Lecture 6

Quasi-linear theory and diffusive transport

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Random walk and diffusion

- Brief summary of Einstein's theory of Brownian motion.
- Assume that individual particle motions are independent and that the displacements of the same particle at different times are also independent.
- This is possible only if time interval between two displacements is not too small. Thus, introduce τ small compared to observation time by sufficiently long for the assumption on independent displacement to be valid.
- In a 1D model, introduce the probability density function (PDF) $\varphi(\Delta)$ so that

$$\frac{dN}{d\Delta} = N\varphi(\Delta)$$

with dN the number of particles N experiencing a displacement between Δ and $\Delta + d\Delta$ over the interval τ.

- Properties of the PDF $\varphi(\Delta)$

$$\int_{-\infty}^{\infty} \varphi(\Delta) d\Delta = 1 \; ; \qquad \text{and} \qquad \varphi(-\Delta) = \varphi(\Delta) \; .$$

- Introduce the 1D particle density $n(x,t)$. By definition

$$n(x, t+\tau) = \int_{-\infty}^{\infty} n(x+\Delta, t) \varphi(-\Delta) d\Delta = \int_{-\infty}^{\infty} n(x+\Delta, t) \varphi(\Delta) d\Delta \; .$$

- Taylor expand:

$$n(x+\Delta, t) = n(x,t) + \Delta \partial_x n(x,t) + (\Delta^2/2) \partial_x^2 n(x,t) + \ldots \; .$$

$$n(x, t+\tau) = n(x,t) + \tau \partial_t n(x,t) + \ldots \; .$$

- Substituting into previous equation:

$$n(x,t)+\tau\partial_t n(x,t)+... = \int_{-\infty}^{\infty} \left(n(x,t) + \Delta\partial_x n(x,t) + (\Delta^2/2)\partial_x^2 n(x,t) + ...\right)\varphi(\Delta)d\Delta \; .$$

- Introducing the average squared displacement

$$\bar{\Delta}^2 = \int_{-\infty}^{\infty} \Delta^2\varphi(\Delta)d\Delta \; ,$$

and the diffusion coefficient $D = \bar{\Delta}^2/(2\tau)$, this yields the diffusion equation

$$\partial_t n(x,t) = \bar{\Delta}^2/(2\tau)\partial_x^2 n(x,t) = D\partial_x^2 n(x,t) \; .$$

E: Derive the diffusion equation step by step. Demonstrate that the Green's function of the diffusion equation is $G(x,t) = (4\pi Dt)^{-1/2}\exp[-x^2/(4Dt)]$.

Propagator of the diffusion equation

- The solution of the diffusion equation with general initial condition $n_0(x) = n(x, t = 0)$ can be obtained from the propagator, or Green's function $G(x, t)$

$$n(x,t) = \int_{-\infty}^{\infty} G(x - x', t) n_0(x') dx' \ .$$

E: Demonstrate that $G(x, t)$ must satisfy the differential equation $\partial_t G = D\partial_x^2 G$ with initial condition $G(x, t = 0) = \delta(x)$.

Hint: Demonstration follows by direct substitution.

E: Do the exercise at the previous page and write $G(x, t)$ explicitly, moving to the Fourier space for representing $\partial_x^2 = -k^2$.

E: Show that the mean-squared displacement is $\langle x^2 \rangle = 2Dt$.

Beyond the diffusion paradigm

- A general characterization of transport can be made by the mean-squared displacement and introducing the transport exponent

$$\left\langle \left(\boldsymbol{r}(t) - \langle \boldsymbol{r}(t) \rangle \right)^2 \right\rangle \sim t^\gamma \; .$$

- Depending on the value of γ, we have the following:
  - $\gamma < 1$: subdiffusive transport
  - $\gamma = 1$: classical diffusion
  - $\gamma > 1$: superdiffusive transport
  - $\gamma = 2$: ballistic transport

- Discussing theories on non-diffusive transport processes is beyond the scope of the present Lecture series.

- General approaches to non-diffusive transport are based on statistical physics and on the introduction of the jump PDF $\psi(x,t)$; e.g. in the theory of continuous time random walk (CTRW; Montroll & Weiss 1965).

- In the case step size and waiting time are independent random variables, $\psi(x,t) = \lambda(x)w(t)$, with $\lambda(x)$ the step size PDF and $w(t)$ the waiting time PDF. Thus,
$$\bar{\Delta}^2 = \int_{-\infty}^{\infty} x^2 \lambda(x) dx \; , \qquad \tau = \int_0^{\infty} t w(t) dt \; .$$

- Another approach is based on the generalized Langevin equation (GLE)
$$\ddot{x}(t) = -\int_0^t \beta(t-t')\dot{x}(t')dt' + \xi(t) \; ,$$
with $\beta(t)$ the memory kernel and $\xi(t)$ a random force such that $\langle \xi(t_1)\xi(t_2) \rangle = C(|t_1 - t_2|) = C(\tau)$ (Kubo 1966).

- The classical Langevin equation is written as (γ represents friction)
$$\ddot{x}(t) = -\gamma \dot{x}(t) + \xi(t) \; .$$

From Dyson to diffusion equation for the beam-plasma problem

- Brief summary from Lecture 4 (p.14).
- Consider the Laplace representation for weakly varying fluctuation spectrum and rewrite Dyson equation and Poisson equation

$$\hat{f}_0(\omega) = \frac{i}{2\pi\omega} F_0 - \frac{e^2}{m^2}\frac{k_0^2}{\omega}\left|\delta\phi_{k_0}(\tau)\right|^2 \frac{\partial}{\partial v}\left[\left(\frac{1}{\omega - \omega_{k_0} + k_0 v} + \frac{1}{\omega + \omega_{k_0}^* - k_0 v}\right)\frac{\partial}{\partial v}\hat{f}_0(\omega - 2i\gamma_{k_0})\right]$$

$$\frac{2}{\omega_p}\left[\frac{\partial}{\partial\tau} + i(\omega_p - \omega_{k_0})\right]\delta\phi_{k_0}(\tau) = \frac{\omega_p^2}{nk_0} i \int dv \int_{-\infty}^{+\infty} e^{-i\omega t}\frac{\delta\phi_{k_0}(\tau)}{\omega + \omega_{k_0} - k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega) d\omega \quad .$$

- Solutions of this system strongly depend on the properties of the fluctuation spectrum; and on the properties of $\hat{f}_0(\omega)$, which can have a broad spectrum (this lecture) or a narrow spectrum (Lecture 4).

- In both cases, we assumed that characteristic nonlinear time is sufficiently short compared to the inverse growth time of the fluctuations $|\gamma_{k_0}|^{-1}$.
- This means (Lecture 4) that on the RHS of the Dyson equation we can take $\hat{f}_0(\omega - 2i\gamma_{k_0}) \rightarrow \hat{f}_0(\omega) \Rightarrow$ Reduced equation is readily rewritten in the t-representation.
- Important physics implications arise when $\tau_{NL} \sim |\gamma_{k_0}|^{-1}$ for a narrow spectrum. This is the phase locking regime (Zonca et al. 2015), of interest for energetic particle interactions with Alfvén waves in fusion plasmas, yielding ballistic transport: e.g., by fishbones and energetic particle modes (EPM) (Lecture 4).

E: Revisit RPs on p. 11 and p. 14 of Lecture 4; and use them as starting point for further analyses and research on superdiffusive and ballistic transport. Can energetic electron transport be ballistic in the beam plasma problem?

Broad spectrum: quasilinear diffusion equation

- Consistent with above discussion, consider the case

$$\frac{1}{\tau_{NL}} \sim |\omega| \ll |\omega_{k_0} - k_0 v| \sim \left| \omega_{k_0} \frac{\Delta k}{k_0} \right| \sim |k_0 \Delta v| \quad .$$

- In this limit ($\omega \to i0^+$)

$$(\omega - \omega_{k_0} + k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v) \;\; ; \qquad (\omega + \omega^*_{k_0} - k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v).$$

- The evolution equation for f_0 becomes (quasilinear diffusion equation)

$$\frac{\partial}{\partial t} f_0(v,t) = \frac{e^2}{m^2} \sum_{k_0} k_0^2 \left| \delta\phi_{k_0}(\tau) \right|^2 \frac{\partial}{\partial v} \left[\pi\delta(\omega_{k_0} - k_0 v) \frac{\partial}{\partial v} f_0(v,t) \right] \quad .$$

E: Derive the evolution equation of $f_0(v,t)$ step by step, symmetrizing $k_0 \leftrightarrow -k_0$.

- The evolution equation of $f_0(v,t)$ is a diffusion equation in v-space, with diffusion coefficient ($\sum_{k_0}$ accounts for the whole k_0 spectrum)

$$D_v = \pi \frac{e^2}{m^2} \sum_{k_0} k_0^2 \left| \delta\phi_{k_0}(\tau) \right|^2 \delta(\omega_{k_0} - k_0 v) \quad .$$

- For consistency with the broad spectrum assumption, we must have

$$\frac{1}{\tau_{NL}} \sim |\omega| \sim \left| \frac{D_v}{(\Delta v)^2} \right| \ll |k_0 \Delta v| \quad ; \quad \Rightarrow \quad \left| \frac{e}{m} \delta\phi_{k_0}(\tau) \right|^{1/2} \ll |\Delta v| \quad .$$

- This is the typical validity limit of quasilinear description.

E: Demonstrate that this condition is equivalent to having $|\gamma_{k_0}|^{-1} \gg \tau_{NL} \gg \omega_B^{-1}$, with ω_B the wave-particle trapping frequency; and that transport takes place because of resonance overlap.

Conservation laws

- Note that we have introduced approximations when deriving the quasilinear diffusion equation on p. 10:

$$(\omega - \omega_{k_0} + k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v) \quad ; \qquad (\omega + \omega^*_{k_0} - k_0 v)^{-1} \to -i\pi\delta(\omega_{k_0} - k_0 v).$$

- More precisely, we have (after shifting $\omega \to \omega + 2i\gamma_{k_0}$ and $\hat{f}_0(\omega - 2i\gamma_{k_0}) \to \hat{f}_0(\omega)$ on p. 8) (Al'tshul' 66):

$$\frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} - \omega_{k_0} + k_0 v} + \frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} + \omega^*_{k_0} - k_0 v} \simeq \frac{(2i\gamma_{k_0} + 2\omega)|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{(\omega + i\gamma_{k_0})^2 - (\omega_{k_0 r} - k_0 v)^2}$$

- Therefore:

$$\frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} - \omega_{k_0} + k_0 v} + \frac{|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}}{\omega + 2i\gamma_{k_0} + \omega^*_{k_0} - k_0 v} \simeq -2i\pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2 e^{-i\omega t}$$

$$+ i\frac{\partial}{\partial\omega_{k_0 r}}\mathcal{P}\left(\frac{1}{\omega_{k_0 r} - k_0 v}\right)\left(e^{-i\omega t}\partial_t|\delta\phi_{k_0}(t)|^2 + 2|\delta\phi_{k_0}(t)|^2\partial_t e^{-i\omega t}\right)$$

- Substituting back into $\hat{f}_0(\omega)$ equation at p.8, symmetrizing $k_0 \leftrightarrow -k_0$, and moving to t-representation:

$$\frac{\partial}{\partial t} f_0 = \frac{e^2}{m^2} \sum_{k_0} k_0^2 \frac{\partial}{\partial v} \left[\pi\delta(\omega_{k_0 r} - k_0 v)) |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 \right.$$
$$\left. - \frac{\partial}{\partial \omega_{k_0 r}} \mathcal{P} \left(\frac{1}{\omega_{k_0 r} - k_0 v} \right) \left(\frac{1}{2} \partial_t |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 + |\delta\phi_{k_0}(t)|^2 \frac{\partial^2}{\partial v \partial t} f_0 \right) \right]$$

- The last two terms are respectively $\sim (\gamma_{k_0}/k_0 \Delta v)$ and $\sim (\tau_{NL}^{-1}/k_0 \Delta v)$ with respect to the quasilinear diffusion on the RHS, derived on p.10. Thus, they are both small.

- It makes sense to consider the adiabatic change of mode amplitude only if it is significant over a characteristic transport time scale.

E: Demonstrate that this condition is equivalent to having $\tau_{NL} \gg |\gamma_{k_0}|^{-1} \gtrsim \omega_B^{-1}$. What is the difference with the ordering considered on p. 11? Is transport always due to resonance overlap?

- Therefore, while the last term is typically dropped, the second last term is kept into account by quasilinear theory for f_0 evolution, which then reads

$$\frac{\partial}{\partial t} f_0 = \frac{e^2}{m^2} \sum_{k_0} k_0^2 \frac{\partial}{\partial v} \left[\pi \delta(\omega_{k_0 r} - k_0 v) |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 \right.$$
$$\left. - \frac{1}{2} \frac{\partial}{\partial \omega_{k_0 r}} \mathcal{P} \left(\frac{1}{\omega_{k_0 r} - k_0 v} \right) \partial_t |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 \right]$$

- Use now Poisson's equation at p. 8 to demonstrate

$$\omega_p^{-1} \partial_t |\delta\phi_{k_0}(t)|^2 = \frac{\omega_p^2}{n k_0} \int dv \pi \delta(\omega_{k_0 r} - k_0 v) |\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0$$

and use this result to demonstrate momentum and energy conservation.

E: Demonstrate that particle conservation follows immediately from the quasilinear diffusion equation above. Why?

- Use the same approach adopted in Lecture 2 (pp. 10-12).
- Momentum conservation (considering only the dominant diffusion term), integrating by parts and using last equation on p. 14

$$\frac{dP_z}{dt} = \int_{-\infty}^{\infty} mv\partial_t f_0 = -\frac{e^2}{m}\sum_{k_0} k_0^2 \int_{-\infty}^{\infty} \pi\delta(\omega_{k_0 r} - k_0 v)|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 dv$$

$$= -\frac{ne^2}{m\omega_p^2}\sum_{k_0} \frac{k_0}{\omega_p}\frac{d}{dt}|E_{k_0}(t)|^2 = -\sum_{k_0}\frac{k_0}{\omega_{k_0}}\frac{d}{dt}\frac{|E_{k_0}(t)|^2}{4\pi}$$

having noted that $\omega_{k_0} = \omega_p$ for Langmuir waves and $|E_{k_0}(t)|^2 = k_0^2|\delta\phi_{k_0}(t)|^2$.

E: Compare this momentum conservation law with that obtained on p. 11 Lecture 2. Are they the same? Why?

- Energy conservation (considering only the dominant diffusion term), integrating by parts and using last equation on p. 14

$$
\begin{aligned}
\frac{d\mathcal{E}}{dt} &= \int_{-\infty}^{\infty} m\frac{v^2}{2}\partial_t f_0 = -\frac{e^2}{m}\sum_{k_0} k_0^2 \int_{-\infty}^{\infty} \pi\delta(\omega_{k_0 r} - k_0 v)v|\delta\phi_{k_0}(t)|^2 \frac{\partial}{\partial v} f_0 dv \\
&= -\frac{ne^2}{m\omega_p^2}\sum_{k_0}\frac{\omega_{k_0}}{\omega_p}\frac{d}{dt}\left|E_{k_0}(t)\right|^2 = -\sum_{k_0}\frac{d}{dt}\frac{\left|E_{k_0}(t)\right|^2}{4\pi}
\end{aligned}
$$

having noted that $\omega_{k_0} = \omega_p$ for Langmuir waves and $\left|E_{k_0}(t)\right|^2 = k_0^2|\delta\phi_{k_0}(t)|^2$.

E: Compare this energy conservation law with that obtained on p. 12 Lecture 2. Are they the same? Why?

□ Some considerations of theoretical and practical importance:

- The present conservation laws are consistent with those derived in Lecture 2, but have been obtained neglecting the principal value term in the quasilinear equation for f_0
- The neglected term represents the non-resonant wave-particle interactions.

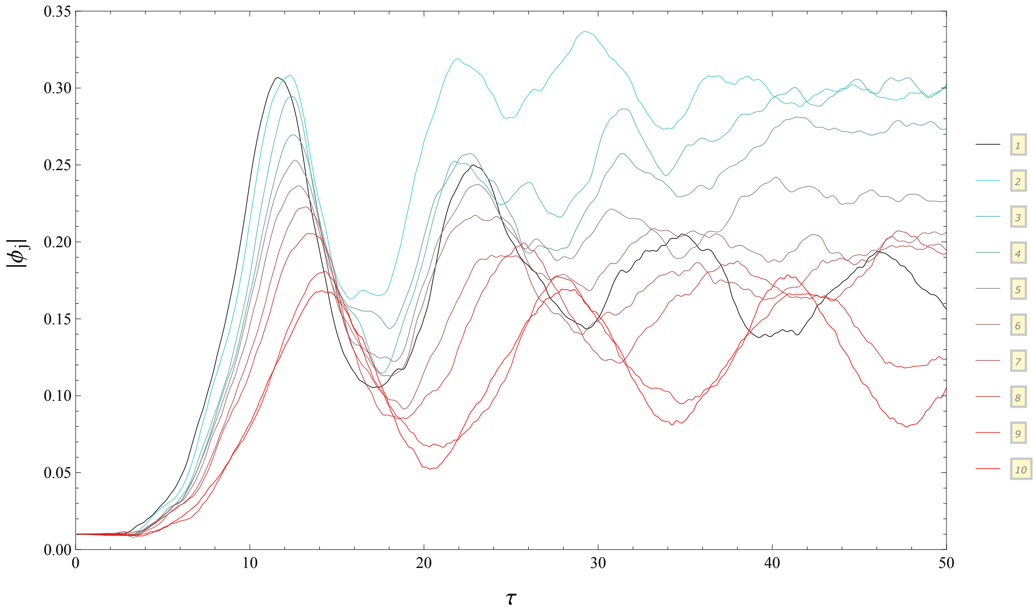
E: Convince yourself that the neglected term is indeed small. Can you quantify how small is small?

E: Show that the neglected non-resonant interaction term is invariant under time reversal. What does that mean? What about the dominant diffusion term kept into account and describing resonant wave-particle interactions?

E: Derive step by step the non-resonant term that was neglected. What is the structure of that term with respect to the analogous term that you could derive for the thermal plasma electrons? What can you conclude about thermal plasma and beam electron non-resonant interactions?

Multi-beam redistribution[1]

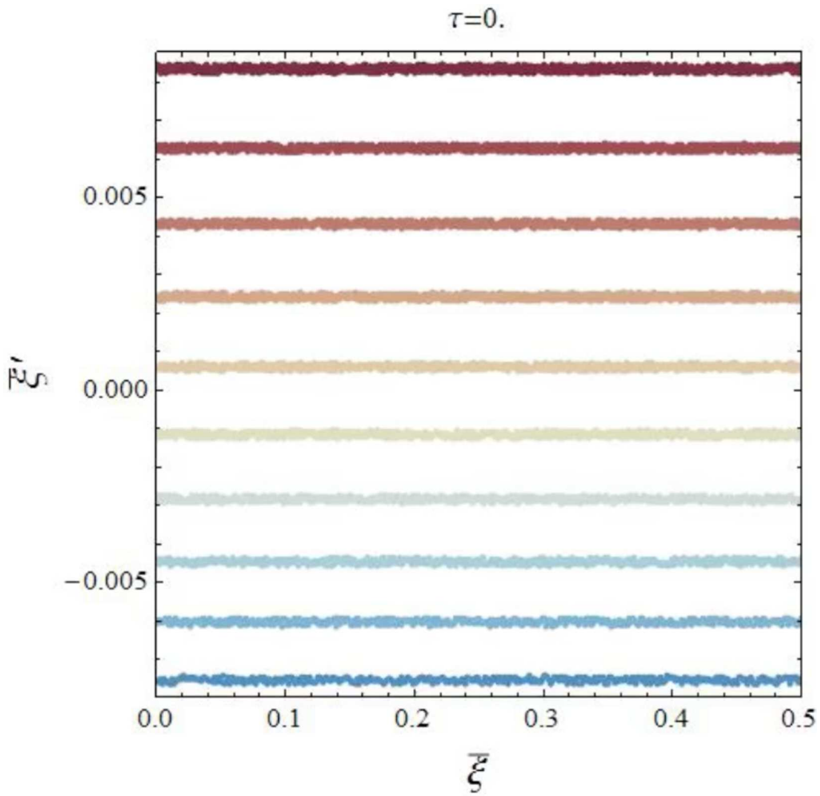
- Choose a set of resonant beams such that $k_j = \omega_p/v_j$. The beam densities σ_j are taken to increase linearly from the slowest beam (n-th) to the fastest. Beam velocities are defined such, $v_j = \Theta_j v_1$, and Θ_j are taken to have $k_{j+1} - k_j = \Delta k =$ const. (other choices are equally good).
- For 10 beams, $\sigma_1 = 0.15$ and $\sigma_{10} = 0.05$; $\ell_1 = 1001$ ($\Theta_1 = 1$) and $\ell_{10} = 1190$ ($\Theta_{10} \simeq 0.841$). Total number of particles is 10^5.



[1]Acknowledgment: based on inputs by Nakia Carlevaro (private communication, 2015).

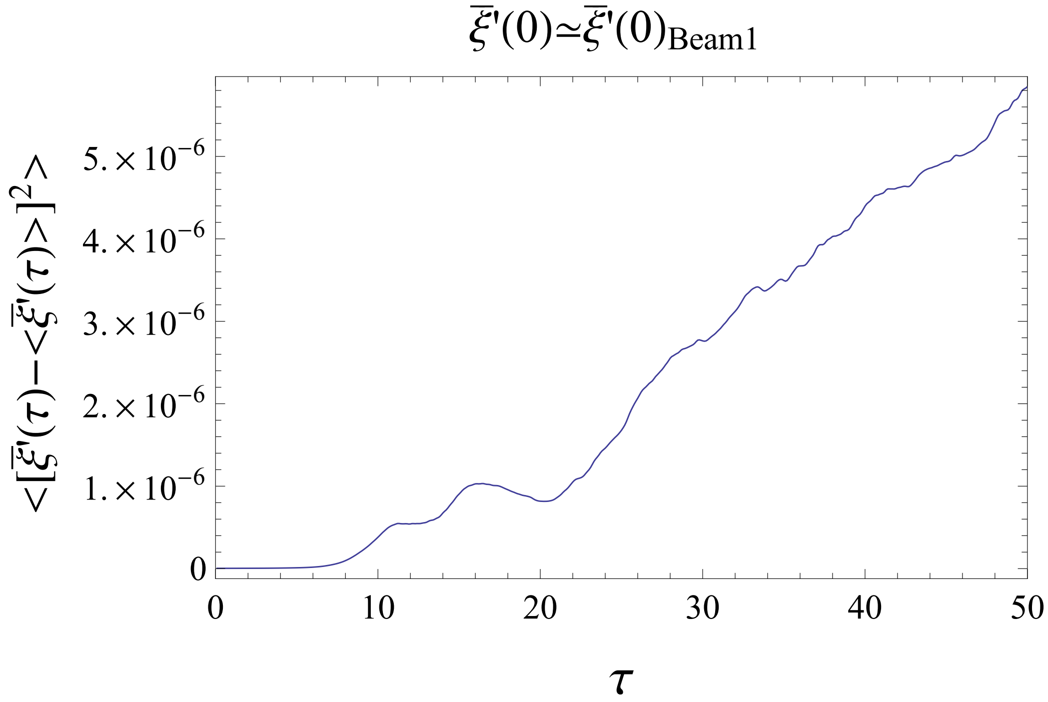
E: Comment about the saturation amplitude and energy transfer between different beam modes in the light of the results with two beams, including sidebands of resonant modes, discussed in Lecture 5

- Evidence of phase space diffusion:

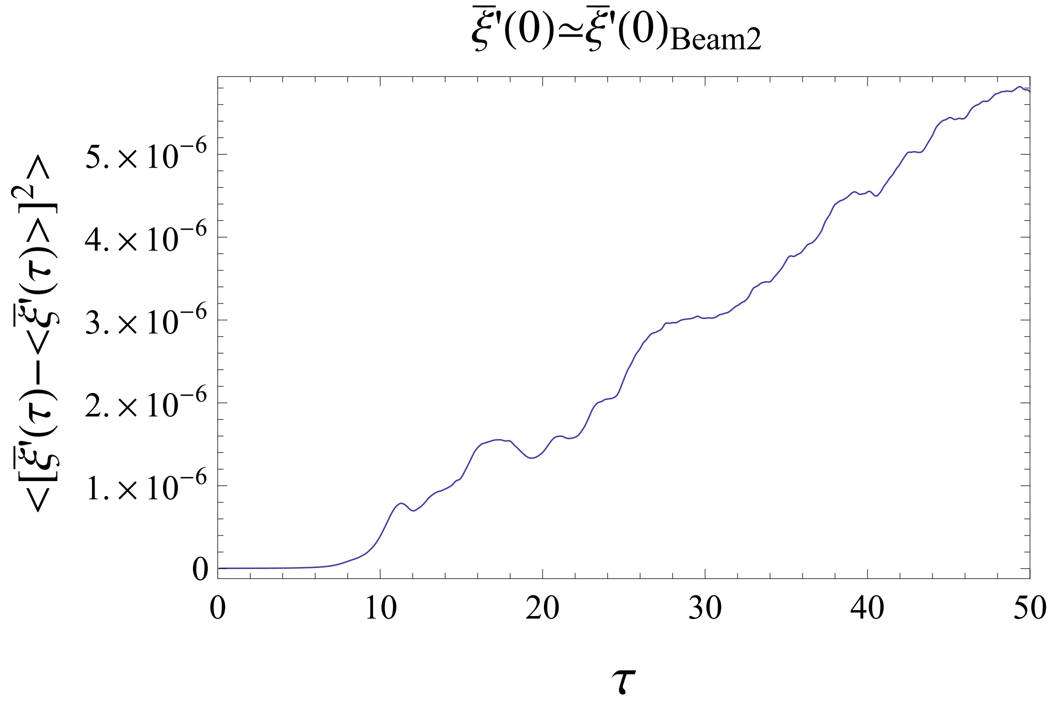


Transport exponent analysis

Beam 1

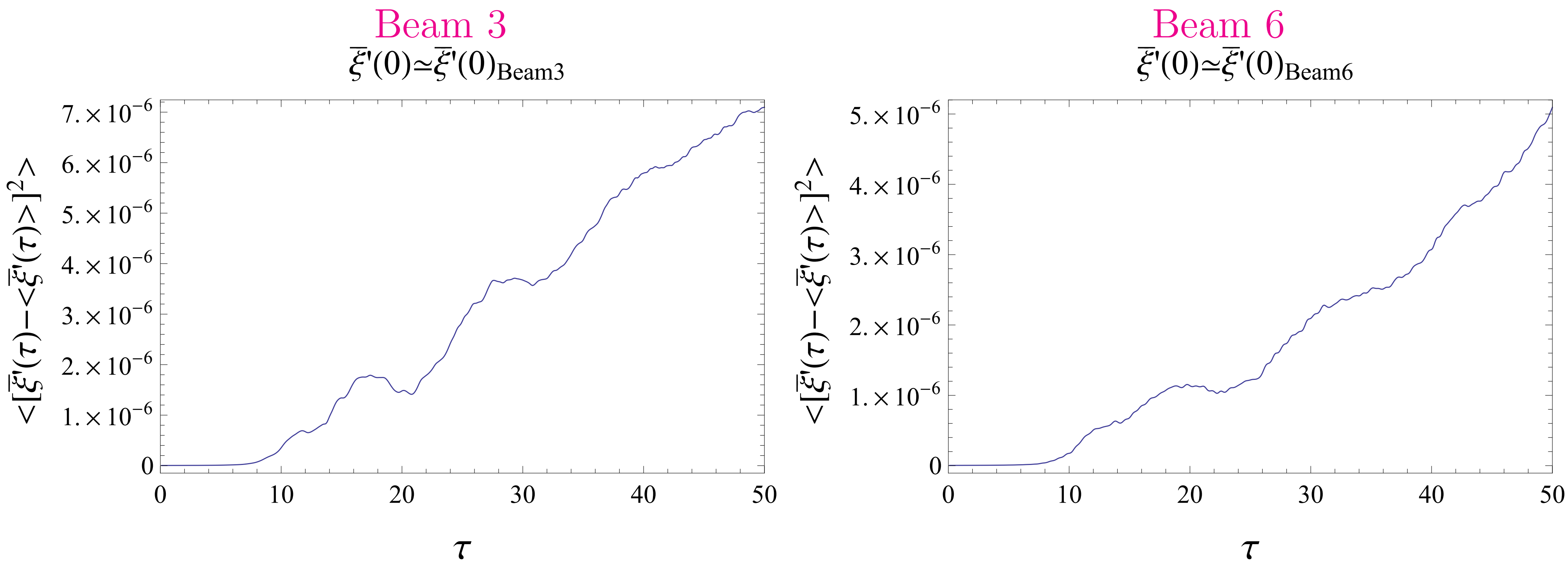


Beam 2



- Ensemble average performed over 10^3 tracers.

E: Can you identify the signature of diffusion? Why?

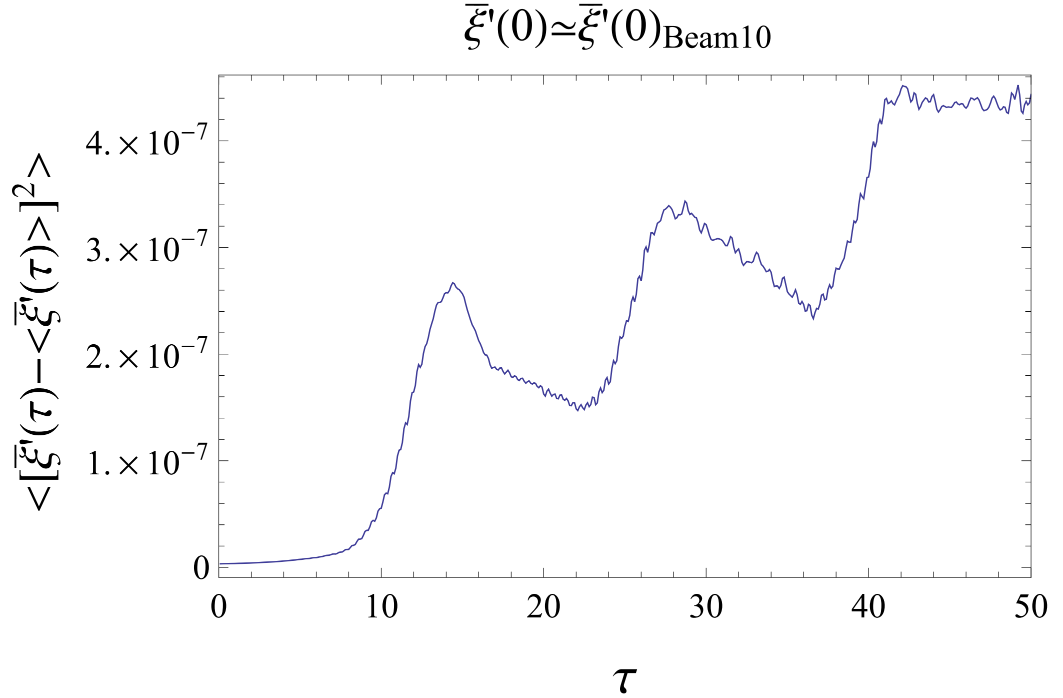


- The diffusion coefficient can be computed from the slope of the mean-squared displacement (cf. p.6).

E: Can you construct the exact diffusion coefficient? (Hint: see problems on p.5). Can you correlate the diffusion coefficient with the spectral intensity? (Hint: see problems on p.11).

E: Are your answers to the problem above consistent with quasilinear diffusion?

- Evidence of residual coherent behavior: note that phase space motion is not completely chaotic and that residual coherent behavior is visible in, e.g., beam 10 nonlinear evolution and ensuing transport.



E: Comment this last result. How can you justify such behavior? Is the fact that beam 10 is characterized by this consistent with your expectations?

References and reading material

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