Lecture 1

Review of basics concepts in Hamiltonian systems and nonlinear charged particle dynamics

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April 25.th, 2016

Introduction to Nonlinear Plasma Physics – Spring 2016,
Part I. Nonlinear Wave-Particle Interactions
April 25– May 6 2016, IFTS – ZJU, Hangzhou

Lecture Plan and Teaching Material

Abstract: This *Introduction to Nonlinear Plasma Physics* is divided into two parts. Part (I) is devoted to *Nonlinear Wave-Particle Interactions* and is lectured by Prof. Fulvio Zonca. Part (II), lectured by Prof. Liu Chen, focuses on *Nonlinear Wave-Wave Interactions*. Prerequisite of the course is a preliminary knowledge of general plasma physics and classical mechanics. General theoretical aspects are treated in Lectures $(1 \div 6)$, which are complemented by corresponding Q&A Sessions.

Main areas that will be explored are:

i) Review of basics concepts in Hamiltonian systems and nonlinear charged particle dynamics (Lecture 1)

ii) Wave particle interactions in magnetized toroidal plasmas. Reduced description of nonlinear dynamics. Effect of plasma non-uniformity and equilibrium geometry. (Lecture 2)

iii) Isolated wave-particle resonances: nonlinear Alfvén Eigenmode dynamics near marginal stability (Lecture 3)

iv) Non-perturbative nonlinear coherent behavior: Energetic Particle Modes and the notion of avalanche. (Lecture 4)

v) Wave-particle interactions with a broad fluctuation spectrum. Quasilinear theory and resonance broadening. (Lecture 5)

vi) Advanced topics in Quasi-linear theory. Thermal plasma effects and the turbulent trapping model. (Lecture 6)

Lecture Notes: Available in electronic form. At the end of each lecture and during the Q&A time, a list of specific reading material is given explicitly.

Exercises and Research Projects: In the lecture notes, Exercises (E) of various difficulty levels are suggested. These are meant to be important part of the lectures themselves. Possible topics for Research Projects (RP) are also indicated, which require significant more in depth analysis and work.

Q&A Sessions: Devoted to detailed discussions and analysis of technical aspects. They are meant to be complementary to general lectures and are a necessary addendum for full appreciation of the material presented in the course.

Lagrangian and Hamiltonian formulation of classical mechanics

- Equations of motion must be equivalent in all coordinate systems and can be obtained from one another by coordinate transformations.
  - Refer to Spring 2011 Lecture 1
  - Generating functions and canonical transformations; e.g., $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$ such that

$$\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{q}} \; ; \; \bar{\boldsymbol{q}} = \frac{\partial F_2}{\partial \bar{\boldsymbol{p}}} \; ; \; \bar{H} = H + \frac{\partial F_2}{\partial t}$$

  - Hamilton-Jacobi equation
  - Notion of integrable and near integrable systems (Spring 2011 Lecture 2)

E: Review the concepts above and describe them in you own words.

The extended phase space

- The Least Action Principle, underlying the derivation of the equations of motion, can be rewritten introducing a parameter ζ, with respect to which all variations are independent; i.e.

$$\delta\left[\int\left(\boldsymbol{p}\frac{d\boldsymbol{q}}{d\zeta}-H(\boldsymbol{p},\boldsymbol{q},t)\frac{dt}{d\zeta}\right)d\zeta\right]=0$$

- From the form of the variational principle, it is intuitive that we can define an extended phase space setting in $2N+2$ dimensions by letting $\bar{p}_i = p_i$, $\bar{q}_i = q_i$ for $i = 1, N$ and $\bar{p}_{N+1} = -H$, $\bar{q}_{N+1} = t$.

- This corresponds to introducing the generating function

$$F_2 = \sum_{i=1}^{N} \bar{p}_i q_i - Ht \ \ ; \ \ \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = H(\boldsymbol{p}, \boldsymbol{q}, t) - H$$

- Equations of motion in the Hamiltonian form are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} \ \ ; \ \ \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i}$$

- $\bar{H}$ is independent of the parameter (time-like) ζ, so that $\bar{H} = \text{const}$, while $t(\zeta) = \zeta + \zeta_0 = \zeta$, for ζ_0 can be arbitrarily set to zero.

The reduced phase space

- Given the concept of the extended phase space, every system can be described as autonomous system, i.e. with an Hamiltonian that does not explicitly depend on time; i.e.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0$$

- Conversely, one could use this equation to solve one of the momenta, say p_N, as a function of $\bar{p}_i = p_i$ and $\bar{q}_i = q_i$ for $i = 1, N-1$ and $\zeta = q_N$ considered as a parameter; $p_N = p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$. Letting $\bar{H} = -p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$, the equations of motion in the $2N-2$ dimensional reduced phase space are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} = -\frac{1}{\dot{q}_N}\frac{\partial H}{\partial q_i} \quad ; \quad \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i} = \frac{1}{\dot{q}_N}\frac{\partial H}{\partial p_i}$$

- If the momentum p_N is a constant, since the original $H(\boldsymbol{p},\boldsymbol{q})$ did not explicitly depend on q_N, then $\bar{H}$ is constant and the reduced phase space can be further reduced.
- A very useful application of the notion of reduced phase space is the definition of a Poincaré surface of section.
- For a 2D system, the motion is bound to occur within the energy surface $H(p_1, p_2, q_1, q_2) = H_0$, which can be written as

$$p_2 = p_2(p_1, q_1, q_2)$$

- If the motion is bounded in the phase space, the motion can repeatedly cross the plane $q_2 = \text{const}$, which is a convenient choice of the surface of section, coinciding with the reduced phase space of the original Hamiltonian system.

- In general, the subsequent crossings of the motion in the (p_1, q_1) surface of section can occur everywhere. However, if a constant of motion exist, in addition to H_0, then $I(p_1, p_2, q_1, q_2) = \text{const}$ in addition to $p_2 = p_2(p_1, q_1, q_2)$. Therefore

$$p_1 = p_1(q_1, q_2)$$

  and the subsequent crossings of the motion in the (p_1, q_1) surface of section must lie on one curve.

- Vice-versa, the fact that the motion in the (p_1, q_1) surface of section lies on a curve can be used as evidence of the existence of a constant of motion in addition to H_0.

- More generally, for $N > 2$ degrees of freedom, we can construct the reduced phase space $p_N = p_N(\boldsymbol{p}, \boldsymbol{q}, q_N)$ and define the surface of section $q_N =$ const. In the reduced phase space, the motion is volume preserving. If one or more constants of motion exists, other than H_0, the system motion will lie on surfaces of dimensionality less than $2N - 2$, where trajectories will be volume preserving.

- If the motion is exactly separable in the (p_i, q_i) coordinates, the motion in each (p_i, q_i) plane is area preserving, a constant of motion (action integral) exists in each degree of freedom and the motion in each (p_i, q_i) plane lies on a smooth curve.

- In general, for a system with N degrees of freedom, the system trajectories projected in the (p_i, q_i) surface of section do not lie on a smooth curve but lie on an annulus of finite area, whose size is related with the nearness to the exact separability of the motion in the (p_i, q_i) coordinates.

Systems with one degree of freedom

- For an autonomous system with one degree of freedom, the Hamiltonian is conserved and is the energy of the system

$$H(p,q) = E$$

- All such systems are integrable. In fact

$$p = p(q,E) \;\; ; \;\; t = \int_{q_0}^{q} \frac{dq}{\dot{q}} = \int_{q_0}^{q} \frac{dq}{\partial H/\partial p}$$

- As notable example of systems with one degree of freedom, one can take the pendulum Hamiltonian.

The pendulum Hamiltonian

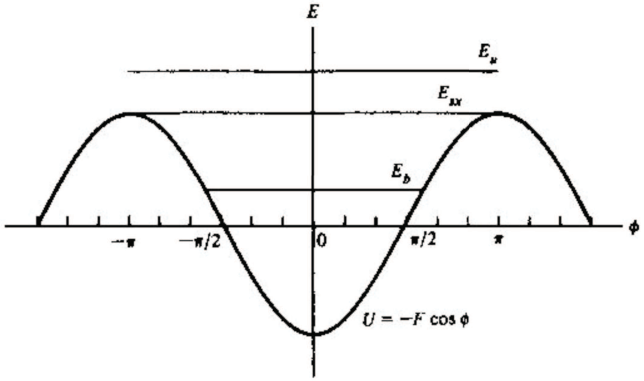
- Given ϕ, the angle from the vertical, and p the angular momentum conjugate to ϕ, the equations of motion of a pendulum of mass m and length h, with $F = mgh$ and $G = (mh^2)^{-1}$, are

$$\dot{p} = -F \sin\phi \;\; ; \;\; \dot{\phi} = Gp$$

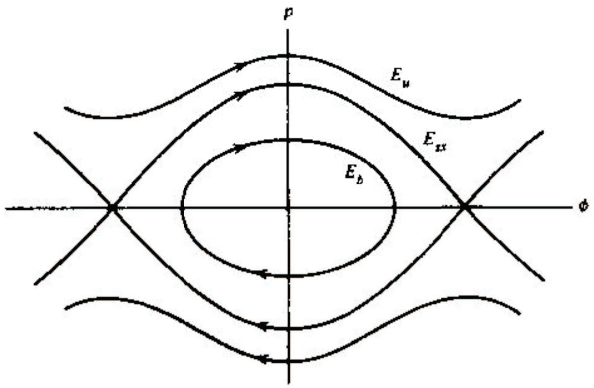
- The Hamiltonian is given by

$$H = \frac{1}{2} G p^2 - F \cos\phi = E$$

- It is useful to analyze the system behavior, comparing energy diagram and phase space diagram



(a)



(b)

Correspondence between energy diagram and phase space diagram for the pendulum Hamiltonian. (*Lichtenberg and Lieberman*, 2010)

For $E > E_{sx} = F$ the motion is unbounded (rotation). For $-F < E < E_{sx} = F$, the motion is bounded (libration). These motion are divided by a separatrix, where the oscillation period is infinite.

- The motion has a stable singular point at $p = 0, \phi = 0$ (trajectories near the singular point remain in its neighborhood) and an unstable singular point at $p = 0, \phi = \pm\pi$ (trajectories near the singular point diverge from it).

- The period is a function of the energy and can be computed in terms of elliptic integrals as

$$T = \oint \frac{d\phi}{\dot{\phi}} = \frac{1}{(2G)^{1/2}} \oint \frac{d\phi}{(E + F\cos\phi)^{1/2}}$$

- The period at the separatrix becomes infinite since both velocity and restoring force vanish at $\phi = \pm\pi$.

- Transformation to action-angle variables (J, θ) is obtained as

$$J(E) = \frac{\alpha}{\pi} \int_0^{\phi_{\max}} \left[\frac{2}{G}(E + F\cos\phi') \right]^{1/2} d\phi'$$

$$\theta(\phi, E) = \left(G \frac{dJ}{dE} \right)^{-1} \int_0^{\phi} \frac{d\phi'}{[(2/G)(E + F\cos\phi')]^{1/2}}$$

- Here, $\alpha = 1, \phi_{\max} = \pi$ for rotations and $\alpha = 2, \cos\phi_{\max} = -E/F$ for librations.

E: Fill in the missing details for deriving action-angle variables. Hint: derive the generating function $F_2 = F_2(\phi, J)$ and note that $J = J(E)$ with the new Hamiltonian given by $\bar{H} = E$.

- Introduce the variables $\kappa^2 = (1 + E/F)/2$ and $\sin\eta = \kappa^{-1}\sin(\phi/2)$ (for $\kappa < 1$), so that $\kappa < 1$ for librations and $\kappa > 1$ for rotations. Then, action-angle coordinates become (*Smith*, 1977, *Rechester*, 1979)

$$J = \frac{8}{\pi}\left(\frac{F}{G}\right)^{1/2}\begin{cases} \boldsymbol{E}(\kappa) - (1-\kappa^2)\boldsymbol{K}(\kappa) \;\; ; \;\; \kappa < 1 \\ (\kappa/2)\boldsymbol{E}(\kappa^{-1}) \;\; ; \;\; \kappa > 1 \end{cases}$$

with $\boldsymbol{K}(\kappa)$ and $\boldsymbol{E}(\kappa)$ the complete elliptic integrals of first and second kind

$$\boldsymbol{K}(\kappa) = \int_0^{\pi/2}\frac{d\eta}{\sqrt{1-\kappa^2\sin^2\eta}} \;\; ; \;\; \boldsymbol{E}(\kappa) = \int_0^{\pi/2}\sqrt{1-\kappa^2\sin^2\eta}\,d\eta$$

- The frequency of the motion can be obtained from $\omega^{-1} = (dJ/dE) = (4F\kappa)^{-1}(dJ/d\kappa)$: i.e., with $\omega_0 = (FG)^{1/2}$ the frequency of deeply trapped librations and noting $\kappa(d\boldsymbol{K}/d\kappa) = \boldsymbol{E}/(1-\kappa^2) - \boldsymbol{K}$, $\kappa(d\boldsymbol{E}/d\kappa) = \boldsymbol{E} - \boldsymbol{K}$

$$\frac{\omega}{\omega_0} = \frac{\pi}{2}\begin{cases} 1/\boldsymbol{K}(\kappa) \;\; ; \;\; \kappa < 1 \\ 2\kappa/\boldsymbol{K}(\kappa^{-1}) \;\; ; \;\; \kappa > 1 \end{cases}$$

- For $\kappa \to 0$, $\omega/\omega_0 \to 1$. Meanwhile, using the limiting expression of $\boldsymbol{K}(\kappa)$ as $\kappa \to 1$, it is clear that

$$\lim_{k\to 1}\frac{\omega}{\omega_0} = \begin{cases} (\pi/2)/\ln\left[\dfrac{4}{(1-\kappa^2)^{1/2}}\right] \;\; ; \;\; \kappa < 1 \\ \pi/\ln\left[\dfrac{4}{(\kappa^2-1)^{1/2}}\right] \;\; ; \;\; \kappa > 1 \end{cases}$$

E: Can you explain why the limiting expression for $\kappa > 1$ is twice that for $\kappa < 1$?

- From previous page expressions, for $\sin\eta = \kappa^{-1}\sin(\phi/2)$ (for $\kappa < 1$) and with $F(\phi,\kappa)$ the incomplete elliptic integral of the first kind, one can also obtain the angle conjugate to J as

$$\theta = \frac{\pi}{2}\begin{cases} F(\eta,\kappa)/\boldsymbol{K}(\kappa) & ; \quad \kappa < 1 \\ 2F(\phi/2,\kappa^{-1})/\boldsymbol{K}(\kappa^{-1}) & ; \quad \kappa > 1 \end{cases}$$

$$F(\phi,\kappa) = \int_0^\phi \frac{d\eta}{\sqrt{1-\kappa^2\sin^2\eta}}$$

E: Fill in the missing details and derive expressions above step by step.

- For the separatrix

$$p_{sx} = \pm(2\omega_0/G)\cos(\phi_{sx}/2) \quad ; \quad \dot{\phi}_{sx} = G p_{sx}$$

- Integrating in time, we have

$$\omega_0 t = \int_0^{\phi_{sx}/2} \frac{d\xi}{\cos\xi} = \ln\tan\left(\frac{\phi_{sx}}{4} + \frac{\pi}{4}\right) \quad ; \quad \phi_{sx} = 4\tan^{-1}\left[\exp(\omega_0 t)\right] - \pi$$

- The importance of the former study of the pendulum Hamiltonian comes from its role in the analysis of near-integrable systems with many degrees of freedom where a resonance between degrees of freedom exists.

- Since this Hamiltonian always emerges from Fourier analysis of the perturbation, in combination of secular perturbation theory and the method of averaging (see Spring 2011 Lectures) it has been said to provide the "universal description of a nonlinear resonance" (*Chirikov*, 1979) and is also called the "standard Hamiltonian" (*Lichtenberg*, 2010).

Removal of resonances: two degrees of freedom

- Near resonances of the unperturbed (integrable) Hamiltonian, resonant denominators appear when computing the first order adiabatic invariant calculated by classical perturbation theory.
- This problem is solve by introducing a canonical transformation to a frame that rotates with the resonant frequency.
- What is left is then the slow oscillation of new variables about their value at resonance.
- Assume a Hamiltonian in action-angle coordinates ($\boldsymbol{n} = (l, m)$)

$$H = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon \sum_{\boldsymbol{n}} H_{\boldsymbol{n}}(\boldsymbol{J}) e^{i\boldsymbol{n}\cdot\boldsymbol{\theta}}$$

- A primary resonance exists if

$$\frac{\omega_2}{\omega_1} = \frac{r}{s} \ : \ r, s \in \mathbb{Z} \ ; \ \omega_1(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_1} \ ; \ \omega_2(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_2}$$

- The notion of resonance can be extended from that of primary resonance to include also the case of secondary resonance created by harmonic frequencies of an island oscillation generated by the primary resonance.

- The resonance is eliminated with the canonical transformation

$$F_2 = (r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2 \hat{J}_2$$

$$J_1 = \frac{\partial F_2}{\partial \theta_1} = r\hat{J}_1 \ ; \ J_2 = \frac{\partial F_2}{\partial \theta_2} = \hat{J}_2 - s\hat{J}_1 \ ; \ \hat{\theta}_1 = \frac{\partial F_2}{\partial \hat{J}_1} = r\theta_1 - s\theta_2 \ ; \ \hat{\theta}_2 = \frac{\partial F_2}{\partial \hat{J}_2} = \theta_2$$

- The new variable $\hat{\theta}_1$ describes the slow deviation from resonance: $\dot{\hat{\theta}}_1 = r\dot{\theta}_1 - s\dot{\theta}_2$
- The choice of θ_2 is arbitrary. However, for computation of higher order resonances, is convenient to choose θ_2 as the slowest of the two original frequencies.
- The new Hamiltonian is then written as

$$\hat{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{l,m} H_{l,m}(\hat{\boldsymbol{J}}) e^{(i/r)[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]}$$

- One can average on the fast $\hat{\theta}_2$ variation in order to eliminate one degree of freedom

$$\bar{H} = \bar{H}_0(\hat{\boldsymbol{J}}) + \epsilon \bar{H}_1(\hat{\boldsymbol{J}}, \hat{\theta}_1) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \left\langle \bar{H}_1(\hat{\boldsymbol{J}}, \hat{\theta}_1) \right\rangle_{\hat{\theta}_2} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{p=-\infty}^{\infty} H_{-pr,ps}(\hat{\boldsymbol{J}}) e^{-ip\hat{\theta}_1}$$

- After averaging, the new Hamiltonian describes a system with only one degree of freedom; thus, it is integrable.

- The averaging is valid near resonance, where $|\dot{\hat{\theta}}_2| \gg |\dot{\hat{\theta}}_1|$.

- Since the new Hamiltonian does not depend on $\hat{\theta}_2$, one readily has $\hat{J}_2 =$ const.

$$\hat{J}_2 = J_2 + \frac{s}{r} J_1 = \text{const.}$$

- For the unperturbed Hamiltonian at the primary resonance, satisfied for a particular $\boldsymbol{J}$, periodic solutions are degenerate in $\boldsymbol{\theta}$ (they exist for all $\boldsymbol{\theta}$).

- The perturbation removes this degeneracy. In fact, periodic solutions exist for a stationary point or set of points in the $\hat{J}_1 - \hat{\theta}_1$ plane, for which

$$\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0 \;\; ; \;\; \frac{\partial \bar{H}}{\partial \hat{\theta}_1} = 0$$

- The Fourier harmonics $H_{-pr,ps}$ generally fall off rapidly as p increases. A good approximation of the Hamiltonian is obtained looking at the integrable motion with $p = 0, \pm 1$ only

$$\bar{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos \theta_1$$

- Fixed points are located at

$$\frac{\partial \bar{H}}{\partial \hat{\theta}_1} = 0 \quad \Rightarrow \quad \hat{\theta}_{10} = 0, \pi$$

- The resonance condition in $\hat{J}_{10}$ is obtained at

$$\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0$$

- At lowest and first order, respectively

$$\frac{\partial \hat{H}_0}{\partial \hat{J}_{10}} = r\frac{\partial H_0}{\partial J_1} - s\frac{\partial H_0}{\partial J_2} = r\omega_1 - s\omega_2 = 0$$

$$\delta \hat{J}_{10}\frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} + \epsilon\frac{\partial H_{0,0}}{\partial \hat{J}_{10}} \pm 2\epsilon\frac{\partial H_{r,-s}}{\partial \hat{J}_{10}} = 0$$

- The Hamiltonian is said to be accidentally degenerate if resonance is satisfied for particular values of J_1, J_2. In this case, the unperturbed Hamiltonian in the new coordinates is function of both actions

$$\hat{H}_0 = \hat{H}_0(\hat{J}_1, \hat{J}_2)$$

- The Hamiltonian is said to be intrinsically degenerate if resonance is satisfied for all values of J_1, J_2. In this case, $H_0 = H_0(sJ_1 + rJ_2)$ and the unperturbed Hamiltonian in the new coordinates is independent of $\hat{J}_1$

$$\hat{H}_0 = \hat{H}_0(\hat{J}_2)$$

- Secondary resonances are almost always accidental.

Accidental degeneracy

- Given the Hamiltonian

$$\bar{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos\theta_1$$

one can estimate the excursions in the $\hat{J}_1 - \hat{\theta}_1$ plane

$$\dot{\hat{J}}_1 = \mathcal{O}(\epsilon H_{r,-s}) \;\; ; \;\; \dot{\hat{\theta}}_1 = \mathcal{O}(1)$$

- It is legitimate to expand about the stationary point in $\hat{J}_1$, i.e. $\Delta\hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$

$$\bar{H}_0(\hat{\boldsymbol{J}}) = \hat{H}_0(\hat{\boldsymbol{J}}_0) + \frac{\partial \hat{H}_0}{\partial \hat{J}_{10}} \Delta\hat{J}_1 + \frac{1}{2}\frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} \Delta\hat{J}_1^2 + \ldots$$

where the linear term must be dropped for the definition of $\hat{J}_{10}$

- Neglecting constant terms and keeping lowest order terms in ϵ and $\Delta\hat{J}_1$, the universal description of the motion near a resonance is given by the *standard Hamiltonian*

$$\Delta\bar{H} = \frac{1}{2}G\Delta\hat{J}_1^2 - F\cos\hat{\theta}_1$$

with the *nonlinear parameter* G and the perturbation strength F given by

$$G(\hat{\boldsymbol{J}}_0) = \frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} \quad ; \quad F(\hat{\boldsymbol{J}}_0) = -2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}_0)$$

- For $GF > 0$, the stable fixed point is at $\hat{\theta}_1 = 0$ and the frequency of librations near the stable point is (see p.16)

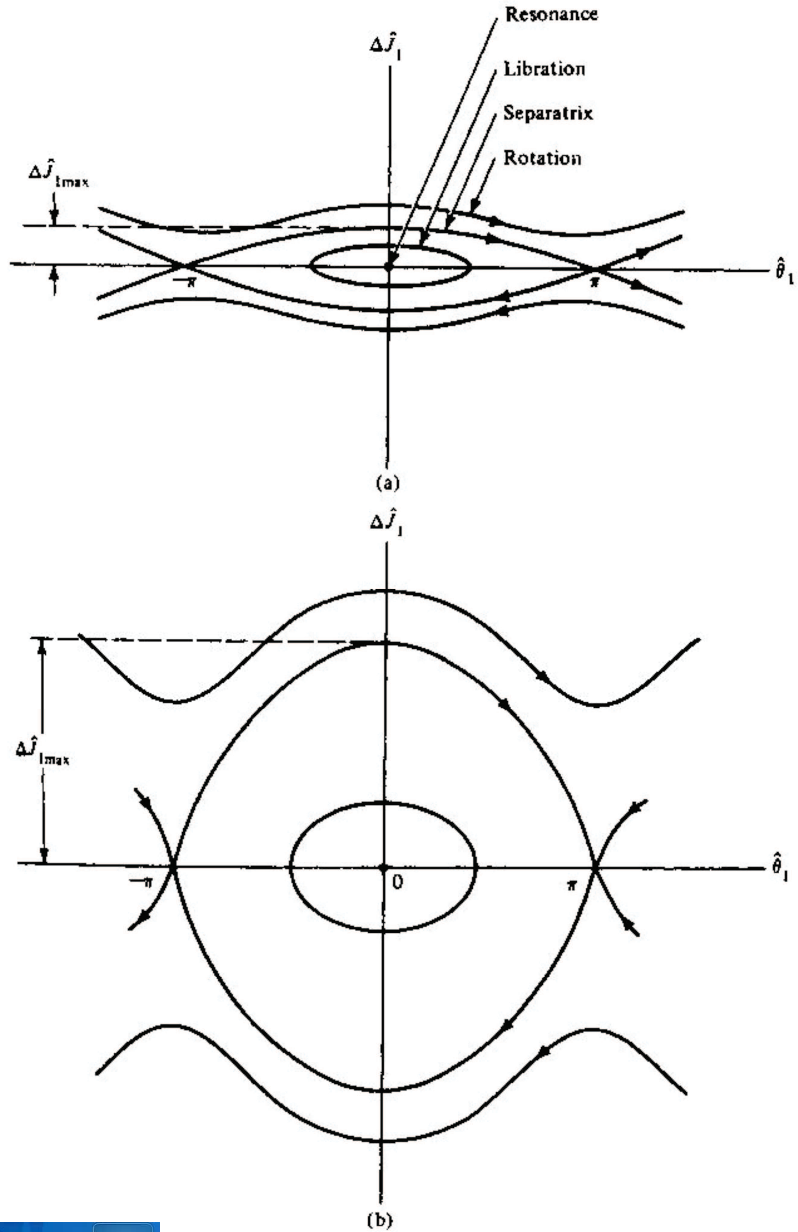
$$\hat{\omega}_1 = (FG)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$

- The maximum excursion $\Delta \hat{J}_{1\max}$ is given by half the separatrix width and is given by (see p.15)

$$\Delta \hat{J}_{1\max} = 2 \left(\frac{F}{G} \right)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$

- Similarly, near the stable fixed point, trajectories are elliptic with

$$\frac{\Delta \hat{J}_1}{\Delta \hat{\theta}_1} = \left(\frac{F}{G} \right)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$



Adiabatic motion near an isolated resonance (*Lieberman and Lichtenberg*, 2010).

(a) accidental degeneracy: maximum excursion in action and libration frequency are $\mathcal{O}(\epsilon^{1/2})$

(b) intrinsic degeneracy: maximum excursion in action is large while libration frequency is low $\mathcal{O}(\epsilon)$ (see Spring 2011 Lecture 4)

Higher order resonances

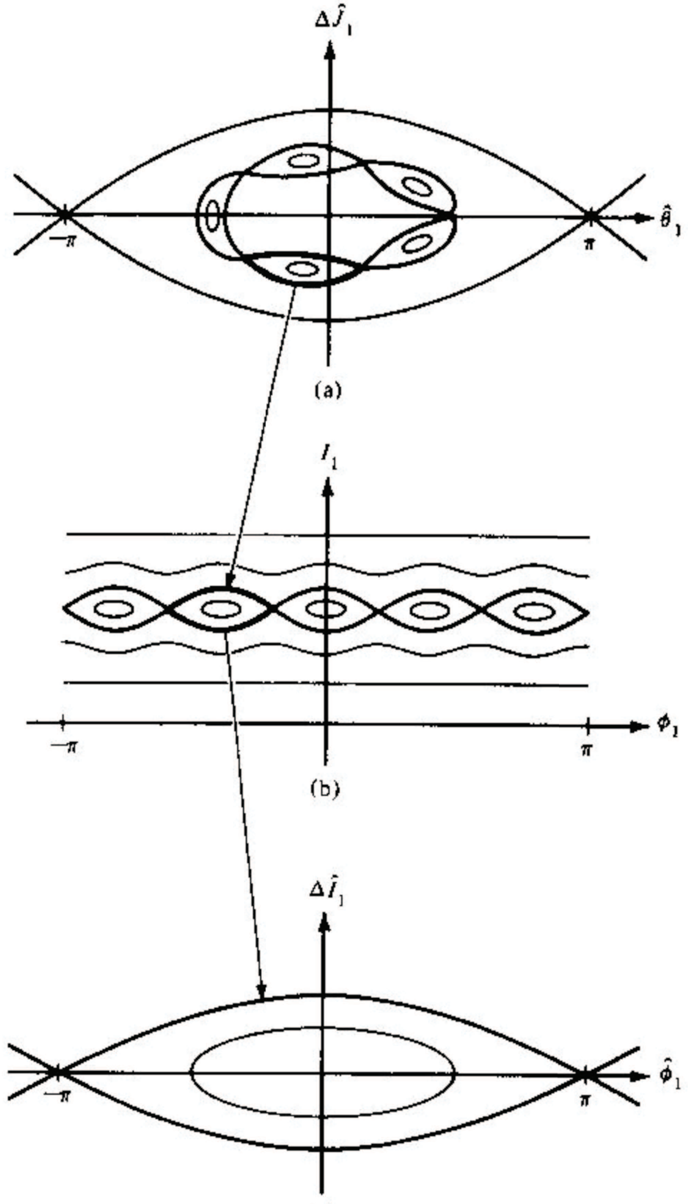
- If ϵ is not sufficiently small, secondary resonances are present in the Hamiltonian

$$\hat{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{l,m} H_{l,m}(\hat{\boldsymbol{J}}) e^{(i/r)[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]}$$

  that destroy the adiabatic invariant $\hat{J}_2$.

- These secondary resonances are between harmonics of the motion in the $\hat{J}_1 - \hat{\theta}_1$ plane with the fundamental frequency ω_2.

- In the adiabatic limit, these resonances give rise to island chains, while the resonances can be treated as previously.



Motion near a secondary resonance (*Lieberman and Lichtenberg*, 2010).

(a) secondary island chain for $5\hat{\omega}_1 = \hat{\omega}_2$

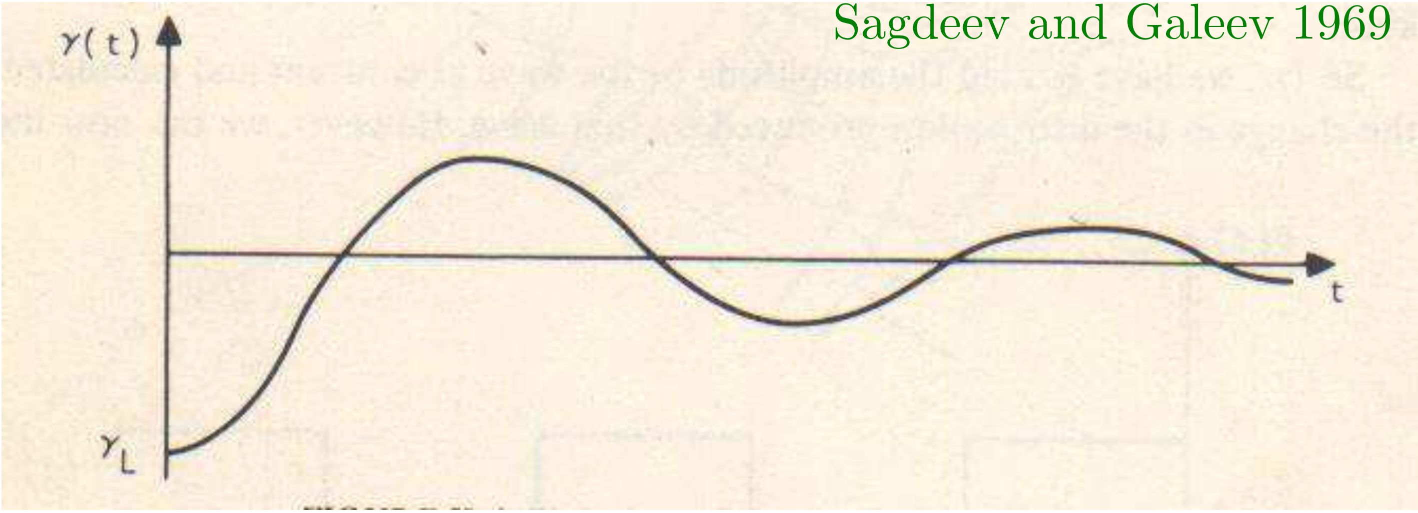
(b) transformation to action-angle coordinates of the unperturbed primary libration

(c) transformation to rotating coordinate system associated with secondary librations (see Spring 2011 Lecture 4)

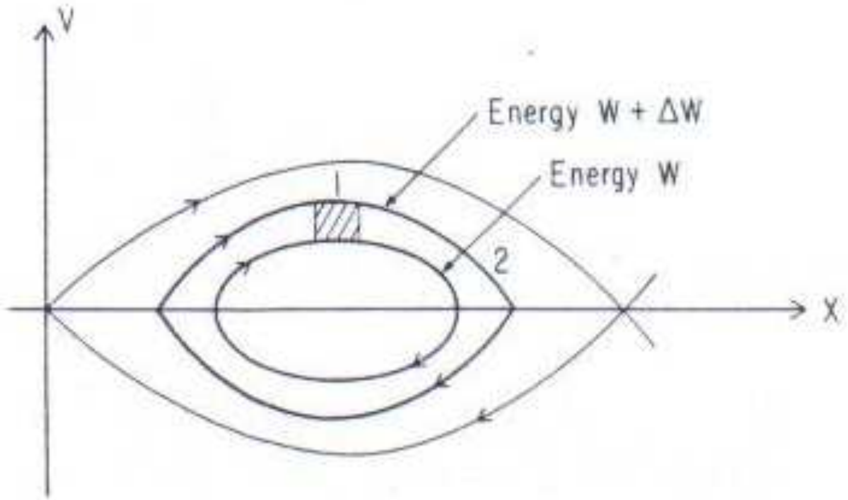
Wave-particle trapping: density flattening by phase mixing

- In the nonlinear beam-plasma system, Langmuir-wave excited by beam particles saturates because of wave-particle trapping, which eventually causes flattening of particle distribution function by phase mixing, demonstrated by oscillation and decay of drive/damping (see Spring 2015 Lectures)

Sagdeev and Galeev 1969



E: Comment the time behavior above of Landau damping in a large amplitude wave and discuss similarity and differences with the saturation of a beam-plasma instability. Hint: see Spring 2015 Lectures

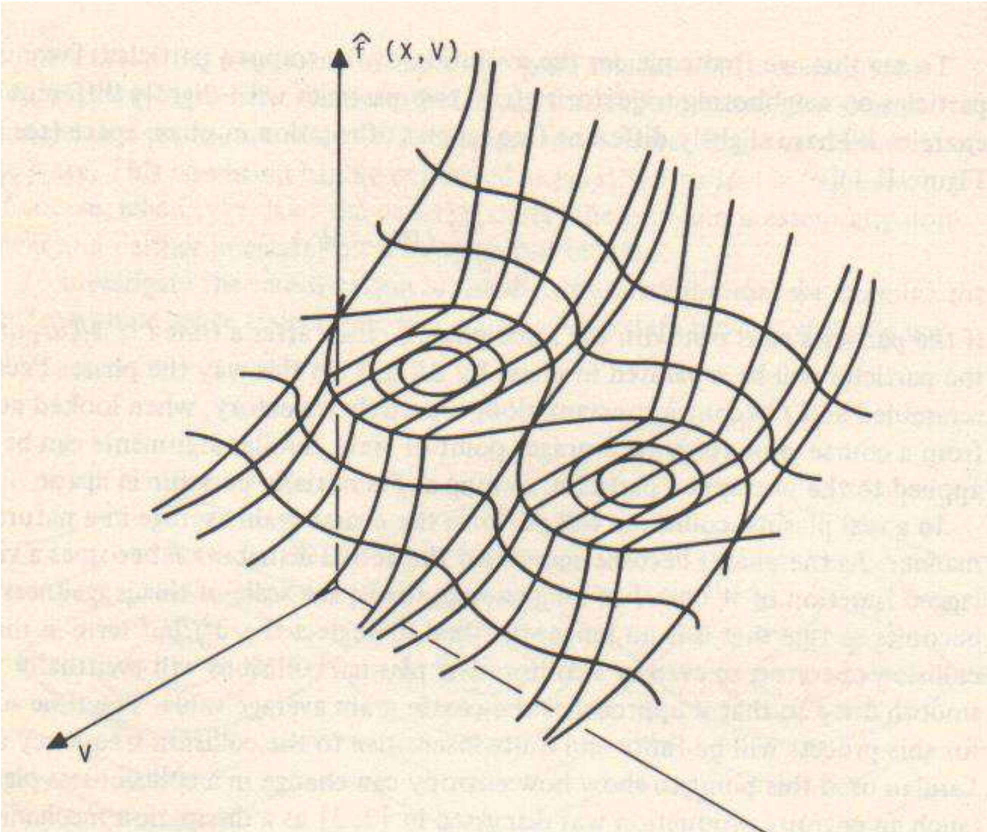


Particle trapping (period τ_B) is what causes the oscillation of the linear damping rate (γ_L). Phase mixing is the collisionless process underlying the vanishing of linear damping on long time scales.

- This calculation of nonlinear Landau damping assumes that the level of fluctuation is sufficiently constant in time. This means that

$$\int_0^\infty \gamma(t)dt \propto \gamma_L \tau_B \ll 1$$

- The two asymptotic limits that can be investigated are $\gamma_L \tau_B \gg 1$ (linear theory) and $\gamma_L \tau_B \ll 1$ (nonlinear damping of constant amplitude wave).



Sagdeev and Galeev 1969

Time asymptotically, the distribution reaches the coarse-grain distribution. With $\Delta W = (\partial W/\partial v)\Delta v = \text{const}$

$$f = \frac{\oint f_0(v)\Delta v d\theta}{\oint \Delta v d\theta}$$

$$f = f_0(0) + \frac{\partial f_0(0)}{\partial v}\frac{\oint d\theta/k}{\oint d\theta/\dot{\theta}}$$

For $\kappa^2 > 1$, $f = f_0(0)$. For $\kappa^2 < 1$

$$f = f_0(0) + \frac{\partial f_0(0)}{\partial v}\frac{\pi}{\kappa \tau_B k F}$$

E: Show that the coarse-grain distribution is continuous at the separatrix $\kappa^2 = 1$ but it has discontinuous derivatives. Show that the time for a complete oscillation diverges approaching the separatrix. What type of divergence is that?

References and reading material

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