condition is obtained when, for some partial amplitudes in the mode structure decomposition, the wave-particle phase is constant along the unperturbed orbits, so that, assuming fluctuations $\propto \exp(-i\omega t)$

$$\omega = \omega(\mu, J, P_\phi) = n\bar{\omega}_d + \ell\omega_b$$

for trapped particles, while, for circulating particles,

$$\omega = \omega(\mu, J, P_\phi) = n\bar{\omega}_d + \ell\omega_b + \left(n\bar{q}(\bar{r}) - m\right)\omega_b \quad .$$

E: Discuss the physics underlying each term in the resonance conditions above.

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