Lecture 5

Nonlinear beam-plasma dynamics with multiple resonances

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Chirikov criterion for resonance overlap

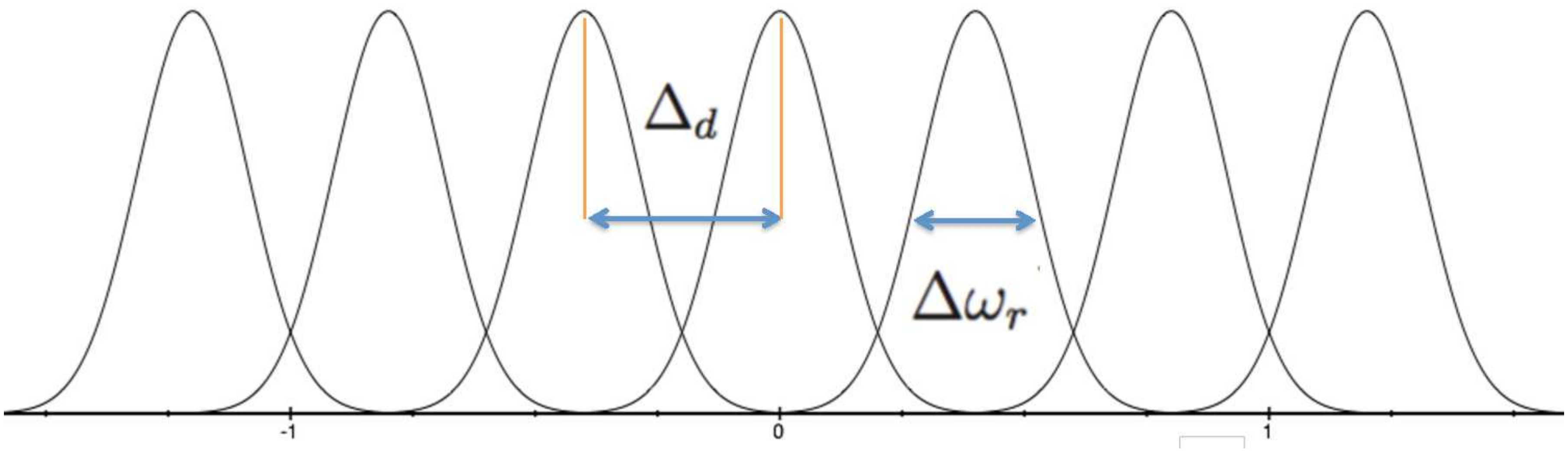
- Chirikov was the first one providing a physical criterion for the onset of chaotic motion in deterministic Hamiltonian systems (Chirikov 1959).
- This criterion was applied to explain puzzling experimental results on plasma confinement in magnetic traps

$$K = \left(\frac{\Delta\omega_r}{\Delta_d} \right)^2 > 1 \; .$$

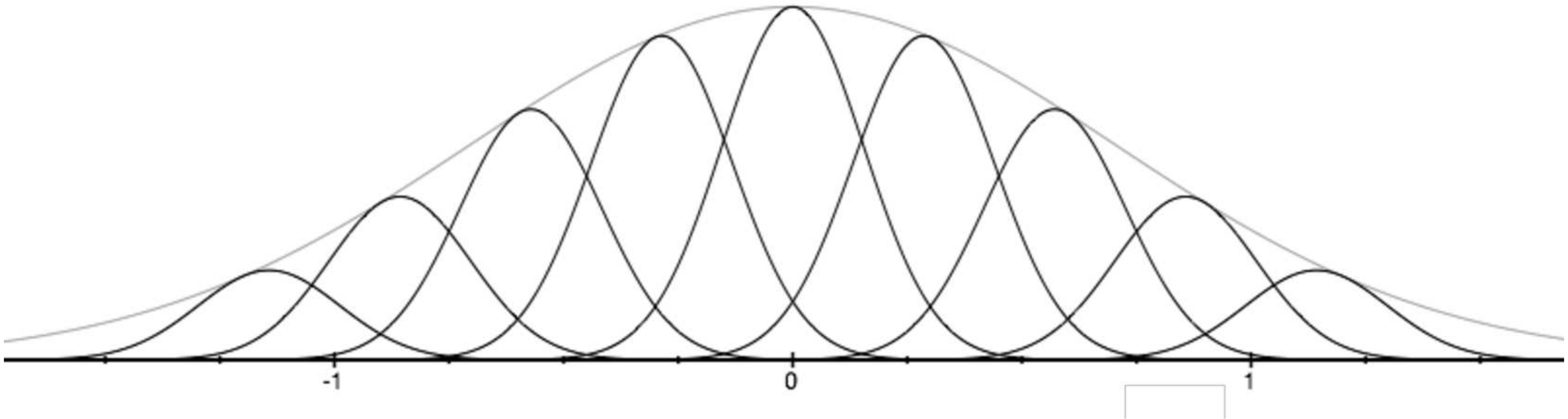
- Here, $\Delta\omega_r$ is the resonance width, $\propto \delta\phi^{1/2}$ when computed in the isolated resonance limit (pendulum approximation; see Lecture 3). Meanwhile, Δ_d is the frequency difference between unperturbed resonances.
- For the beam-plasma system $\Delta\omega_r \simeq 3\omega_p(n_B/2n)^{1/3}$ (see Lecture 3).

The multi-bleam plasma system

- If multiple beams are characterized by $\Delta_d > \Delta\omega_r$, the beams behave as independent and the nonlinear dynamics is equivalent to n-uncoupled oscillators (pendulums).



- If $\Delta\omega_r > \Delta_d$, resonances are coupled (Chirikov 1959). Since nonlinear saturation of the simple beam-plasma system implies beam-heating (see Lecture 2), it is possible to assume that n beams are "cold" (Shapiro 1963).



- The cold mutli-beam plasma system is a good paradigm for a warm-beam nonlinearly interacting with background plasma. However, one needs to study the system of n beams coupled with $m > n$ self-consistent nonlinear oscillators, due to the generation of near-resonant sidebands (see later; Carlevaro 2015).

Vlasov and O'Neil formulations

- The warm-beam plasma interaction can be simulated with the same Vlasov-Poisson equations adopted in Lecture 2 by simply accounting for a spectrum of Langmuir waves, $\epsilon(\omega_j, k_j) = 0$, with $\omega_j \simeq \omega_p$ and $j = 1, ..., m$.
- The approach based on n beams coupled with $m > n$ self-consistent non-linear oscillators must be derived as extension of the original approach by O'Neil et al 1971 (Lecture 3).
- Derivation follows Carlevaro 2015. Advantages:
  - Spectrum of m Langmuir waves is not selected arbitrarily: the minimal choice is of $m = n$ plasma waves degenerated with the corresponding beam modes $\omega_p = k_j v_{Dj}$, $j = 1, ..., n$
  - Control on the nonlinear dynamics through spectral density (location of resonances) and spectral intensity (saturation amplitudes of the oscillators in the uncoupled limit)
  - Vlasov and multi-beam approach a la O'Neil coincide in the mean field limit (Klimontovich, see Lecture 4)

Nonlinear evolution of n-resonant beams

- Adopt a 1D model with the thermal plasma treated as linear dielectric medium and n cold electron beams. Assume that the system is periodic with finite size L.
- The motion along the x direction is periodic, and is labeled by x_1, x_2, ..., x_n for each beam, respectively.
- Decomposing the single beams in N_1, N_2, ..., N_n charge sheets located at x_{1i}, x_{2i}, ..., x_{ni}, the system can be discretized using the following total charge density

$$\rho_B(x,t) = -e\, n_B \left(\frac{L\sigma_1}{N_1} \sum_{i=1}^{N_1} \delta(x - x_{1i}) + ... + \frac{L\sigma_n}{N_n} \sum_{i=1}^{N_n} \delta(x - x_{ni}) \right) ,$$

where n_B is the total suprathermal particle number density while $\sigma_1 n_B$, ..., $\sigma_n n_B$ are the number densities of each beam.

- The Langmuir wave scalar potential can be expressed using the Fourier representation as

$$\varphi(x,t) = \sum_{j=1}^{m} \left(\varphi_j(k_j,t)e^{ik_jx-i\omega_pt} + c.c.\right) ,$$

where $\varphi_j(k_j,t)$ are assumed to be slowly varying fields and we have explicitly indicated the time dependence relative to the rapid oscillations associated to the plasma frequency.

- The trajectories of charge sheets are derived from the equation of motion $\ddot{x} = e\nabla\varphi/m_e$ as

$$\ddot{x}_{1i} = \frac{ie}{m_e}\sum_{j=1}^{m} k_j\varphi_j e^{ik_jx_{1i}-i\omega_pt} + c.c. ,$$

$$\vdots$$

$$\ddot{x}_{ni} = \frac{ie}{m_e}\sum_{j=1}^{m} k_j\varphi_j e^{ik_jx_{ni}-i\omega_pt} + c.c. .$$

- We now write Poisson's equation in Fourier space, by projecting on the $\sim e^{ik_jx}$ plane wave

$$k_j^2 \epsilon(\omega_j, k_j) \varphi_j e^{-i\omega_p t} = 4\pi \, \rho_{Bj}(k_j, t) \; ,$$

$$\rho_{Bj}(k_j, t) = \frac{1}{L} \int_0^L dx \; \rho_B(x, t) e^{-ik_jx} = -e \, n_B \left(\frac{\sigma_1}{N_1} \sum_{i=1}^{N_1} e^{-ik_jx_{1i}} + \ldots + \frac{\sigma_n}{N_n} \sum_{i=1}^{N_n} e^{-ik_jx_{ni}} \right) .$$

- Using the fact that m (nearly degenerate) Langmuir waves satisfy

$$\epsilon(\omega_j, k_j) \simeq \epsilon(\omega_p, k_j) + \frac{2}{\omega_p}(i\partial_t + \omega_p) \simeq \frac{2}{\omega_p}(i\partial_t + \omega_p) \; ,$$

the Poisson's equation becomes

$$\frac{2}{\omega_p} \dot{\varphi}_j = i \, \frac{4\pi e n_B}{k_j^2} \left(\frac{\sigma_1}{N_1} \sum_{i=1}^{N_1} e^{-ik_jx_{1i} + i\omega_p t} + \ldots + \frac{\sigma_n}{N_n} \sum_{i=1}^{N_n} e^{-ik_jx_{1i} + i\omega_p t} \right) .$$

Nonlinear equations in dimensionless form

- Introduce the nonlinear displacement ζ of charged particle sheets, separating the unperturbed (free streaming) motion

$$x_{1i} = v_1 t + \zeta_{1i}(t) \; , \quad x_{2i} = v_2 t + \zeta_{2i}(t) \; , \quad \ldots \; , \quad x_{ni} = v_n t + \zeta_{ni}(t) \; .$$

- Introduce the scaled variables (see Lecture 2)

$$k_j = 2\pi \ell_j / L \; , \quad \bar{\zeta}_{ni} = 2\pi \zeta_{ni} / L \; , \quad \tau = t \omega_p \bar{\eta} \; , \quad \phi_j = [\varphi_j \, e k_j^2]/[m \bar{\eta}^2 \omega_p^2] \; ,$$

with $\bar{\eta} = (n_B/2n_p)^{1/3}$ and the frequency mismatches (Note: $\beta_{jj} \equiv 0$)

$$\beta_{1j} = [k_j v_1 - \omega_p]/\omega_p \bar{\eta} \; , \quad \ldots \; , \quad \beta_{nj} = [k_j v_n - \omega_p]/\omega_p \bar{\eta} \; .$$

E: Discuss the connection of these normalizations with those used in Lecture 3. What is the physical meaning of frequency mismatch? Why do we have $\beta_{jj} \equiv 0$?

□ Final form of nonlinear dimensionless equations

$$\bar{\zeta}''_{1i} = i \sum_{j=1}^{m} \ell_j^{-1} \phi_j \, e^{i\ell_j \bar{\zeta}_{1i} + i\beta_{1j}\tau} + c.c. \; ,$$

$$\vdots$$

$$\bar{\zeta}''_{ni} = i \sum_{j=1}^{m} \ell_j^{-1} \phi_j \, e^{i\ell_j \bar{\zeta}_{ni} + i\beta_{nj}\tau} + c.c. \; ,$$

$$\phi'_j = i \left(\frac{\sigma_1}{N_1} \sum_{i=1}^{N_1} e^{-i\ell_j \bar{\zeta}_{1i} - i\beta_{1j}\tau} + \ldots + \frac{\sigma_n}{N_n} \sum_{i=1}^{N_n} e^{-i\ell_j \bar{\zeta}_{ni} - i\beta_{nj}\tau} \right) .$$

E: Derive these equations step by step.

Conservation laws and linear dispersion relation

- Conservation laws are derived and discussed in Lecture 3 for the single beam case. Adopting the same approach, one can derive the momentum conservation

$$\sum_{j=1}^{m} \frac{|\phi_j|^2}{\ell_j} + \frac{d}{d\tau} \left[\frac{\sigma_1}{N_1} \sum_{i=1}^{N_1} \bar{\zeta}_{1i} + \ldots + \frac{\sigma_n}{N_n} \sum_{i=1}^{N_n} \bar{\zeta}_{ni} \right] = 0 \; .$$

- Meanwhile, energy conservation can be expressed as

$$\sum_{j=1}^{m} \frac{|\phi_j|^2}{\bar{\eta}\ell_j^2} + 2 \sum_{j=1}^{m} \Omega_{jRe} \frac{|\phi_j|^2}{\ell_j^2}$$

$$+ \frac{1}{2} \left[\frac{\sigma_1}{N_1} \sum_{i=1}^{N_1} \left(\bar{\zeta}'_{1i} + \frac{2\pi}{L} \frac{v_1}{\bar{\eta}\omega_p} \right)^2 + \ldots + \frac{\sigma_n}{N_n} \sum_{i=1}^{N_n} \left(\bar{\zeta}'_{ni} + \frac{2\pi}{L} \frac{v_n}{\bar{\eta}\omega_p} \right)^2 \right] = 0$$

- In these equations, we have used the definition (see Lecture 3)

$$\phi_j(\tau) = \phi_j(0) \exp\left(-i \int_0^\tau \Omega_j(\tau')d\tau'\right)$$

- We have also made use of the identity

$$i \sum_{j=1}^{m} \frac{\left(\phi_j^* \phi_j'' - \phi_j \phi_j''^*\right)}{\ell_j^2} = 2\frac{d}{d\tau} \sum_{j=1}^{m} \Omega_{jRe} \frac{|\phi_j|^2}{\ell_j^2}$$

E: Derive these conservation laws step by step and show that they reduce to the conservation laws discussed in Lecture 3 for the case of a single beam.

E: Why do you need to invoke momentum conservation to recover the single beam "energy" conservation?

E: Recall that nonlinear displacements ζ are expressed in the (different) beam moving frames. Recall the normalization condition of the fields ϕ_j. Can you articulate why the dependences on ℓ_j are those you should expect and why they are consistent with the conservation laws of Lecture 2?

- The linear dispersion relation is obtained noting $\phi'_j = -i\Omega_j\phi_j$ and that, denoting beams by the subscript a, by perturbation expansion

$$\bar{\zeta}_{ai} = \bar{\zeta}^0_{ai} + \bar{\zeta}'^0_{ai}\,\tau - i\sum_{j=1}^{m} \frac{\ell_j^{-1}\phi_j\ \exp[i\ell_j\bar{\zeta}^0_{ai} + i\ell_j\bar{\zeta}'^0_{ai}\tau + i\beta_{aj}\tau]}{(\Omega_j - \beta_{aj} - \ell_j\bar{\zeta}'^0_{ai})^2} + c.c.\ .$$

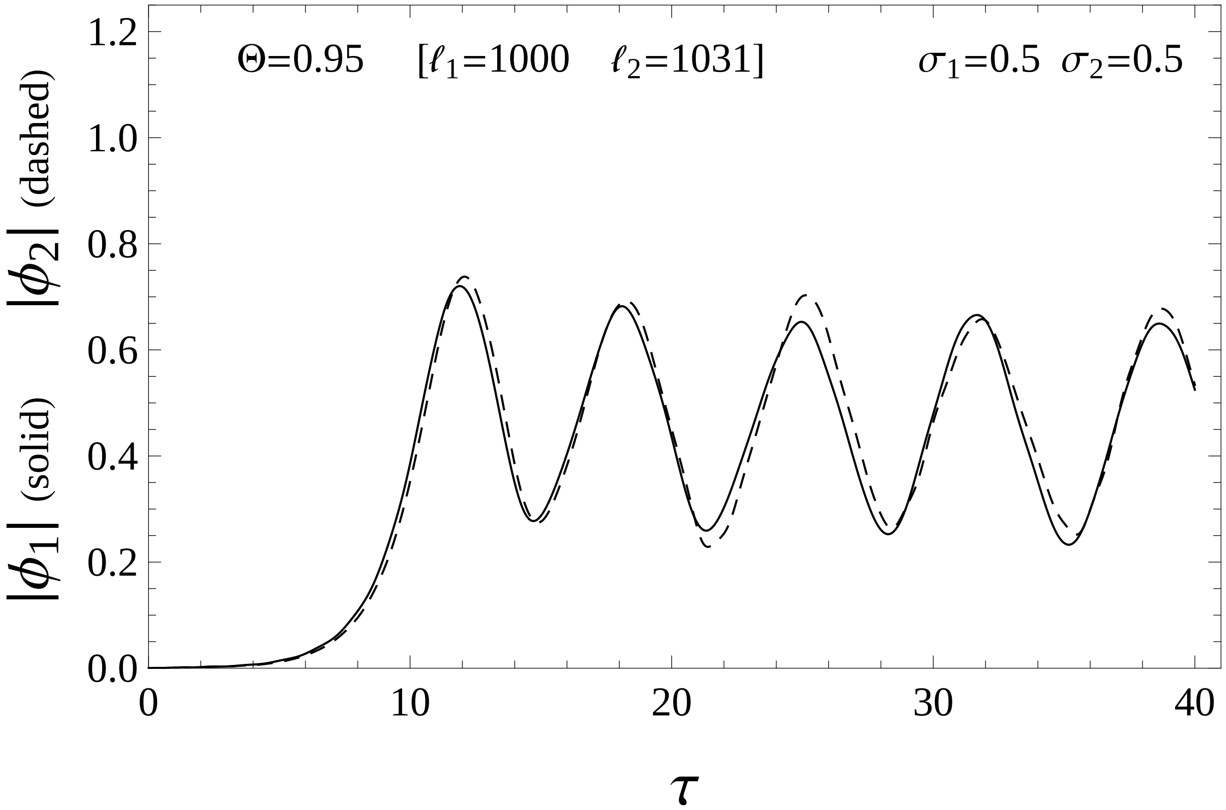
- Choosing $\bar{\zeta}'^0_{ai} = 0$ and noting $\sum_i e^{i\ell_j\bar{\zeta}^0_{ai}} = 0$ for uniformly distributed particles (see Hands on Session 3), the dispersion relation reduces to

$$\Omega_j = \frac{\sigma_1}{(\Omega_j - \beta_{1j})^2} + \frac{\sigma_2}{(\Omega_j - \beta_{2j})^2} + \ldots + \frac{\sigma_n}{(\Omega_j - \beta_{nj})^2}\ .$$

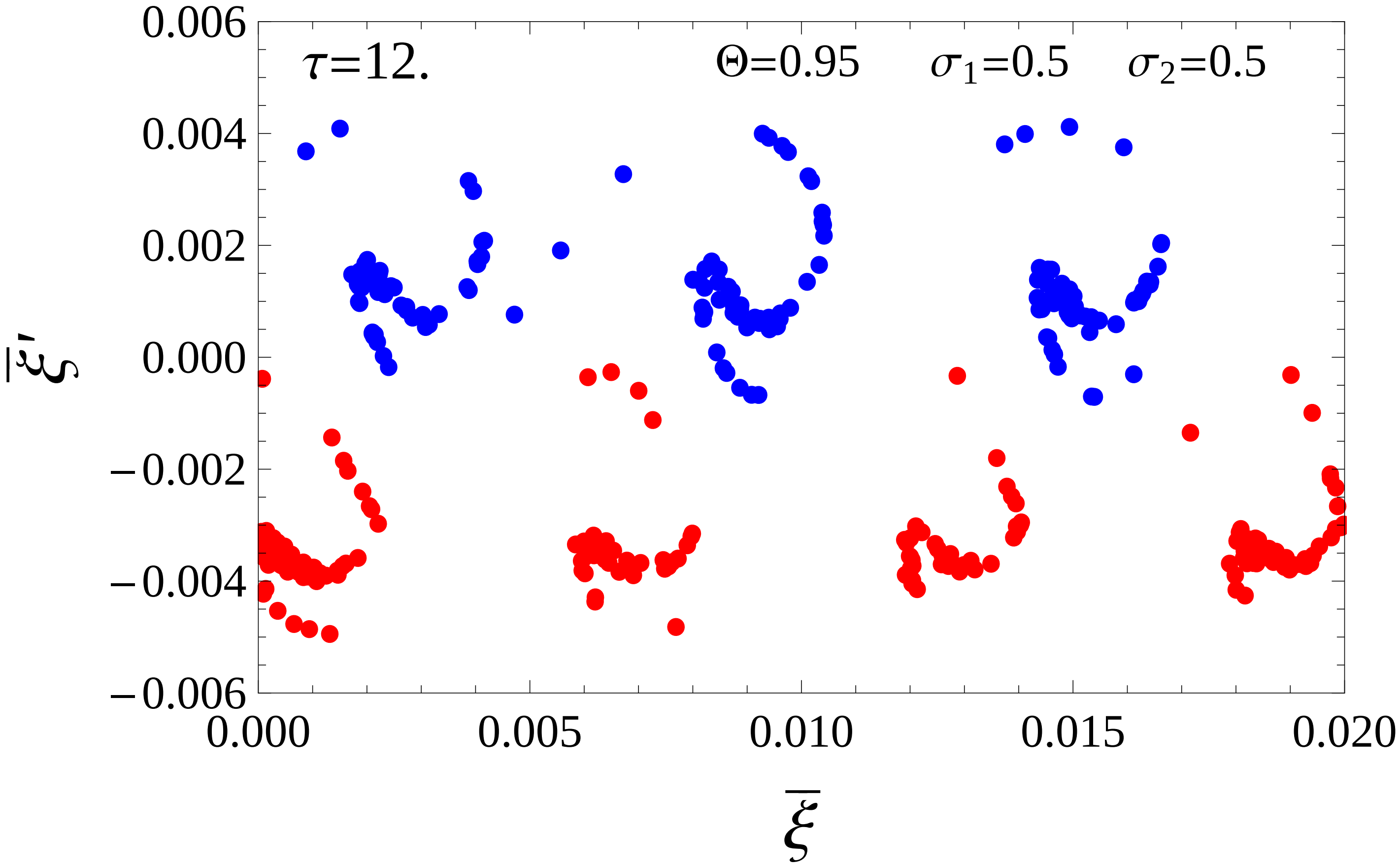
E: Derive the dispersion relation step by step.

Numerical simulation results for two beams

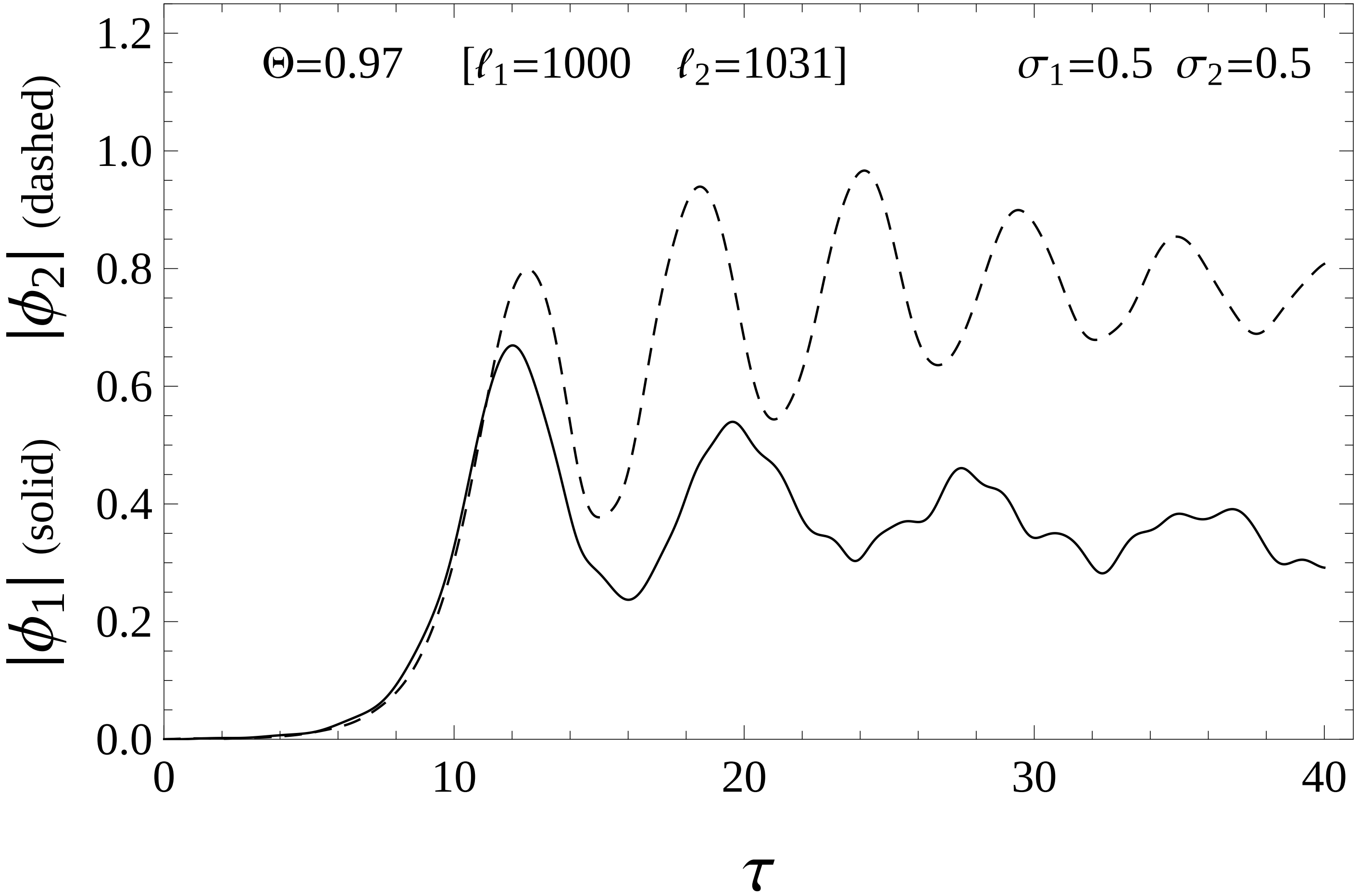
- Consider two beams with $\Theta \equiv v_2/v_1 < 1$ and, by definition $k_2/k_1 = \ell_2/\ell_1 = 1/\Theta > 1$. Fixed parameter $\eta = 0.01$. Thus, natural beam resonance width is (see p. 2) $\Delta\omega_r/\omega \simeq \Delta v/v \simeq 0.03\sigma^{1/3}$.



- Sufficiently separated beams behave independently, as expected.

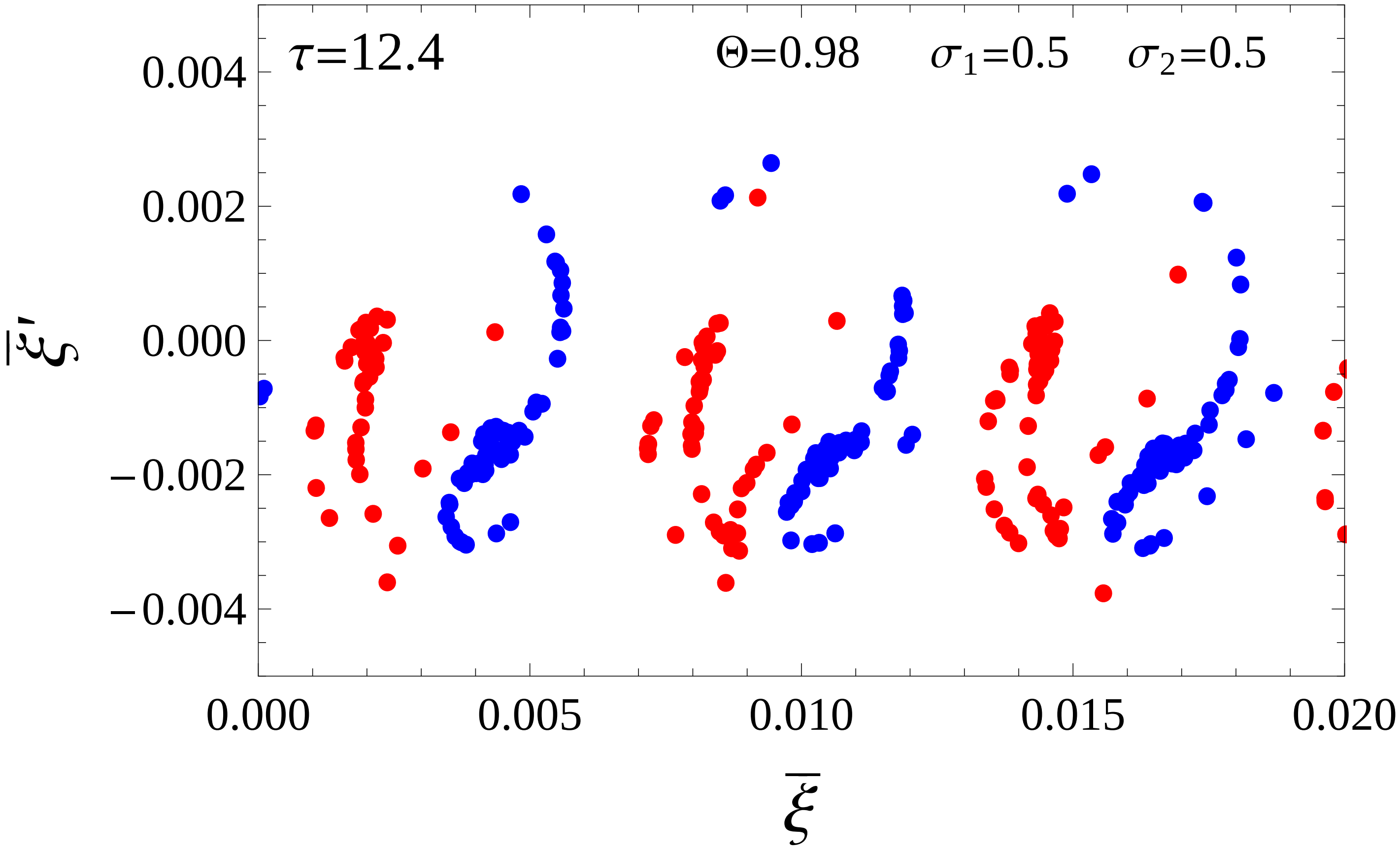


- Evidence of interacting beams. (i): faster beam mode is depleted

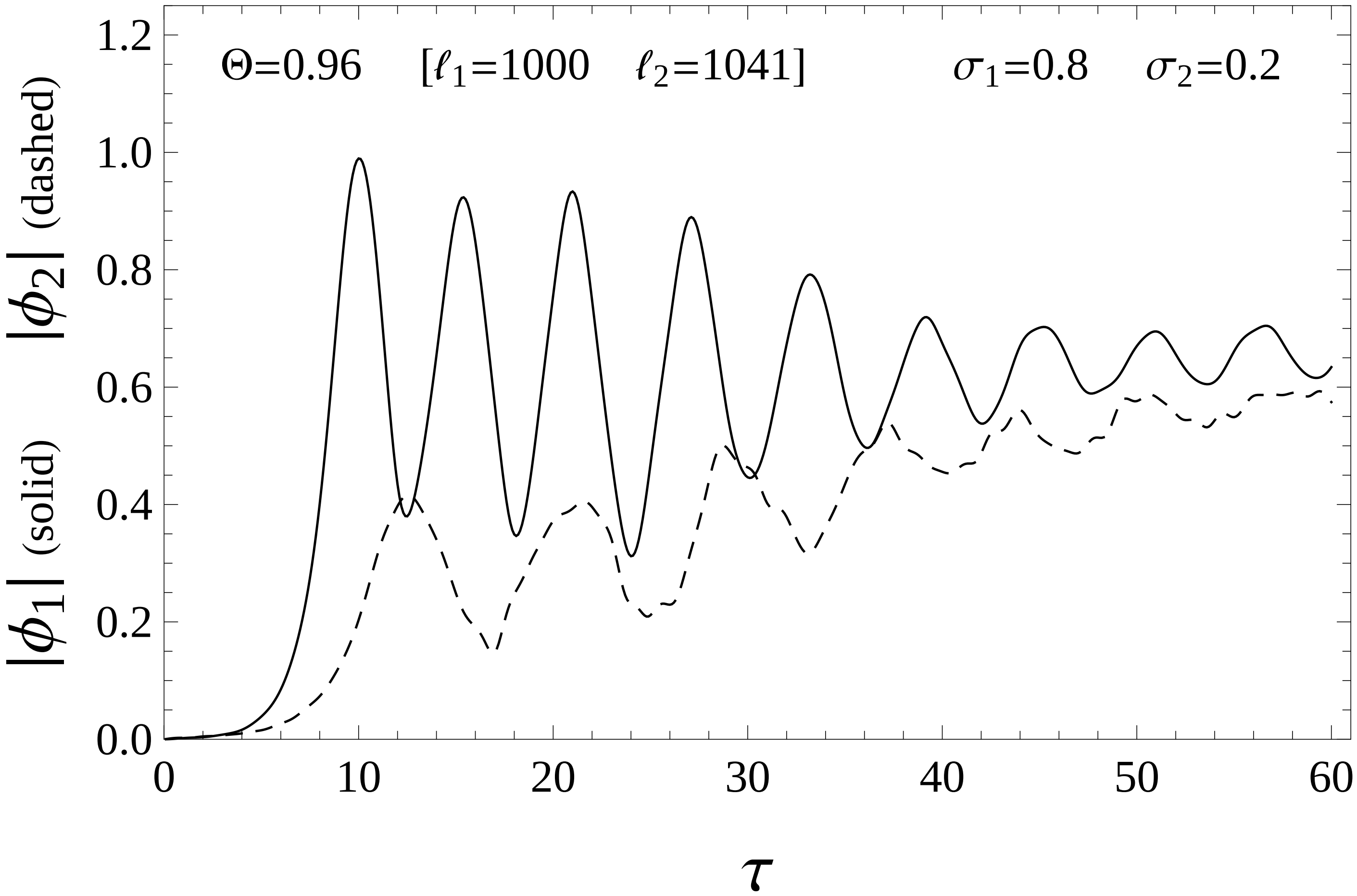


E: Explain why, in your opinion, the faster beam mode is depleted of energy, and not the slower one.

- Evidence of interacting beams. (ii)



- Evidence of interacting beams. (iii): $\sigma_1 > \sigma_2$; unbalanced beams



E: With $\sigma_1 > \sigma_2$, it is still the faster beam mode that is depleted of energy. Can you explain why? Is the ratio of saturated amplitudes what you expect?

Back to using Lagrangian Coherent Structures

- Use of LCS introduced in Hands on Session 1 and Lecture 4 and computed from FTLE.

$$\sigma(\bar{\xi}, \bar{\xi}', \tau, \Delta\tau) = \ln\left(\delta_{\Delta\tau}/\delta_\tau\right)/\Delta\tau \ .$$

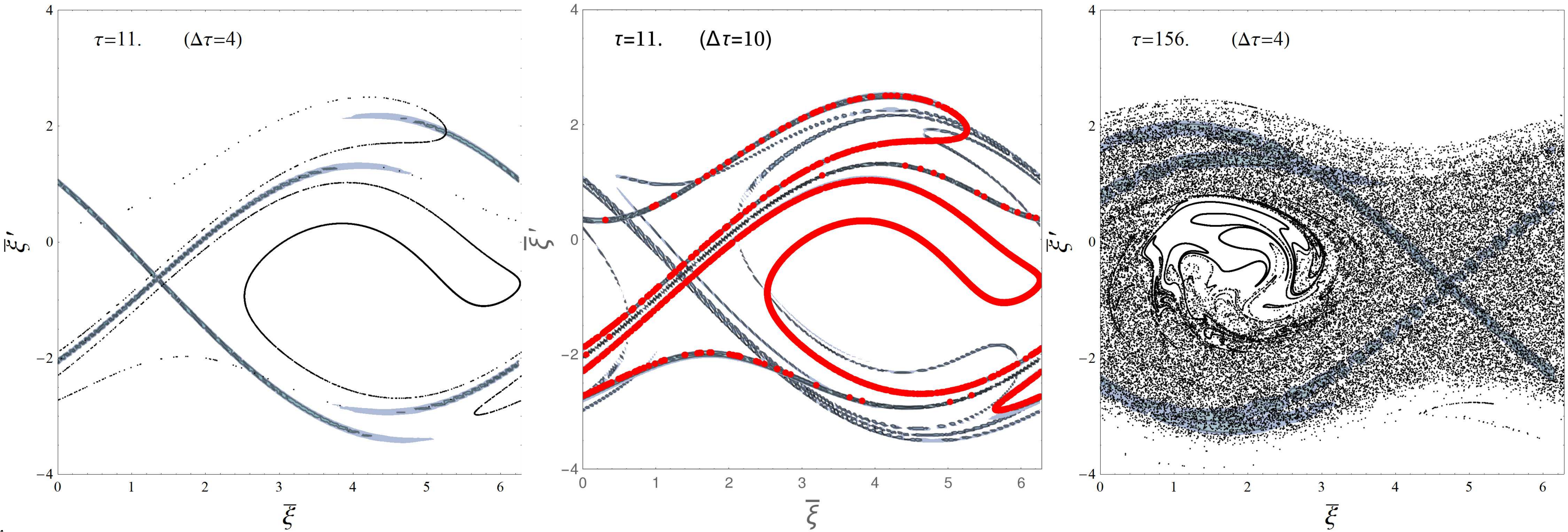
- When considering forward time evolution $\Delta\tau > 0$, the curves where the FTLE field is peaked define a repulsive transport barrier.

- When considering backward time evolution $\Delta\tau < 0$, the curves where the FTLE field is peaked represent an attractive transport barrier.

- Approximated 1D structures which correspond to the LCSs can be built by plotting the maximum values of $\sigma(\bar{\xi}, \bar{\xi}', \tau, \Delta\tau)$ as extracted from a contour plot of the reduced phase-space surface of section (see Lecture 1).

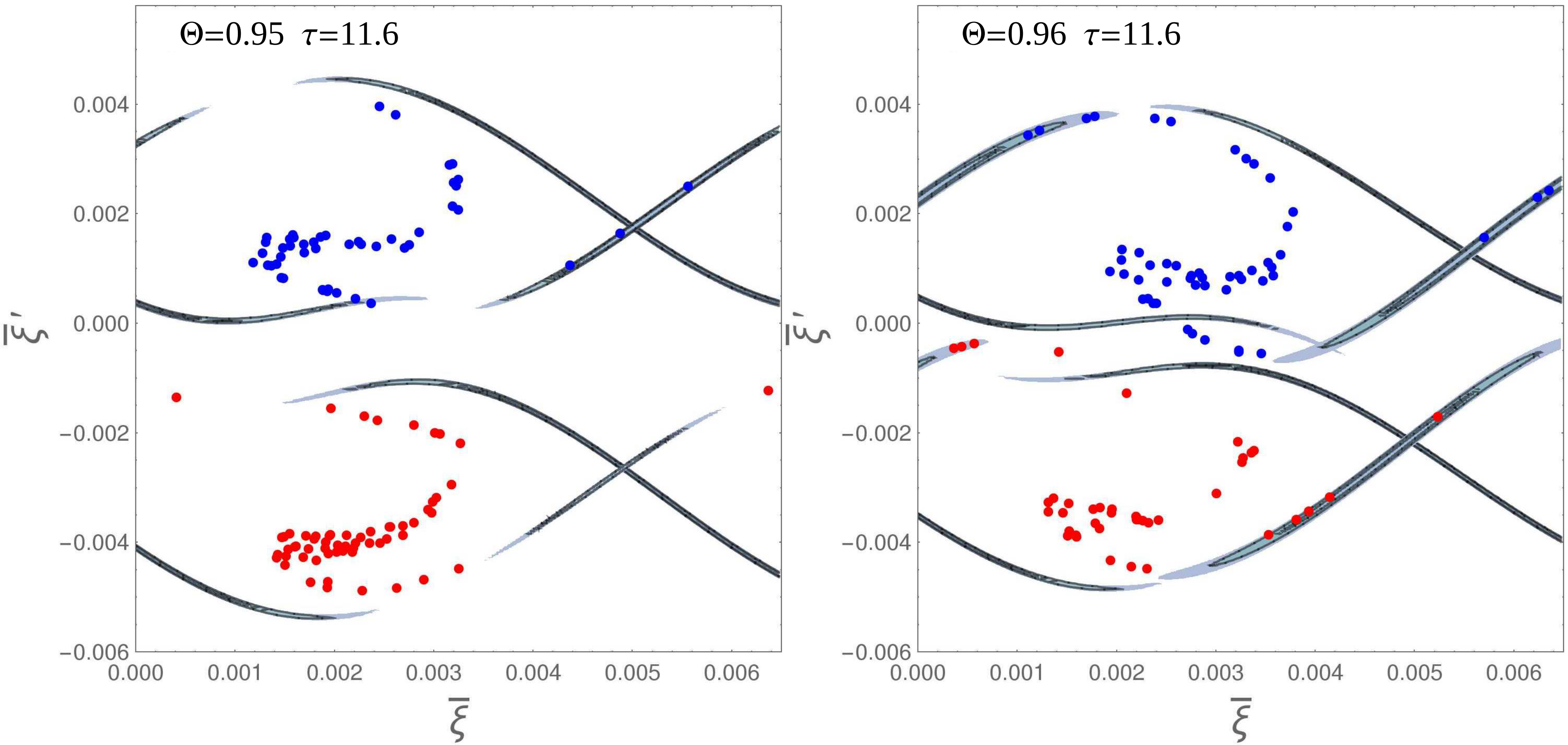
E: Comment about the sign of FTLE σ for forward and backward integration in time. Can you justify the definition of repulsive and attractive barriers?

- Evidence that LCS computed from FTLE describe transport barriers on a finite amount of time Δt, as they evolve in time t.

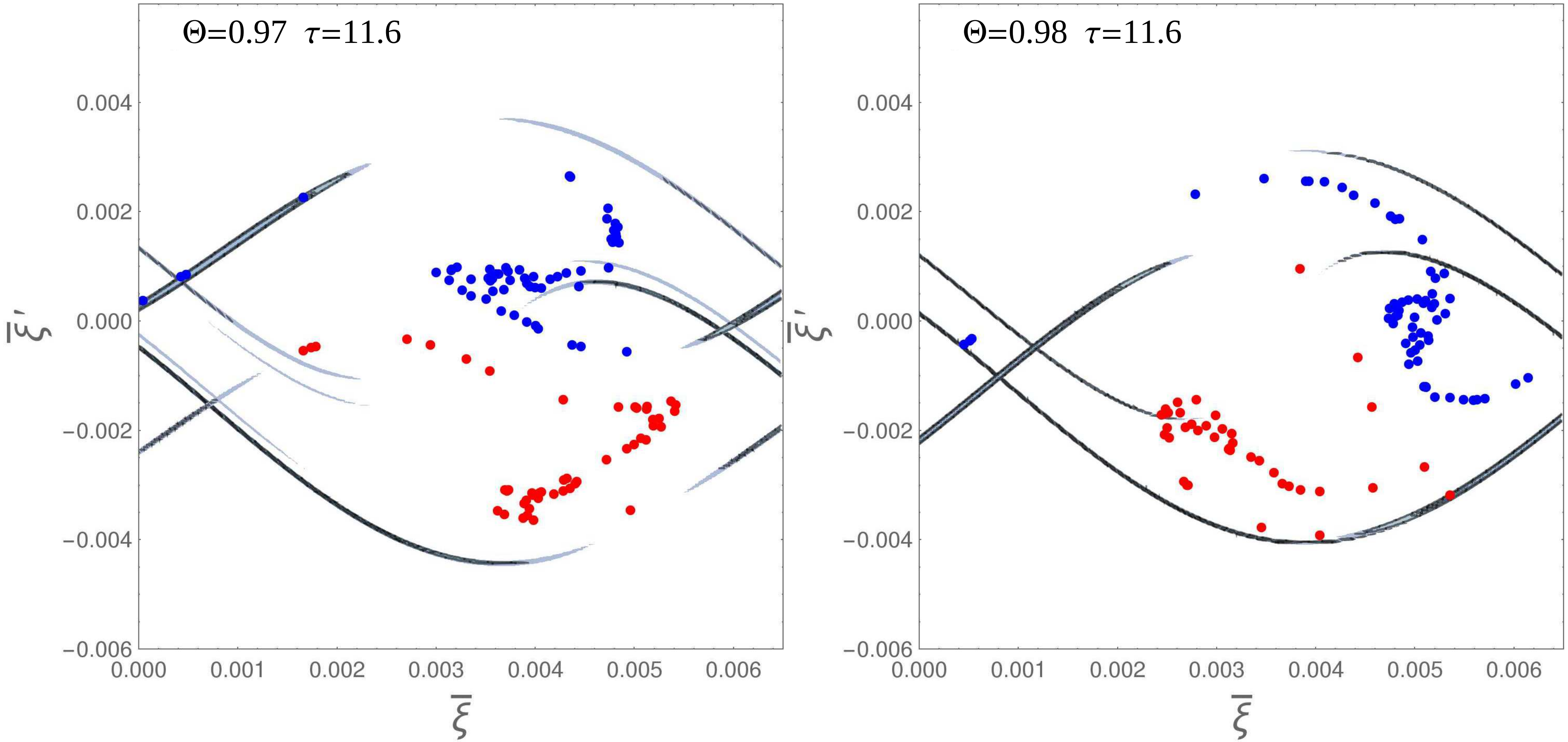


- This property can be used to determine the criterion for non-negligible interaction between two cold beams.

- The value of Θ controls the beam merging: (i) independent beams



- The value of Θ controls the beam merging: (ii) interacting beams

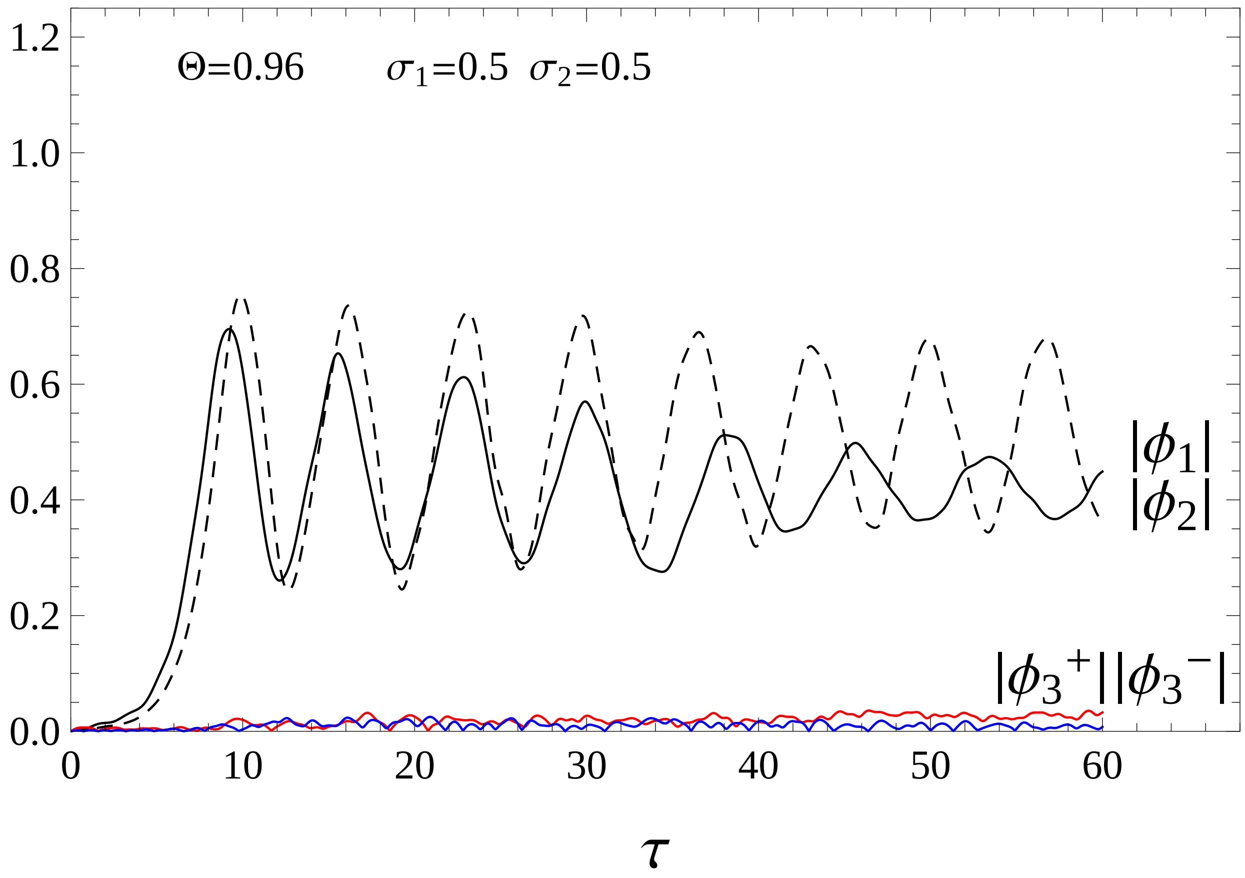


- The criterion for interacting beams is consistent with $|1 - \Theta| \simeq \Delta\omega_r/\omega \simeq \Delta v/v \simeq 0.03$ (see p. 2 and p. 14).

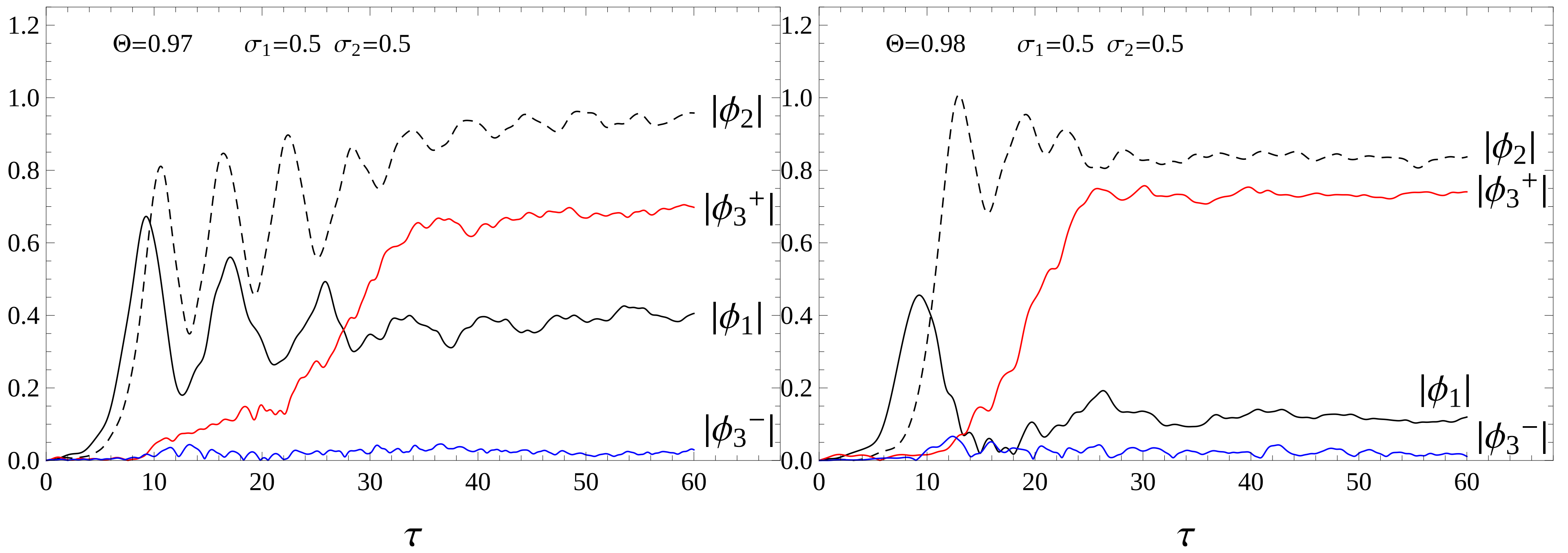
Sideband generation in the beam-plasma system

- Consider the system of two beams and two nearly degenerate Langmuir waves, discussed so far.
- In addition to k_1 and k_2 modes ($k_1 < k_2$; $\Delta k = k_2 - k_1 > 0$), we have the obvious beat wave generation at $k_3^- = k_1 - \Delta k = 2k_1 - k_2$ and $k_3^+ = k_2 + \Delta k = 2k_2 - k_1$.
- Once the mechanism for driving nearest sidebands is identified, the process can repeat nonlinearly and generate a whole nonlinear spectrum of modes, stemming from the original two, resonantly driven by the two driving suprathermal beams.
- Efficient excitation of sideband modes is clearly not independent of the properties of driving suprathermal particle source.
- Investigate sideband generation mechanism for two beams assuming independent/interacting beams.

- Inefficient sideband generation: (i) independent beams



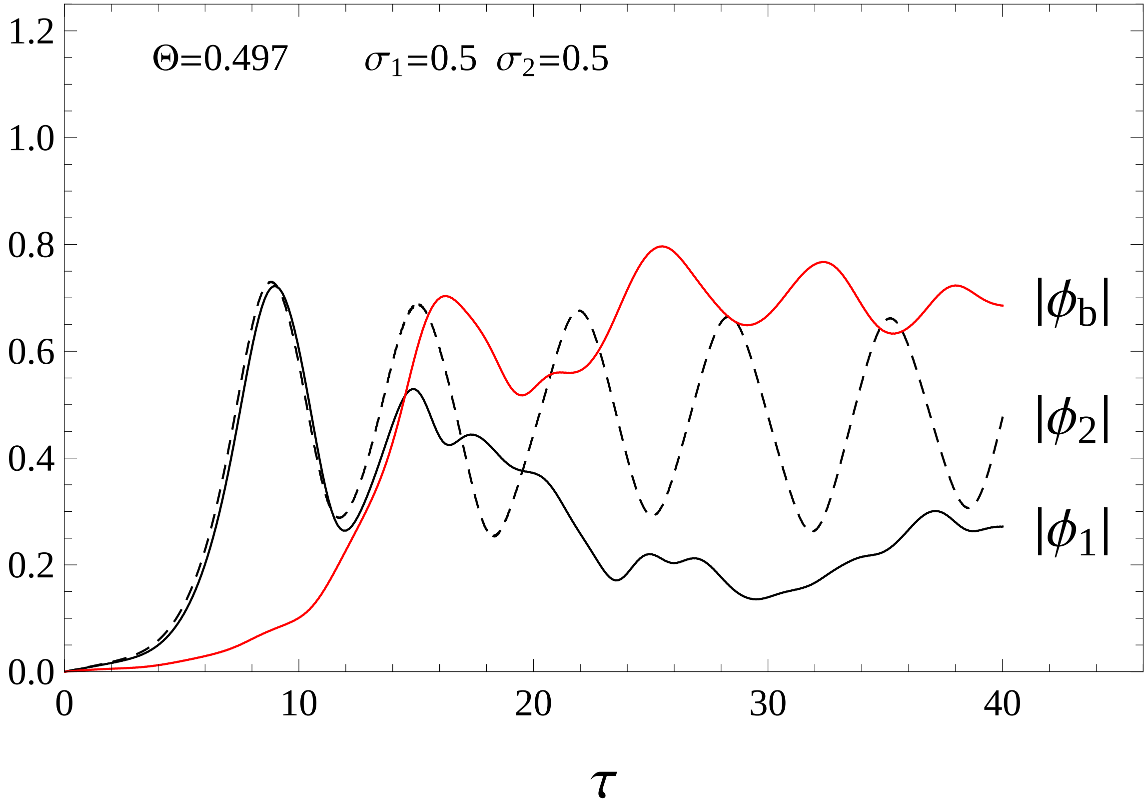
- Efficient sideband generation: (ii) interacting beams



- The k_1 mode is depleted of energy in amount depending on the initial beam velocity difference. The amplitude of the k_2 mode is weakly affected by this process.

E: Explain these numerical results on the basis of your understanding of conservation laws discussed in this lecture as well as Lecture 2 and Lecture 3.

- Efficient sideband generation: (iii) independent beams



- This special case occurs when $\Theta \to 1/2^-$; i.e. $k_2 \to 2k_1^+$ and $k_b = k_2 - k_1 \to k_1^+$. The "beat" k_b mode is driven at low frequency, but it may be seed of a near-resonant high frequency.

References and reading material

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