Lecture 4

Physics processes in the nonlinear beam-plasma dynamics

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Brief summary of previous lectures

- Brief summary of Lecture 2 and Lecture 3.
- The most unstable beam-plasma mode is nearly degenerate with the Langmuir wave.
- For a weak energetic beam, $n_B/n \ll 1$ and $v_D \gg v_T$, the thermal plasma can be considered as a linear dielectric medium.
- Thermal plasma nonlinearity is $\sim (n_B/n)^{1/3}$ higher order with respect to energetic particle nonlinearity.
- Resonant excitation and nonlinear saturation can be considered for a nearly monochromatic wave: $\omega_0 = \omega_p + \delta\omega$, $k_0 = (\omega_p/v_D) + \delta k$, $\delta k/k_0 \sim \delta\omega/\omega_0 \sim (n_B/n)^{1/3}$.
- Saturation occurs by wave particle trapping followed by phase mixing and harmonic generation. Conservation laws, can be derived and demonstrated.

Comparing Vlasov and O'Neil formulations

- The Vlasov-Poisson system (Lecture 2) is the mean field limit of the Hamiltonian formulation treated a la Klimontovich (Klimontovich 1967), i.e., the study of the "particle" density evolution in the phase space interpreted as that of charge sheets (O'Neil etal 1971) (Lecture 3).
- TheVlasov-Poisson equations are exactly recovered taking the continuum or mush limit, i.e., artificially increasing the number of particles keeping the charge density constant.
- In fact, the mush limit consists in constructing an ideal substance where the effect of collisions (discreteness) is vanishingly small ($K \to \infty$)

$$n \to Kn; \quad m \to m/K; \quad q \to q/K; \quad T \to T/K;$$

$$v_T \to v_T; \quad nm \to nm; \quad nq \to nq\,; \quad nT \to nT;$$

$$\omega_p \to \omega_p; \quad \lambda_D \to \lambda_D; \quad (q/\lambda_D^2) \to K^{-1}(q/\lambda_D^2)\,; \quad (n\lambda_D^3) \to K(n\lambda_D^3).$$

The Klimontovich equation

- The Klimontovich description of the dynamics of a system of N particles is exact (Klimontovich 1967). It introduces the phase space density

$$N_s(\boldsymbol{x},\boldsymbol{v},t) = \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{X}_i(t))\delta(\boldsymbol{v} - \boldsymbol{V}_i(t)) \; ;$$

where $\boldsymbol{X}_i(t)$ and $\boldsymbol{V}_i(t)$ are the positions and velocities of the i-th particle.

- The Klimontovich description is exact, since motion of all particles is known at each time. The function $N_s(\boldsymbol{x},\boldsymbol{v},t)$, in one dimension satisfies

$$\frac{\partial}{\partial t} N_s + v \frac{\partial}{\partial x} N_s + \frac{q}{m} E^m \frac{\partial}{\partial v} N_s = 0 \; .$$

which is the continuity equation for $N_s(\boldsymbol{x},\boldsymbol{v},t)$.

- Here, $V_i = \dot{X}_i$ and $\dot{V}_i = (q/m)E^m(X_i, t)$ are the equations of motion in the "microscopic" field E^m.

E: Derive the Klimontovich equation by taking explicitly $\partial_t N_s$ and using the definition of N_s. Repeat the derivation in 3D: what is the form of the Klimontovich equation? (Hint: see, e.g., Nicholson 1983).

E: What is the physical meaning of $E^m(X_i, t)$ in the O'Neil formulation of the beam-plasma problem? Note that the thermal plasma is treated as a linear dielectric medium and recall the definition of polarization charge and free charge, and the connection of $\boldsymbol{D}$ and $\boldsymbol{E}$ (see Lecture 2).

- Introducing ensemble averaging; and $f_s(x, v, t) = \langle N_s \rangle$, $E = \langle E^m \rangle$, with $N_s = f_s + \delta N_s$, $E^m = E + \delta E$ (note that δN_s and δE are fluctuations in the statistical sense)

$$\partial_t f_s + v\partial_x f_s + (q/m)E\partial_v f_s = -(q/m)\langle \delta E \partial_v \delta N_s \rangle \ .$$

E: Show that the right hand side tends to zero in the mush limit (Hint: see, e.g., Nicholson 1983; use the central limit theorem). Repeat the exercise in 3D.

Eulerian versus Lagrangian descriptions[1]

- Introduce Z as generic notation for phase space coordinates. Generally $Z \equiv (\boldsymbol{x}, \boldsymbol{p})$. More specifically it may represent our 1D beam-plasma phase space or the phase space in gyro-centre coordinates, when reducing Vlasov equation to the Gyrokinetic equation.

- Numerical treatment must rely on a discretization of the particle distribution function

$$f(Z,t) = \int f(Z',t)\delta(Z - Z')dZ' \simeq \sum_{l=1}^{N} \Delta_l f(Z_l,t)\delta(Z - Z_l)$$

- Eulerian and Lagrangian approaches differ in the way of treating this discretization.

[1] Acknowledgment: Based on inputs by Sergio Briguglio, private communication 2015.

- Eulerian (Vlasov) approach (not investigated in this Lecture series)
  - fixed phase-space discretization

$$Z_l = Z_{l0} \; , \qquad \Delta_l = \Delta_{l0} \; , \qquad \forall l$$

  - finite-difference evolution of the distribution function

$$f(Z_l, t + \Delta t) = f(Z_l, t) - \Delta t \dot{Z}^i \partial_{Z^i} f(Z_l, t)$$

- Lagrangian - Particle In Cell (PIC) approach (Hands on Session 2)
  - phase-space grid points $Z_l(t)$ ("macro-particles") evolve according to equations of motion
  - volume elements $\Delta_l(t)$ evolve as

$$\dot{\Delta}_l = \Delta_l \left(\partial_{Z^i} \dot{Z}^i \right)_{t, Z_l(t)}$$

Renormalized particle response and the Dyson equation

- Introduce Laplace transform and inverse as usual (v dependence is implicit):

$$\delta f_k(t) = \int_{-\infty}^{+\infty} e^{-i\omega t} \delta \hat{f}_k(\omega) d\omega \quad ,$$

$$\delta \hat{f}_k(\omega) = \frac{1}{2\pi} \int_0^{+\infty} e^{i\omega t} \delta f_k(t) dt$$

- Assuming vanishing initial conditions for δf_{k0}, the Laplace solution of the Vlasov equation is

$$\delta \hat{f}_{k_0}(\omega) = \frac{e}{m} \frac{k_0}{\omega - k_0 v} \int \delta \hat{\phi}_{k0}(\omega') \frac{\partial}{\partial v} \hat{f}_0(\omega - \omega') d\omega' \quad .$$

E: Derive this result step by step. What is the physics described by $\hat{f}_0(\omega - \omega')$?

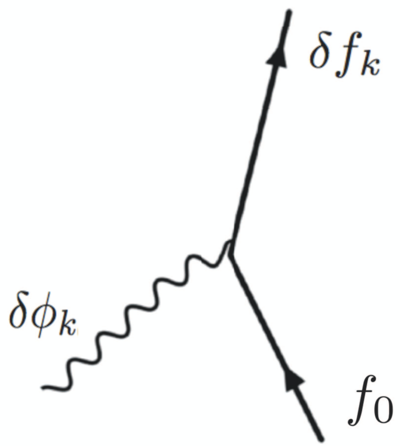
- By direct substitution into the Poisson equation, one can obtain,

$$\frac{2}{\omega_p}\frac{\partial}{\partial t}\delta\phi_{k_0} = \frac{\omega_p^2}{nk_0} i \int dv \iint_{-\infty}^{+\infty} e^{-i\omega t} \frac{\delta\hat{\phi}_{k0}(\omega')}{\omega - k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega - \omega') d\omega d\omega' \ .$$

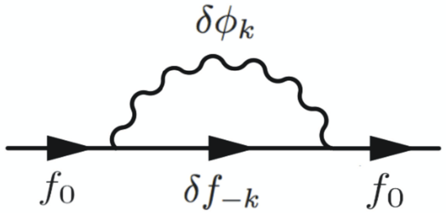
- We can close this equation only with the expression of $\hat{f}_0(\omega)$, which can be again computed from the $k = 0$ Vlasov equation with initial condition F_0 $\Rightarrow$ Expression for the phase space zonal structure!

$$\hat{f}_0(\omega) = \frac{i}{2\pi\omega} F_0 + \frac{e}{m}\frac{k_0}{\omega} \int_{-\infty}^{+\infty} \left[\delta\hat{\phi}_{k_0}(\omega') \frac{\partial}{\partial v} \delta\hat{f}_{-k_0}(\omega - \omega') - \delta\hat{\phi}_{-k_0}(\omega') \frac{\partial}{\partial v} \delta\hat{f}_{k_0}(\omega - \omega') \right] d\omega' \ .$$

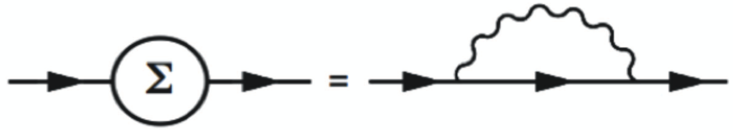
E: Derive this result step by step. Can you introduce collisions in the equation for phase space zonal structures? [Hint: Spring 2010 Lecture Notes]

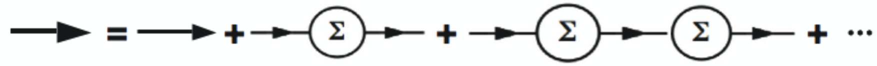


Generation of the distribution δf_k due to the interaction of f_0 with the field $\delta\phi_k$, corresponding to the solution of the Poisson Equation.



Nonlinear distortion of f_0 due to emission and absorption of the field $\delta\phi_k$.





The diagram of the process is defined in the top frame, while the solution of the "Dyson" equation corresponds to the summation of all terms in the Dyson series (bottom).

- Explicit form of the Dyson equation is obtained by substitution of the expression of $\delta\hat{f}_{k_0}(\omega)$ $\Rightarrow$ Renormalized solution for the phase space zonal structure.

$$\hat{f}_0(\omega) = \frac{i}{2\pi\omega}F_0 - \frac{e^2}{m^2}\frac{k_0^2}{\omega}\iint_{-\infty}^{+\infty}\left[\delta\hat{\phi}_{k_0}(\omega')\delta\hat{\phi}_{-k_0}(\omega'')\frac{\partial}{\partial v}\left(\frac{1}{\omega-\omega'+k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega-\omega'-\omega'')\right)\right.$$
$$\left.+\delta\hat{\phi}_{-k_0}(\omega')\delta\hat{\phi}_{k_0}(\omega'')\frac{\partial}{\partial v}\left(\frac{1}{\omega-\omega'-k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega-\omega'-\omega'')\right)\right]d\omega' d\omega'' \ .$$

- This equation forms a close set with the Poisson equation (repeated below)

$$\frac{2}{\omega_p}\frac{\partial}{\partial t}\delta\phi_{k_0} = \frac{\omega_p^2}{nk_0}i\int dv\iint_{-\infty}^{+\infty}e^{-i\omega t}\frac{\delta\hat{\phi}_{k0}(\omega')}{\omega-k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega-\omega')d\omega d\omega' \ .$$

RP: Investigate the self-consistent nonlinear beam-plasma problem by numerical solution of these coupled equations. Add collisions and other forms of "external" particle scatterings: what is the effect on the mode nonlinear dynamics? Can you identify adiabatic and non-adiabatic regimes?

Weakly varying fluctuation spectrum

- Assume that ϕ_{k_0} amplitudes are weakly varying in time about a given frequency $\omega_{k_0} = \omega_{k_0 r} + i\gamma_{k_0}$. Then

$$\delta\hat{\phi}_{k_0}(\omega) = \frac{i}{2\pi}\frac{\delta\phi_{k_0}}{\omega - \omega_{k_0}} \quad , \qquad \delta\hat{\phi}_{-k_0}(\omega) = \frac{i}{2\pi}\frac{\delta\phi^*_{k_0}}{\omega + \omega^*_{k_0}}$$

- These expressions can be extended to weakly varying amplitudes and frequencies, provided that $|\dot{\omega}_{k_0 r}| \ll |\gamma_{k_0}\omega_{k_0 r}|$ and $|\dot{\gamma}_{k_0}| \ll |\gamma^2_{k_0}|$. This is, *e.g.*, the example of nearly periodic shear Alfvén waves, with frequency chirping that can be non-adiabatic.

E: Show that, assuming $|\gamma_{k_0}/\omega_{k_0}| \ll 1$, the correct constraint is $|\dot{\omega}_{k_0 r}| \ll |\gamma_{k_0}\omega_{k_0 r}|$ and not $|\dot{\omega}_{k_0 r}| \ll |\omega^2_{k_0 r}|$ that applies in the general case $|\gamma_{k_0}/\omega_{k_0}| \sim 1$. Show also that it must be $|\dot{\gamma}_{k_0}| \ll |\gamma^2_{k_0}|$. [Hint: the relevant nonlinear time scale is $\tau_{NL} \sim |\gamma_{k_0}|^{-1}$].

E: Show that this representation applies for non-adiabatic frequency sweeping modes; *i.e.*, $|\dot{\omega}_{k_0r}| \sim \omega_B^2$, with ω_B the wave particle trapping frequency.

- Introduce the slow and fast time variables τ and t, with $\tau \simeq t$ but $|\partial_t| \gg |\partial_\tau|$ and $d_t = \partial_t + \partial_\tau$. Furthermore,

$$\delta\phi_{k_0}(t) \equiv \lim_{\tau\to t} \delta\phi_{k_0}(t,\tau) \equiv \lim_{\tau\to t} \delta\phi_{k_0}(\tau) \exp\left(-i\omega_{k_0}(\tau)t\right)$$

$$\delta\phi_{k_0}(\tau) \equiv \delta\bar{\phi}_{k_0} \exp\left[-i\int_0^\tau \omega_{k_0}(t')dt' + i\omega_{k_0}(\tau)\tau\right]$$

- From these definitions:

$$(\partial_t + \partial_\tau)\delta\phi_{k_0}(t,\tau) = \left[-i\omega_{k_0}(\tau) + i\dot{\omega}_{k_0}(\tau)(\tau - t)\right]\delta\phi_{k_0}(t,\tau).$$

- Thus, this representation admits the Laplace transforms shown in the previous page for a sufficiently short time $|t-\tau| \lesssim \tau_{NL} \simeq |\gamma_{k_0}|^{-1}$.

E: Reconsider the two previous exercises and solve them again in the light of these results.

- Consider now the Laplace representation for weakly varying fluctuation spectrum and rewrite Dyson equation and Poisson equation (note: shift the frequency integration in ω to leave $\hat{f}_0(\omega)$ explicit)

$$\hat{f}_0(\omega) = \frac{i}{2\pi\omega}F_0 - \frac{e^2}{m^2}\frac{k_0^2}{\omega}\left|\delta\phi_{k_0}(\tau)\right|^2 \frac{\partial}{\partial v}\left[\left(\frac{1}{\omega-\omega_{k_0}+k_0 v}+\frac{1}{\omega+\omega_{k_0}^*-k_0 v}\right)\frac{\partial}{\partial v}\hat{f}_0(\omega-2i\gamma_{k_0})\right]$$

$$\frac{2}{\omega_p}\left[\frac{\partial}{\partial\tau}+i(\omega_p-\omega_{k_0})\right]\delta\phi_{k_0}(\tau) = \frac{\omega_p^2}{nk_0}i\int dv\int_{-\infty}^{+\infty} e^{-i\omega t}\frac{\delta\phi_{k_0}(\tau)}{\omega+\omega_{k_0}-k_0 v}\frac{\partial}{\partial v}\hat{f}_0(\omega)d\omega \ .$$

E: Use Cauchy's residue theorem and carry out the contour integrals in frequency space to demonstrate equations above.

RP: Recall that not only $\delta\phi_{k_0}(\tau)$ but also $\omega_{k_0}(\tau)$ can be a function of time. Can you solve above equations for the beam plasma problem. What is the solution you obtain? What happens for increasing beam strength?

- Solutions of this system strongly depend on the properties of the fluctuation spectrum; and on the properties of $\hat{f}_0(\omega)$, which can have a broad spectrum or a narrow spectrum.

- Broad spectrum: the characteristic (nonlinear) time of $\hat{f}_0(\omega)$ relaxation is sufficiently long compared to the inverse spectral width of the fluctuations $\sim (\omega_{k_0}/k_0)\Delta k$. At the same time, though it remains sufficiently short compared to the inverse growth time of the fluctuations $|\gamma_{k_0}|^{-1}$.

- Narrow spectrum: the characteristic (nonlinear) time of $\hat{f}_0(\omega)$ is sufficiently short compared to the inverse spectral width of the fluctuations. Thus, the spectrum can be considered nearly periodic or monochromatic.

- In both cases, we assume that characteristic nonlinear time is sufficiently short compared to the inverse growth time of the fluctuations $|\gamma_{k_0}|^{-1}$. This means that on the RHS of the Dyson equation we can take $\hat{f}_0(\omega - 2i\gamma_{k_0}) \to \hat{f}_0(\omega) \Rightarrow$ Reduced equation is readily rewritten in the t-representation.

E: Justify the last statement and rewrite the reduced equation in the t-representation.

- Therefore, we assume that the fluctuation amplitude is not changing significantly during the nonlinear dynamics.

E: What happens if $\tau_{NL} \sim |\gamma_{k_0}|^{-1}$? Can you discuss the relationship of your answer with the RP at p.14?

- In the case of narrow spectrum and $\tau_{NL} \sim |\gamma_{k_0}|^{-1}$, we enter in the regime of phase-locking. Two examples of this for toroidal plasmas are analyzed in Spring 2014 Lecture 5 (fishbones) and Lecture 6 (EPMs).

Narrow spectrum: wave-particle trapping

- Introduce the notation [Al'tshul' 66]

$$\alpha^2 \equiv \sqrt{2}|ek_0^2\delta\phi_{k_0}/m| = \sqrt{2}\omega_B^2 \quad ; \qquad x = k_0(v - \omega_{k_0}/k_0)/\alpha \quad .$$

$$\Psi(\omega, x) = \frac{\alpha^2}{\omega^2 - \alpha^2 x^2}\frac{\partial}{\partial x}\hat{f}_0(\omega, x) \quad .$$

- For $|\gamma_{k_0}|^{-1} \gg \omega_B^{-1} \sim \tau_{NL}$, the Dyson equation at p. 14 becomes

$$\frac{\partial^2}{\partial x^2}\Psi(\omega, x) + \left(\frac{\omega^2}{\alpha^2} - x^2\right)\Psi(\omega, x) = \frac{i}{2\pi\omega}\frac{d}{dx}F_0 \quad ,$$

- Introduce the functions $||\psi_\ell(x)|| = 1$, expressed in term of Hermite polynomials $H_\ell(x)$

$$\psi_\ell(x) \equiv (2^\ell \ell! \pi^{1/2})^{-1/2} e^{-x^2/2} H_\ell(x) \quad .$$

- The solution $\hat{f}_0(\omega, x)$ can be expressed as series

$$\hat{f}_0(\omega, x) = \frac{i}{2\pi\omega}\bar{F}_0(0) - \frac{i\alpha^2}{2\pi\omega}\sum_{\ell=0}^{\infty}\frac{\beta_\ell d\psi_\ell(x)/dx}{\omega^2 - (2\ell+1)\alpha^2} \quad ,$$

$$\beta_\ell \equiv \int_{-\infty}^{\infty}(dF_0(0)/dx)\psi_\ell(x)dx \quad .$$

E: Derive this equation step by step.

- Moving to the t-representation [Al'tshul' 66]

$$f_0(u, t) = F_0(v) + \frac{\alpha}{k_0}\sum_{\ell=0}^{\infty}\frac{\beta_\ell}{(2\ell+1)}\frac{d}{dv}\psi_\ell\left[\frac{(k_0 v - \omega_{k_0})}{\alpha}\right]\left[1 - \cos\left(\sqrt{2\ell+1}\alpha t\right)\right] \quad .$$

E: Derive this equation step by step.

- This equation describes the oscillations of particles that are trapped in the wave, which, however, do not decay in time as expected in consequence of phase mixing.
- Thus, this approach does not describe the relaxation to the coarse grain particle distribution function (see Lecture 3).
- It was pointed out by [O'Neil 65] that the reason for this stands in the assumption of negligible harmonic generation at $k = \ell k_0 (\ell \geq 2)$ in both $\delta\phi_k$ and δf_k, which breaks down on long time scales.

E: Comment the interpretation provided by [O'Neil 65].

The use of kinetic Poincaré plots

- In Lecture 1, we have introduced Poincaré plots as portraits of the reduced phase space defined on a surface of section.
- Following the approach by O'Neil etal 1971, we can represent the phase space behavior of a number of self-consistently moving particles, in order to represent nonlinear dynamics.
- Note that this is not a Poincaré plot in the strict sense (see above and Lecture 1).
- More generally, White 2012 introduces the notion of kinetic Poincaré plot to represent nonlinear dynamics of a number of tracers (not generally self-consistent), on a suitable surface of section periodically crossed and defined in the phase-space of one single-particle moving in toroidal geometry, given two invariants of motion.

- One invariant is μ: thus, the extended phase space is reduced to 6D, since the dynamics in typically independent of the gyrophase (see pp. 15-16 of Lecture 1).

$$\bar{H} = -\mu(P_\theta, P_\phi, H, \theta, \phi, t)$$

- The reduced 6D phase space can be further reduced, since the reduced phase space Hamiltonian is constant. We can parameterize the system coordinates as a function of one periodic degree of freedom; i.e. the poloidal angle at the outboard midplane

$$P_\theta = P_\theta(P_\phi, H, \phi, t)$$

- Using the second of assumed two invariants, $I = I(P_\theta, P_\phi, H, \theta, \phi, t)$, to independently solve for P_θ, we then have

$$P_\phi = P_\phi(H, \phi, t)$$

- Note that, if the system is autonomous (does not depend on t), H is constant and $P_\phi = P_\phi(\phi) =$ const, if the system is axisymmetric. In the presence of fluctuations it is in general not so.

E: What happens if the system is periodic in t?

E: Make an example to illustrate your previous answer. If you have an instability, what differences do you expect?

- The approach by White 2012 is further extended by Briguglio et al. 2014, who investigates the nonlinear dynamics of a number of self-consistent tracers moving in toroidal geometry, given two invariants of motion. Tracers are self-consistent since they are computed in the "given" fields of a fully self-consistent numerical simulation, stored previously.

- Kinetic Poincaré plots can be made more insightful by coloring the tracers with physical information (Briguglio et al. 2014), e.g. initial position and/or wave-particle power transfer.

From Briguglio et al. 2014

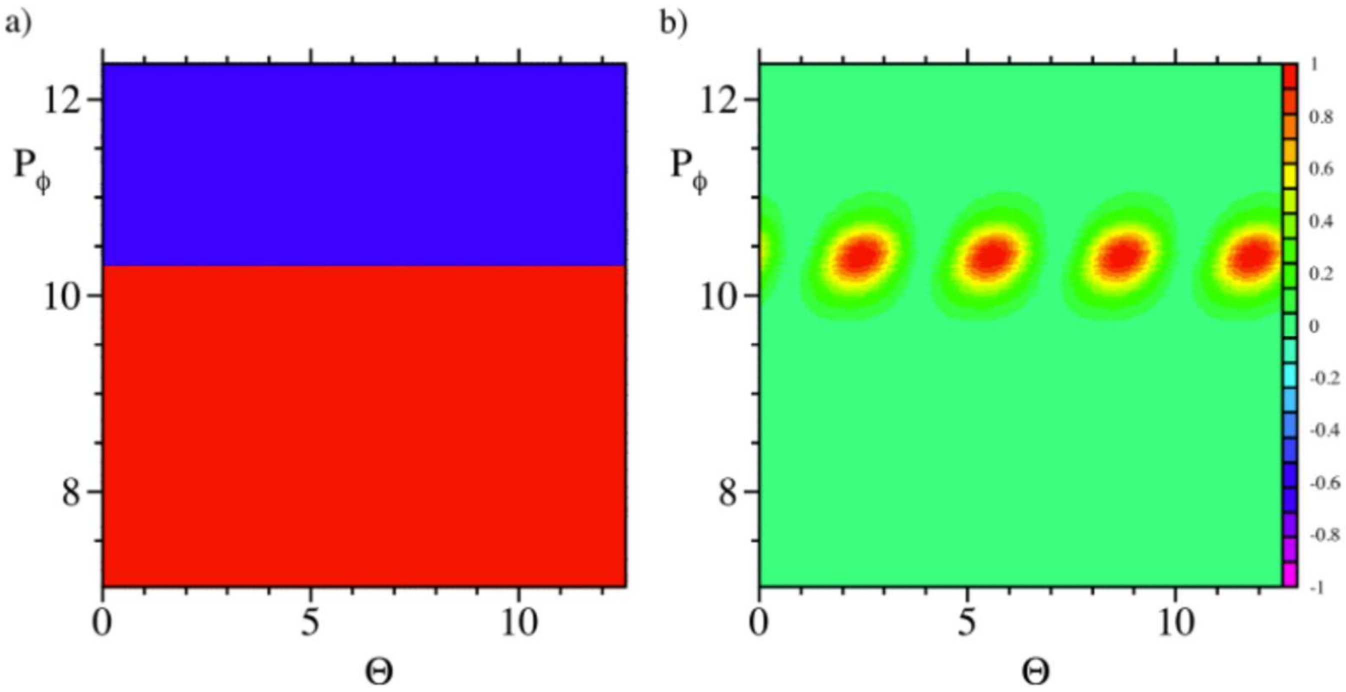


FIG. 11. Test particle markers in the (Θ, P_ϕ) plane during the linear phase of the mode evolution. (a) Each marker is coloured according to the birth P_ϕ value of the particle (red for $P_\phi < P_{\phi\,\mathrm{res}}$, blue otherwise); (b) each marker is coloured according to the macroparticle power transfer (time-varying color).

- Similar technique can be used for the beam-plasma problem: see Hands on Session 4.

E: What is the main difference of the beam-plasma system and the nonlinear Alfvén wave problem discussed above?

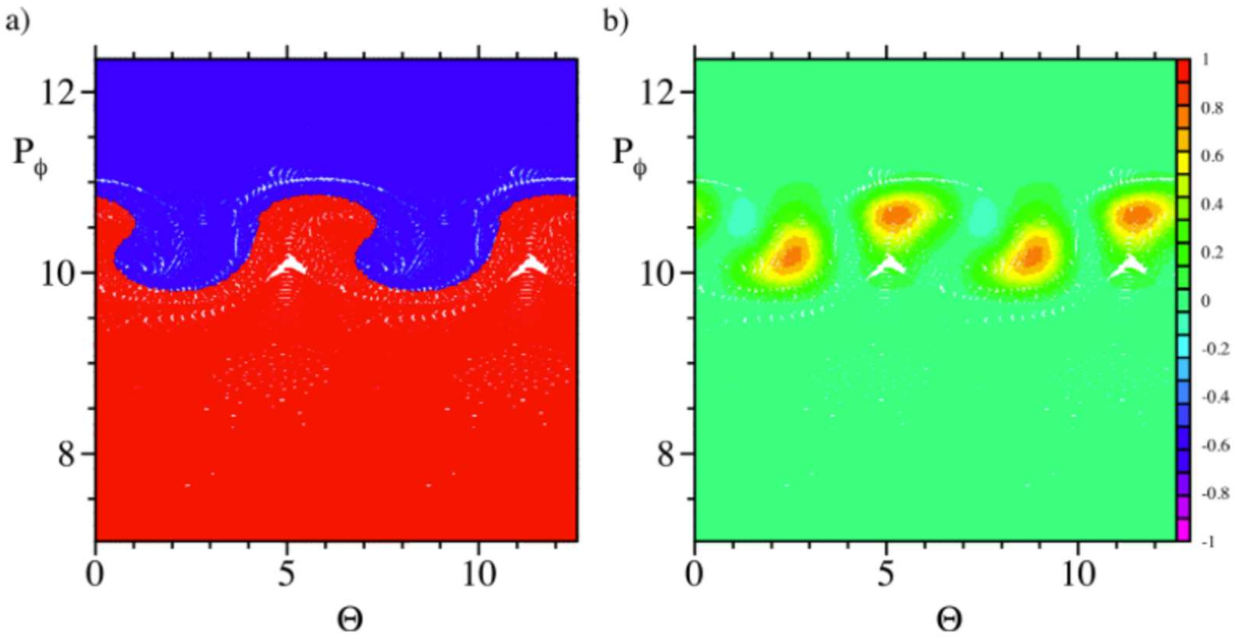


FIG. 12. Same as Fig. 11, for the nonlinear phase of the mode evolution ($t = 1470\tau_{A0}$).

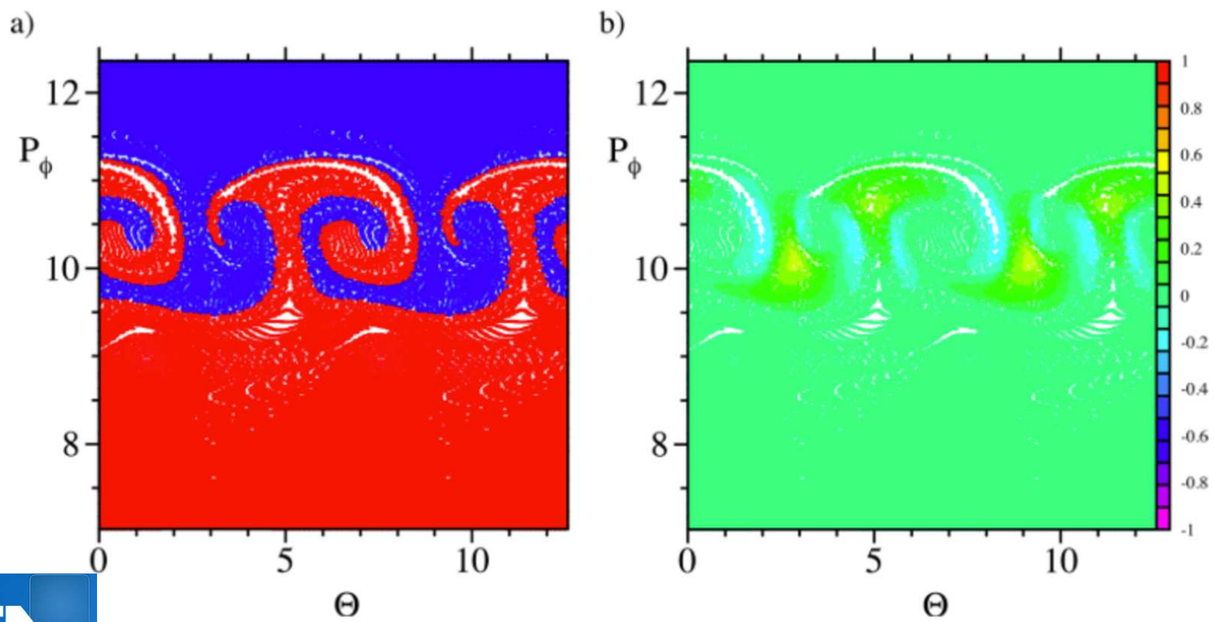
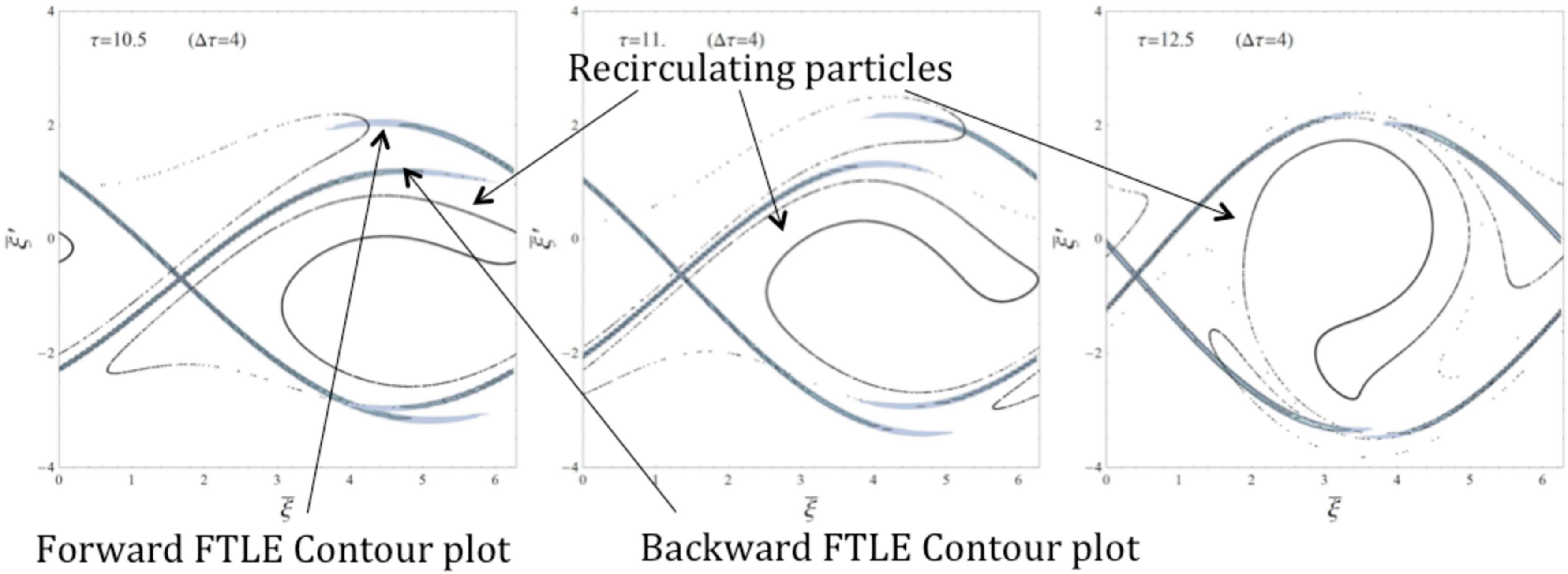


FIG. 13. Same as Fig. 12, at $t = 1640\tau_{A0}$.

The use of Lagrangian Coherent Structures

- Lagrangian Coherent Structures (LCS) have been introduced as representation of barrier between phase-space regions that do not mix on a given time scale (see Hands on Session 1).



- In Hands on Session 1, we have computed LCS starting from finite time Lyapunov exponents (FTLE).
- Illustration of how to practically compute LCS from FTLE is given in Hands on Session 4.
- Usefulness of application of LCS to the beam plasma problem is illustrated in Lecture 5.

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