Lecture 5

Wave-particle interactions with a broad fluctuation spectrum. Quasilinear theory and resonance broadening.

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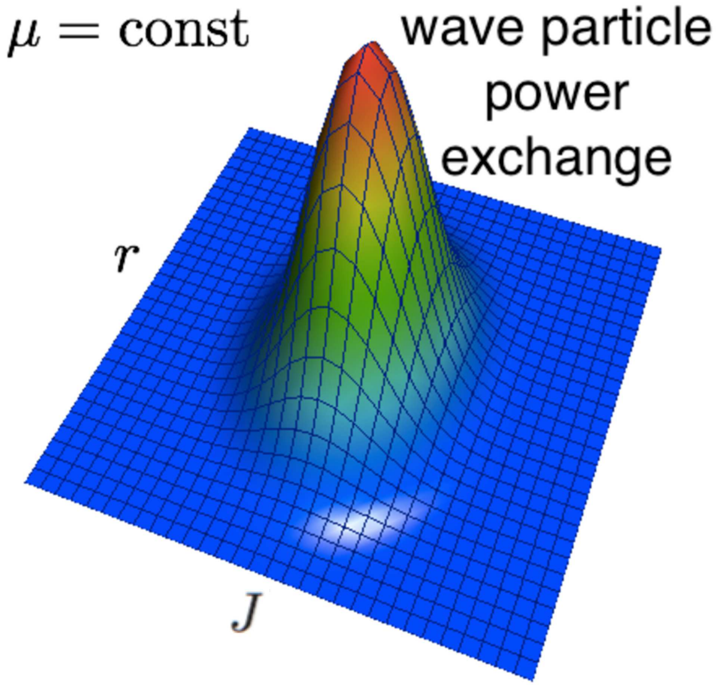
From fishbone to bump-on-tail paradigm

- Use theoretical framework developed in Lecture 1 and Lecture 2.
- This allows to construct reduced dynamic description of a time dependent non-uniform plasma with one degree of freedom in the corresponding reduced phase space
  - gyrokinetic transport theory: μ conservation ($|\omega| \ll |\Omega|$)
  - identification of additional (nonlinear) invariant of motion $|\omega| \sim |n\bar{\omega}_{dk}| \ll \omega_b \Rightarrow J = \text{const} \Rightarrow$ fishbone paradigm: (Lecture 4) $\Rightarrow$ neglect finite Larmor and magnetic drift orbit width
- The system can be further reduced to the description of a time dependent uniform plasma with one degree of freedom (e.g., bump-on-tail paradigm) when the nonlinear particle displacement is small compared to the mode characteristic width [Berk & Breizman 1990] (Lecture 3).

- Mapping $r \leftrightarrow v$ for each $(\mu, J) =$ const pair [Zonca et al., NJP 2015]

$$-\frac{m}{e}\frac{nc}{d\psi/dr}\frac{\partial}{\partial r} \leftrightarrow k_0\frac{\partial}{\partial v} \; , \qquad n\bar{\omega}_{dk} - \omega_k \simeq n\bar{\omega}_{dk0}\frac{(r-r_0)}{L_{dk0}} \leftrightarrow k_0 v - \omega_p \; .$$

- Use the freedom of choice of beam density to preserve overall (integral) wave-power exchange at each $(\mu, J) =$ const, keeping each $(\mu, J) =$ const pair independent for reconstruction of the relaxed distribution function in higher dimensionality $\Rightarrow$ finite temporal scale for validity of reduced dynamic description (Lecture 2) .



- Mapping of $r \leftrightarrow v$ allows deriving Dyson equation for the beam-plasma system, given that of the fishbone paradigm (p.18 Lecture 4).

$$\begin{aligned}\hat{F}_0(\omega) &= \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}\bar{F}_0(0) - \frac{e^2k_0^2}{m^2\omega}\frac{\partial}{\partial v} \\ &\times \left\{ \left[\frac{\partial_v \hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega - \omega_0(\tau) + k_0 v} + \frac{\partial_v \hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega + \omega_0^*(\tau) - k_0 v} \right] \left| \delta\bar{\phi}_{k0}(\tau) \right|^2 \right\} \quad .\end{aligned}$$

E: Discuss this solution in comparison with the Dyson equation of the "fishbone paradigm" (p.18 Lecture 4).

- The Dyson equation is closed by the wave evolution (Poisson) equation

$$\frac{2}{\omega_p}\left(\frac{\partial}{\partial\tau} + i(\omega_p - \omega_0(\tau))\right)\delta\bar{\phi}_{k0}(\tau) = \frac{\omega_p^2}{nk_0} i \int dv \int_{-\infty}^{+\infty} e^{-i\omega\tau} \frac{\delta\bar{\phi}_{k0}(\tau)}{\omega + \omega_0(\tau) - k_0 v}\frac{\partial}{\partial v}\hat{F}_0(\omega) d\omega$$

E: Discuss how this formulation contains and extends the approach of Lecture 3 (see RP on p. 22 of Lecture 3).

- Solutions of this system strongly depend on the properties of the fluctuation spectrum; and on the properties of $\hat{F}_0(\omega)$, which can have a broad spectrum (this lecture) or a narrow spectrum (Lecture 4).

- Behaviors crucially depend on the characteristic nonlinear time compared to the inverse growth time of the fluctuations $|\gamma_{k_0}|^{-1} \sim \gamma_0^{-1}$.

- If nonlinear time is sufficiently short, and/or the growth rate sufficiently weak (Lecture 3), on the RHS of the Dyson equation we can take $\hat{F}_0(\omega - 2i\gamma_{k_0}) \to \hat{F}_0(\omega) \Rightarrow$ Reduced equation is readily rewritten in the t-representation: $\hat{F}_0(\omega) = (2\pi)^{-1} \int_0^\infty e^{i\omega t} F_0(t) dt$.

- Important physics implications arise when $\tau_{NL} \sim |\gamma_{k_0}|^{-1}$ for a narrow spectrum. This is the phase locking regime [Zonca et al. 2015], of interest for EP interactions with Alfvén waves in fusion plasmas, yielding ballistic transport: e.g., by fishbones and energetic particle modes (EPM) (Lecture 4).

Broad spectrum: quasilinear diffusion equation

- Consistent with above discussion, consider the case

$$\frac{1}{\tau_{NL}} \sim |\omega| \ll |\omega_{k_0} - k_0 v| \sim \left| \omega_{k_0} \frac{\Delta k}{k_0} \right| \sim |k_0 \Delta v| \quad .$$

- In this limit ($\omega \to i0^+$)

$$(\omega - \omega_{k_0} + k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v) \quad ; \qquad (\omega + \omega^*_{k_0} - k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v).$$

- The evolution equation for F_0 becomes (quasilinear diffusion equation)

$$\frac{\partial}{\partial t} F_0(v,t) = \frac{e^2}{m^2} \sum_{k_0} k_0^2 \left| \delta\phi_{k_0}(\tau) \right|^2 \frac{\partial}{\partial v} \left[\pi\delta(\omega_{k_0} - k_0 v) \frac{\partial}{\partial v} F_0(v,t) \right] \quad .$$

E: Derive the evolution equation of $F_0(v,t)$ step by step, symmetrizing $k_0 \leftrightarrow -k_0$.

- The evolution equation of $F_0(v,t)$ is a diffusion equation in v-space, with diffusion coefficient ($\sum_{k_0}$ accounts for the whole k_0 spectrum)

$$D_v = \pi \frac{e^2}{m^2} \sum_{k_0} k_0^2 \left| \delta\phi_{k_0}(\tau) \right|^2 \delta(\omega_{k_0} - k_0 v) \quad .$$

- For consistency with the broad spectrum assumption, we must have

$$\frac{1}{\tau_{NL}} \sim |\omega| \sim \left| \frac{D_v}{(\Delta v)^2} \right| \ll |k_0 \Delta v| \quad ; \quad \Rightarrow \quad \left| \frac{e}{m} \delta\phi_{k_0}(\tau) \right|^{1/2} \ll |\Delta v| \quad .$$

- This is the typical validity limit of quasilinear description.

E: Demonstrate that this condition is equivalent to having $|\gamma_{k_0}|^{-1} \gg \tau_{NL} \gg \omega_B^{-1}$, with ω_B the wave-particle trapping frequency; and that transport takes place because of resonance overlap.

Conservation laws

- Note that we have introduced approximations when deriving the quasilinear diffusion equation on p. 6:

$$(\omega-\omega_{k_0}+k_0v)^{-1} \to -i\pi\delta(\omega_{k_0}-k_0v) \quad ; \qquad (\omega+\omega^*_{k_0}-k_0v)^{-1} \to -i\pi\delta(\omega_{k_0}-k_0v).$$

- More precisely, we have (after shifting $\omega \to \omega + 2i\gamma_{k_0}$ and $\hat{F}_0(\omega - 2i\gamma_{k_0}) \to \hat{F}_0(\omega)$ on p. 4) [Al'tshul' 66]:

$$\frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} - \omega_{k_0} + k_0 v} + \frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} + \omega^*_{k_0} - k_0 v} \simeq \frac{(2i\gamma_{k_0} + 2\omega)|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{(\omega + i\gamma_{k_0})^2 - (\omega_{k_0 r} - k_0 v)^2}$$

- Therefore:

$$\frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} - \omega_{k_0} + k_0 v} + \frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} + \omega^*_{k_0} - k_0 v} \simeq -2i\pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}$$

$$+i\frac{\partial}{\partial\omega_{k_0 r}}\mathcal{P}\left(\frac{1}{\omega_{k_0 r} - k_0 v}\right)\left(e^{-i\omega t}\partial_t|\delta\phi_{k_0}(t)|^2 + 2|\delta\phi_{k_0}(t)|^2\partial_t e^{-i\omega t}\right)$$

- Substituting back into $\hat{F}_0(\omega)$ equation at p.4, symmetrizing $k_0 \leftrightarrow -k_0$, and moving to t-representation:

$$\frac{\partial}{\partial t}F_0 = \frac{e^2}{m^2}\sum_{k_0} k_0^2 \frac{\partial}{\partial v}\left[\pi\delta(\omega_{k_0r} - k_0 v))|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v}F_0 \right.$$
$$\left. - \frac{\partial}{\partial \omega_{k_0 r}}\mathcal{P}\left(\frac{1}{\omega_{k_0 r} - k_0 v}\right)\left(\frac{1}{2}\partial_t|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v}F_0 + |\delta\phi_{k_0}(t)|^2\frac{\partial^2}{\partial v \partial t}F_0\right)\right]$$

- The last two terms are respectively $\sim (\gamma_{k_0}/k_0\Delta v)$ and $\sim (\tau_{NL}^{-1}/k_0\Delta v)$ with respect to the quasilinear diffusion on the RHS, derived on p.6. Thus, they are both small.

- It makes sense to consider the adiabatic change of mode amplitude only if it is significant over a characteristic transport time scale.

E: Demonstrate that this condition is equivalent to having $\tau_{NL} \gg |\gamma_{k_0}|^{-1} \gtrsim \omega_B^{-1}$. What is the difference with the ordering considered on p. 7? Is transport always due to resonance overlap?

- Therefore, while the last term is typically dropped, the second last term is kept into account by quasilinear theory for F_0 evolution, which then reads

$$\frac{\partial}{\partial t}F_0 = \frac{e^2}{m^2}\sum_{k_0} k_0^2 \frac{\partial}{\partial v}\left[\pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v}F_0 \right.$$
$$\left. -\frac{1}{2}\frac{\partial}{\partial \omega_{k_0 r}}\mathcal{P}\left(\frac{1}{\omega_{k_0 r} - k_0 v}\right)\partial_t|\delta\phi_{k_0}(t)|^2\frac{\partial}{\partial v}F_0\right]$$

- Use now Poisson's equation at p. 4 to demonstrate

$$\omega_p^{-1}\partial_t|\delta\phi_{k_0}(t)|^2 = \frac{\omega_p^2}{nk_0}\int dv\pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2\frac{\partial}{\partial v}F_0$$

and use this result to demonstrate momentum and energy conservation.

E: Demonstrate that particle conservation follows immediately from the quasilinear diffusion equation above. Why?

- Momentum conservation (considering only the dominant diffusion term), integrating by parts and using last equation on p. 10

$$
\begin{aligned}
\frac{dP_z}{dt} &= \int_{-\infty}^{\infty} mv\partial_t F_0 = -\frac{e^2}{m}\sum_{k_0} k_0^2 \int_{-\infty}^{\infty} \pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} F_0 dv \\
&= -\frac{ne^2}{m\omega_p^2}\sum_{k_0} \frac{k_0}{\omega_p}\frac{d}{dt}|E_{k_0}(t)|^2 = -\sum_{k_0} \frac{k_0}{\omega_{k_0}}\frac{d}{dt}\frac{|E_{k_0}(t)|^2}{4\pi}
\end{aligned}
$$

having noted that $\omega_{k_0} = \omega_p$ for Langmuir waves and $|E_{k_0}(t)|^2 = k_0^2|\delta\phi_{k_0}(t)|^2$.

E: Compare this momentum conservation law with that obtained on p. 11 of Spring 2015 Lecture 2. Are they the same? Why?

- Energy conservation (considering only the dominant diffusion term), integrating by parts and using last equation on p. 10

$$\frac{d\mathcal{E}}{dt} = \int_{-\infty}^{\infty} m\frac{v^2}{2}\partial_t F_0 = -\frac{e^2}{m}\sum_{k_0} k_0^2 \int_{-\infty}^{\infty} \pi\delta(\omega_{k_0 r} - k_0 v) v |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} F_0 dv$$

$$= -\frac{ne^2}{m\omega_p^2}\sum_{k_0}\frac{\omega_{k_0}}{\omega_p}\frac{d}{dt}|E_{k_0}(t)|^2 = -\sum_{k_0}\frac{d}{dt}\frac{|E_{k_0}(t)|^2}{4\pi}$$

having noted that $\omega_{k_0} = \omega_p$ for Langmuir waves and $|E_{k_0}(t)|^2 = k_0^2|\delta\phi_{k_0}(t)|^2$.

E: Compare this energy conservation law with that obtained on p. 12 of Spring 2015 Lecture 2. Are they the same? Why?

E: Show that the average kinetic energy density of thermal electron oscillations is the same as the average electric field energy density for Langmuir waves. How does this relate with the energy conservation above?

- Some considerations of theoretical and practical importance:
  - The present conservation laws are consistent with those derived in Spring 2015 Lecture 2, but have been obtained neglecting the principal value term in the quasilinear equation for F_0
  - The neglected term represents the non-resonant wave-particle interactions.

E: Convince yourself that the neglected term is indeed small. Can you quantify how small is small?

E: Show that the neglected non-resonant interaction term is invariant under time reversal. What does that mean? What about the dominant diffusion term kept into account and describing resonant wave-particle interactions?

E: Derive step by step the non-resonant term that was neglected. What is the structure of that term with respect to the analogous term that you could derive for the thermal plasma electrons? What can you conclude about thermal plasma and beam electron non-resonant interactions? (see also Lecture 6)

Resonance broadening

- When particles are subject to fluctuations or undergo collisions, it is important to evaluate the effect of such interactions on the resonance condition $\omega = kv$. This study leads to the description of resonance broadening [Dupree 1966].
- Consider the Vlasov equation for a reference mode ("test wave" [Dupree 1966])

$$i\left(\boldsymbol{k}_0 \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}_0}\right) F_{\boldsymbol{k}_0} + \frac{e}{m} \boldsymbol{E}_{\boldsymbol{k}_0} \cdot \frac{\partial F_0}{\partial \boldsymbol{v}} + \frac{e}{m} \sum_{\boldsymbol{k}} \boldsymbol{E}_{\boldsymbol{k}_0 - \boldsymbol{k}} \cdot \frac{\partial F_{\boldsymbol{k}}}{\partial \boldsymbol{v}} e^{i \Delta\omega_{\boldsymbol{k}_0, \boldsymbol{k}} t} = 0$$

with $\Delta\omega_{\boldsymbol{k}_0,\boldsymbol{k}} = \omega_{\boldsymbol{k}_0} - \omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}_0 - \boldsymbol{k}}$. The same equation can be written for all fluctuations $(\omega_{\boldsymbol{k}}, \boldsymbol{k})$.

- The nonlinear term above will contain a term depending explicitly on $F_{\boldsymbol{k}_0}$, because $F_{\boldsymbol{k}}$ nonlinearly depends on $F_{\boldsymbol{k}_0}$ through a similar nonlinear term. This suggests that a renormalization procedure is necessary.
- Solve formally for $F_{\boldsymbol{k}}$

$$F_{\boldsymbol{k}} = \frac{e}{m}\frac{i}{(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})}\left(\boldsymbol{E}_{\boldsymbol{k}}\cdot\frac{\partial F_0}{\partial\boldsymbol{v}} + \boldsymbol{E}_{\boldsymbol{k}-\boldsymbol{k}_0}\cdot\frac{\partial F_{\boldsymbol{k}_0}}{\partial\boldsymbol{v}}e^{i\Delta\omega_{\boldsymbol{k},\boldsymbol{k}_0}t}\right.$$
$$\left. + \sum_{\boldsymbol{k}'\neq\boldsymbol{k}_0}\boldsymbol{E}_{\boldsymbol{k}-\boldsymbol{k}'}\cdot\frac{\partial F_{\boldsymbol{k}'}}{\partial\boldsymbol{v}}e^{i\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}t}\right)$$

- This expression can be substituted back into the equation for $F_{\boldsymbol{k}_0}$ that yields

$$i\left(\boldsymbol{k}_0\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}_0}\right)F_{\boldsymbol{k}_0} - \frac{\partial}{\partial\boldsymbol{v}}\cdot\left(\boldsymbol{D}\cdot\frac{\partial F_{\boldsymbol{k}_0}}{\partial\boldsymbol{v}}\right) = -\frac{e}{m}\boldsymbol{E}_{\boldsymbol{k}_0}\cdot\frac{\partial F_0}{\partial\boldsymbol{v}} + \text{NL terms}$$

- NL terms on the RHS contain either F_0 or $F_{\boldsymbol{k}}$ with $\boldsymbol{k} \neq \boldsymbol{k}_0$. Thus they can be neglected in the theory of resonance broadening.

$$\begin{aligned}\text{NL terms} &= \frac{e^2}{m^2} \sum_{\boldsymbol{k}} e^{i\Delta\omega_{\boldsymbol{k}_0,\boldsymbol{k}} t} \boldsymbol{E}_{\boldsymbol{k}_0-\boldsymbol{k}} \cdot \frac{\partial}{\partial \boldsymbol{v}} \left[\frac{i}{(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})} \right.\\ &\left. \times \left(\boldsymbol{E}_{\boldsymbol{k}} \cdot \frac{\partial F_0}{\partial \boldsymbol{v}} + \sum_{\boldsymbol{k}' \neq \boldsymbol{k}_0} \boldsymbol{E}_{\boldsymbol{k}-\boldsymbol{k}'} \cdot \frac{\partial F_{\boldsymbol{k}'}}{\partial \boldsymbol{v}} e^{i\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'} t} \right) \right]\end{aligned}$$

E: Derive the expression above and explain which, if any, are the terms analyzed earlier for the investigation of the quasilinear theory.

E: Discuss whether the expression above, neglected here, contains merely a renormalization of F_0 (Dyson equation) or other physics processes are included.

- The resonance broadening effect is contained in the 3rd "diffusive"-like term on the RHS on p. 15 [Dupree 1966]. The real part of $\boldsymbol{D}$ contains the actual resonance broadening, while the imaginary part of $\boldsymbol{D}$ contains the real resonance frequency shift, which we neglect hereafter.

$$\frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\boldsymbol{D} \cdot \frac{\partial F_{\boldsymbol{k}_0}}{\partial \boldsymbol{v}} \right) = -\frac{e^2}{m^2} \sum_{\boldsymbol{k}} \boldsymbol{E}_{\boldsymbol{k}_0 - \boldsymbol{k}} \cdot \frac{\partial}{\partial \boldsymbol{v}} \left[\frac{i}{(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}})} \boldsymbol{E}_{\boldsymbol{k} - \boldsymbol{k}_0} \cdot \frac{\partial F_{\boldsymbol{k}_0}}{\partial \boldsymbol{v}} \right]$$

E: Derive the expression above and discuss the various terms.

E: Can you recognize which term is responsible for the resonance broadening and which for the resonance frequency shift?

- Note that even the resonances in the expression above are, in general, "broadened". Thus, we have a chain of resonance broadening. For simplicity, here, we consider only the broadening of the reference mode, neglecting others.

- For simplicity, assume e.s. fluctuations, $\boldsymbol{E}_{\boldsymbol{k}} \parallel \boldsymbol{k}$,

$$D_{ij} = \frac{e^2}{m^2}\mathbb{I}\mathrm{m}\sum_{\boldsymbol{k}} \frac{k_i k_j}{k^2} \left|\boldsymbol{E}_{\boldsymbol{k}}\right|^2 \left[\frac{1}{((\boldsymbol{k}+\boldsymbol{k}_0)\cdot\boldsymbol{v} - (\omega_{\boldsymbol{k}}+\omega_0))}\right]$$

- For the 1D beam-plasma system, resonance broadening can be described by (dropping NL terms on RHS)

$$i\left(k_0 v - \omega_0\right) F_{k_0} - D\frac{\partial^2}{\partial v^2} F_{k_0} = -\frac{e}{m} E_{k_0} \frac{\partial F_0}{\partial v}$$

$$D = \pi \frac{e^2}{m^2}\mathbb{I}\mathrm{m}\sum_{k} \left|E_k\right|^2 \delta\left[(k+k_0)v - (\omega_k+\omega_0)\right]$$

E: Derive these equations step by step.

- The first equation is most easily solved in Fourier space

$$F_{k_0}(v) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{F}_{k_0}(w) e^{+ivw} dw \; , \qquad \bar{F}_{k_0}(w) = \int_{-\infty}^{+\infty} F_{k_0}(v) e^{-ivw} dv$$

$$\left(k_0 \frac{\partial}{\partial w} + i\omega_0 \right) \bar{F}_{k_0} - Dw^2 \bar{F}_{k_0} = \frac{e}{m} E_{k_0} \overline{\left(\frac{\partial F_0}{\partial v} \right)}$$

- Assuming $k_0 > 0$ for simplicity ($k_0 < 0$ is obtained in the same way), this first order ordinary differential equation has solution

$$\begin{aligned} \bar{F}_{k_0} \quad &= \quad -\frac{e}{mk_0} E_{k_0} e^{-i\frac{\omega_0}{k_0}w + \frac{Dw^3}{3k_0}} \\ &\quad \times \int_w^\infty \left(e^{i\frac{\omega_0}{k_0}w' - \frac{Dw'^3}{3k_0}} \int_{-\infty}^{\infty} e^{-iv'w'} \frac{\partial F_0(v')}{\partial v'} dv' \right) dw' \end{aligned}$$

- The phase factor is rapidly cut-off because of the diffusion term. For this reason, it is possible to approximate $Dw^3/(3k_0) \simeq (D/3k_0)^{1/3}w$ [Dupree 1966]. Expanding around $w' = w$,

$$i\left(\frac{\omega_0}{k_0} - v'\right)w' - \frac{Dw'^3}{3k_0} \simeq i\left(\frac{\omega_0}{k_0} - v'\right)w - \left(\frac{D}{3k_0}\right)^{1/3}w - \left[i\left(v' - \frac{\omega_0}{k_0}\right) + \left(\frac{D}{3k_0}\right)^{1/3}\right](w' - w)$$

E: Comment about the adopted method for approximating the integral above.

- By direct substitution, this gives

$$\bar{F}_{k_0} = -\frac{e}{mk_0}E_{k_0}\int_{-\infty}^{\infty}\frac{e^{-iv'w}}{i\left(v' - \frac{\omega_0}{k_0}\right) + \left(\frac{D}{3k_0}\right)^{1/3}}\frac{\partial F_0(v')}{\partial v'}dv'$$

$$\Rightarrow F_{k_0} = -\frac{e}{m}E_{k_0}\frac{1}{i\left(k_0 v - \omega_0\right) + \left(Dk_0^2/3\right)^{1/3}}\frac{\partial F_0(v)}{\partial v}$$

- From the last equation, one readily sees the effect of resonance broadening due to finite amplitude fluctuations. From p. 18

$$D = \pi \frac{e^2}{m^2} \mathbb{I}\mathrm{m} \sum_k |E_k|^2 \, \delta \left[(k + k_0) v - (\omega_k + \omega_0) \right]$$

- Two fundamental time scales enter the problem:
  - time scale for a particle to diffuse over $\sim k_0^{-1}$: $(D k_0^2/3)^{-1/3}$
  - autocorrelation time of fluctuations: $|k_0 \Delta v|^{-1}$, necessary for relative phase of two waves to randomize due to different phase velocities
  - further discussions in Lecture 6

E: Derive and discuss the two time scales just mentioned above.

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