Lecture 1

Basic concepts in Hamiltonian mechanics and Hamiltonian mapping

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Lecture Plan and Teaching Material

Abstract: Prerequisite of the course is a preliminary knowledge of general plasma physics and classical mechanics. General theoretical aspects are treated in Lectures $(1 \div 6)$, which are complemented by corresponding Hands On Sessions for practical demonstration based on numerical simulations. Hands On Sessions are also meant to serve as Q&A time.

Main areas that will be explored are:

i) Basic concepts in Hamiltonian mechanics and Hamiltonian mapping (Lecture 1)
The concept and use of Lagrangian Coherent Structures (Hands On Session 1)

ii) The nonlinear beam-plasma system: Vlasov formulation (Lecture 2)
Numerical solution of the Vlasov-Poisson problem (Hands On Session 2)

iii) The nonlinear beam-plasma system: O'Neil formulation (Lecture 3)
Numerical solution of the O'Neil problem (Hands On Session 3)

iv) Physics processes in the nonlinear beam-plasma dynamics (Lecture 4)

Use of kinetic Poincaré plots and Lagrangian Coherent Structures for illustrating nonlinear physics (Hands On Session 4)

v) Nonlinear beam-plasma dynamics with multiple resonances (Lecture 5)
Resonance overlap and stochastization of particle orbits (Hands On Session 5)

vi) Quasi-linear theory and diffusive transport (Lecture 6)
Characterization of transport (Hands On Session 6)

Lecture Notes: Available in electronic form. At the end of each lecture and during the Q&A time, a list of specific reading material is given explicitly.

Exercises: In the lecture notes, Exercises (E) of various difficulty levels are suggested. These are meant to be important part of the lectures themselves.

Hands On Sessions: Also available in electronic form and prepared by Teaching Assistants (Chen Zhao, Yuan Quan and Lai Po-Yen). Active participation to practical demonstrations is crucial for actual understanding of technical aspects involved with numerical simulations.

Lagrangian and Hamiltonian formulation of classical mechanics

- Equations of motion must be equivalent in all coordinate systems and can be obtained from one another by coordinate transformations.
- Introduce vector position and velocity $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ and the Lagrangian function

$$L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) = T(\dot{\boldsymbol{q}}, \boldsymbol{q}) - U(\boldsymbol{q}, t)$$

  where $T(\dot{\boldsymbol{q}}, \boldsymbol{q})$ is the kinetic energy and $U(\boldsymbol{q}, t)$ is the potential energy.
- Equations of motion in the Lagrangian form are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

E: Give a physical reason why $T(\dot{\boldsymbol{q}}, \boldsymbol{q})$ does not depend explicitly on time. Can you give an example in which the potential energy can be function of $\dot{\boldsymbol{q}}$, i.e. it can be expressed in the form $U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$?

E: Show that the Euler-Lagrange form of the equations of motions coincide with Newton's law. What is the general form of the force field?

E: Show that the Euler-Lagrange form of the equations of motions can be derived from the Least Action Principle, i.e. the variational form $\delta \int L dt = 0$.

- The Hamiltonian function is defined by

$$H(\boldsymbol{p}, \boldsymbol{q}, t) \equiv \sum_i \dot{q}_i p_i - L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$$

where $\dot{\boldsymbol{q}}$ is considered as a function of $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{p}, \boldsymbol{q})$.

- By definition one has

$$
\begin{aligned}
dH &= \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \\
&= \sum_i \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\
&= \sum_i \left(p_i - \frac{\partial L}{\partial \dot{q}_i} \right) d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) dq_i - \frac{\partial L}{\partial t} dt
\end{aligned}
$$

- This is consistent if and only if equations of motion in the Hamiltonian form are satisfied

$$
\dot{q}_i = \frac{\partial H}{\partial p_i} \;\; ; \;\; \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \left(p_i = \frac{\partial L}{\partial \dot{q}_i} \right) \;\; ; \;\; \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = -\frac{dH}{dt}
$$

- Any set of variables $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfying equations of motion in the Hamiltonian form are called canonical and p_i and q_i are called canonical conjugate variables.

E: Show that, given the Hamiltonian form of equations of motion, $H = T + U$. What properties of T do you need to use? Show that, if H does not depend on time explicitly, then H is conserved and the energy is conserved.

E: Show that, given the Hamiltonian H is the Legendre transform of the Lagrangian L. Note that the construction of the Hamiltonian is always possible because T is a quadratic form of $\dot{q}_i$, i.e. $T = (1/2)\sum_{i,j} a_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$.

E: Show that, for particles with charge e and mass m, the canonical momentum in an e.m. field is $\boldsymbol{p} = m\boldsymbol{v} + (e/c)\boldsymbol{A}$. Hint: recall that a general force can be given by the expression $\boldsymbol{F} = -\partial_{\boldsymbol{q}} U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) + (d/dt)\partial_{\dot{\boldsymbol{q}}} U(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$

Generating functions and canonical transformations

- Canonical transformations preserve the Hamiltonian structure of the equations of motions. Thus, they must satisfy the same Least Action Principle

$$\delta \left[\int_{t_1}^{t_2} \left(\boldsymbol{p}\dot{\boldsymbol{q}} - H(\boldsymbol{p}, \boldsymbol{q}, t) \right) dt \right] = 0$$

- The integrands, expressed in old $\boldsymbol{p}, \boldsymbol{q}$ and new $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ coordinates, should differ at most of a complete differential, i.e.

$$\boldsymbol{p} d\boldsymbol{q} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = \bar{\boldsymbol{p}} d\bar{\boldsymbol{q}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$$

with $F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$ being the generating function of the canonical transformation from $\boldsymbol{p}, \boldsymbol{q}$ and new $\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}$ coordinates.

- From the differential forms above, one readily derives

$$\boldsymbol{p} = \frac{\partial F_1}{\partial \boldsymbol{q}} \;\; ; \;\; \bar{\boldsymbol{p}} = -\frac{\partial F_1}{\partial \bar{\boldsymbol{q}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_1}{\partial t}$$

- Introduce the generating function $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$ via the Legendre transform $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t) = F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) + \bar{\boldsymbol{p}}\bar{\boldsymbol{q}}$ and substitute dF_2 into the differential forms

$$\boldsymbol{p} d\boldsymbol{q} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = -\bar{\boldsymbol{q}} d\bar{\boldsymbol{p}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$$

One derives readily

$$\boldsymbol{p} = \frac{\partial F_2}{\partial \boldsymbol{q}} \;\; ; \;\; \bar{\boldsymbol{q}} = \frac{\partial F_2}{\partial \bar{\boldsymbol{p}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_2}{\partial t}$$

- Introduce the generating function $F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t)$ via the Legendre transform $F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t) = F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) - \boldsymbol{p}\boldsymbol{q}$ and substitute dF_3 into the differential forms

$$-\boldsymbol{q}d\boldsymbol{p} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = \bar{\boldsymbol{p}}d\bar{\boldsymbol{q}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t)$$

One derives readily

$$\boldsymbol{q} = -\frac{\partial F_3}{\partial \boldsymbol{p}} \;\; ; \;\; \bar{\boldsymbol{p}} = -\frac{\partial F_3}{\partial \bar{\boldsymbol{q}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_3}{\partial t}$$

- Introduce the generating function $F_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t)$ via the Legendre transform $F_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t) = F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t) + \bar{\boldsymbol{p}}\bar{\boldsymbol{q}}$ and substitute dF_4 into the differential forms

$$-\boldsymbol{q}d\boldsymbol{p} - H(\boldsymbol{p}, \boldsymbol{q}, t)dt = -\bar{\boldsymbol{q}}d\bar{\boldsymbol{p}} - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)dt + dF_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t)$$

One derives readily

$$\boldsymbol{q} = -\frac{\partial F_4}{\partial \boldsymbol{p}} \;\; ; \;\; \bar{\boldsymbol{q}} = \frac{\partial F_4}{\partial \bar{\boldsymbol{p}}} \;\; ; \;\; \bar{H} = H + \frac{\partial F_4}{\partial t}$$

Hamilton - Jacobi equation

- With canonical transformation theory, one can formally solve the equations of motion. For example, choosing $\bar{H} = 0$, the new canonical coordinates will be constants of motion; i.e., they can be considered as initial values of the untransformed coordinates. Thus, the equations for coordinate transformation give positions and momenta at any time of the considered motion.

- In this case, $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$ is called Hamilton's principal function and can be determined from (note that $\bar{\boldsymbol{p}}$ plays the role of a parameter)

$$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}, t\right) + \frac{\partial F_2}{\partial t} = 0$$

- For an autonomous Hamiltonian, $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}})$ is called Hamilton's characteristic function and, setting $H = E$, can be determined from the Hamilton - Jacobi equation, i.e.

$$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}\right) = E$$

- As noted before, the solution of the Hamilton-Jacobi equation gives the Hamilton's characteristic function for generating the canonical transformation such that all p_i are constants and $q_i = \omega_i t + \beta_i$.
- Computing Hamilton's principal or characteristic functions is generally difficult unless the corresponding equations are separable. Integrability and separability of the Hamilton-Jacobi equation are linked in a subtle way.
- The Hamilton-Jacobi formalism is generally very useful for obtaining approximate (series) solutions of systems that are nearly separable or nearly-integrable.

E: Discuss examples of integrable (separable) Hamiltonian systems on the basis of your knowledge.

The extended phase space

- The Least Action Principle, underlying the derivation of the equations of motion, can be rewritten introducing a parameter ζ, with respect to which all variations are independent; i.e.

$$\delta \left[\int \left(\boldsymbol{p} \frac{d\boldsymbol{q}}{d\zeta} - H(\boldsymbol{p}, \boldsymbol{q}, t) \frac{dt}{d\zeta} \right) d\zeta \right] = 0$$

- From the form of the variational principle, it is intuitive that we can define an extended phase space setting in $2N + 2$ dimensions by letting $\bar{p}_i = p_i$, $\bar{q}_i = q_i$ for $i = 1, N$ and $\bar{p}_{N+1} = -H$, $\bar{q}_{N+1} = t$.

- This corresponds to introducing the generating function

$$F_2 = \sum_{i=1}^{N} \bar{p}_i q_i - Ht \;\; ; \;\; \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = H(\boldsymbol{p}, \boldsymbol{q}, t) - H$$

- Equations of motion in the Hamiltonian form are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} \;\; ; \;\; \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i}$$

- $\bar{H}$ is independent of the parameter (time-like) ζ, so that $\bar{H} = \text{const}$, while $t(\zeta) = \zeta + \zeta_0 = \zeta$, for ζ_0 can be arbitrarily set to zero.

The reduced phase space

- Given the concept of the extended phase space, every system can be described as autonomous system, i.e. with an Hamiltonian that does not explicitly depend on time; i.e.

$$H(\boldsymbol{p}, \boldsymbol{q}) = H_0$$

- Conversely, one could use this equation to solve one of the momenta, say p_N, as a function of $\bar{p}_i = p_i$ and $\bar{q}_i = q_i$ for $i = 1, N-1$ and $\zeta = q_N$ considered as a parameter; $p_N = p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$. Letting $\bar{H} = -p_N(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, q_N)$, the equations of motion in the $2N-2$ dimensional reduced phase space are

$$\frac{d\bar{p}_i}{d\zeta} = -\frac{\partial \bar{H}}{\partial \bar{q}_i} = -\frac{1}{\dot{q}_N}\frac{\partial H}{\partial q_i} \quad ; \quad \frac{d\bar{q}_i}{d\zeta} = \frac{\partial \bar{H}}{\partial \bar{p}_i} = \frac{1}{\dot{q}_N}\frac{\partial H}{\partial p_i}$$

- If the momentum p_N is a constant, since the original $H(\boldsymbol{p}, \boldsymbol{q})$ did not explicitly depend on q_N, then $\bar{H}$ is constant and the reduced phase space can be further reduced.
- A very useful application of the notion of reduced phase space is the definition of a Poincaré surface of section.
- For a 2D system, the motion is bound to occur within the energy surface $H(p_1, p_2, q_1, q_2) = H_0$, which can be written as

$$p_2 = p_2(p_1, q_1, q_2)$$

- If the motion is bounded in the phase space, the motion can repeatedly cross the plane $q_2 = \text{const}$, which is a convenient choice of the surface of section, coinciding with the reduced phase space of the original Hamiltonian system.

- In general, the subsequent crossings of the motion in the (p_1, q_1) surface of section can occur everywhere. However, if a constant of motion exist, in addition to H_0, then $I(p_1, p_2, q_1, q_2) = \text{const}$ in addition to $p_2 = p_2(p_1, q_1, q_2)$. Therefore

$$p_1 = p_1(q_1, q_2)$$

  and the subsequent crossings of the motion in the (p_1, q_1) surface of section must lie on one curve.

- Vice-versa, the fact that the motion in the (p_1, q_1) surface of section lies on a curve can be used as evidence of the existence of a constant of motion in addition to H_0.

- More generally, for $N > 2$ degrees of freedom, we can construct the reduced phase space $p_N = p_N(\boldsymbol{p}, \boldsymbol{q}, q_N)$ and define the surface of section $q_N =$ const. In the reduced phase space, the motion is volume preserving. If one or more constants of motion exists, other than H_0, the system motion will lie on surfaces of dimensionality less than $2N - 2$, where trajectories will be volume preserving.

- If the motion is exactly separable in the (p_i, q_i) coordinates, the motion in each (p_i, q_i) plane is area preserving, a constant of motion (action integral) exists in each degree of freedom and the motion in each (p_i, q_i) plane lies on a smooth curve.

- In general, for a system with N degrees of freedom, the system trajectories projected in the (p_i, q_i) surface of section do not lie on a smooth curve but lie on an annulus of finite area, whose size is related with the nearness to the exact separability of the motion in the (p_i, q_i) coordinates.

Action-angle variables

- In this Lecture, it was already pointed out that separability and integrability are concepts related in a subtle way. Assume we can write the generating function F_2 as

$$F_2 = \sum_i S_i(q_i, \alpha_1 ... \alpha_N)$$

where the constants α_i are the new momenta associated with the N constants of motion.

- Note that, for a system with N-degrees of freedom, with $N > 1$, N constants of motion that decouple the N degrees of freedom can be found if the Hamilton-Jacobi equation is completely separable in some coordinate system (for $N = 1$ this is obvious for the constants coincides with the energy).

- If the Hamiltonian can be written in the separated form (not the most general one)

$$H = \sum_i H_i\left(\frac{\partial S_i}{\partial q_i}, q_i\right)$$

then the Hamilton-Jacobi equation can be separated into N conditions

$$H_i\left(\frac{\partial S_i}{\partial q_i}, q_i\right) = \alpha_i \quad ; \quad \sum_i \alpha_i = H_0$$

- In general, the new momenta can be taken as generic independent functions of the α_i, i.e. $J_i = J_i(\boldsymbol{\alpha})$ and $\alpha_i = \alpha_i(\boldsymbol{J})$. Thus

$$F_2 = \sum_i S_i(q_i, \boldsymbol{\alpha}(\boldsymbol{J})) = \sum_i \bar{S}_i(q_i, \boldsymbol{J}) \quad ; \quad \bar{H}(\boldsymbol{J}) = \sum_i \alpha_i(\boldsymbol{J})$$

- A particular choice of $\boldsymbol{J}$ is useful for completely separable periodic systems, i.e. system for which in every degree of freedom
  - p_i and q_i are periodic functions of t with the same period: the motion is called libration
  - p_i is a periodic function of q_i: the motion is called rotation
- The periods in each degree of freedom need not be the same. If the periods are not in the ratio of rational numbers the motion is called conditionally periodic.
- The convenient choice of J_i are via the action integrals (constants of motion)

$$J_i = \frac{1}{2\pi} \oint p_i dq_i$$

- The conjugate coordinates θ_i are given by

$$\dot{\theta}_i = \frac{\partial \bar{H}}{\partial J_i} = \omega_i \quad ; \quad \theta_i = \omega_i t + \beta_i$$

- The conjugate coordinates θ_i are angles in the sense that, after one period

$$\Delta\theta_i = \oint d\theta_i = \oint \frac{\partial}{\partial q_i} \frac{\partial \bar{S}}{\partial J_i} dq_i = \frac{\partial}{\partial J_i} \oint \frac{\partial \bar{S}}{\partial q_i} dq_i = \frac{\partial}{\partial J_i} \oint p_i dq_i = 2\pi$$

- From this condition, the period of the motion is $T = 2\pi/\omega_i$ and ω_i is the angular frequency of the motion.
- When studying near-integrable systems, the integrable part of the system is usually represented in action-angle variables, while the further response is investigated using perturbation theory or other methods.

General notion of mapping

- Visualization of multiply periodic motions is important for increased understanding.
- Topological considerations that can be made, naturally yield to a set of difference equations, that can be seen as *mapping* of the dynamical trajectory onto a subset of the system phase space.
- Not only visualization (and understanding) are often simpler via mapping equations. Physics properties, such as numerical calculation of stochastic behaviors, are often simpler when considered for mappings.
- Conversely, regular trajectories are very conveniently described in terms of differential equations.

The twist mapping

- On pp. 16-17, we discussed the representation of trajectories in a system with two degrees of freedom by the surface of section plots. Say this is the $J_1 - \theta_1$ plane at $\theta_2 =$ const. with no loss of generality; so drop the subscript 1 as well for simplicity of notation.

- After a time $\Delta t = 2\pi/\omega_2$ the "initial" J_n, θ_n coordinates will advance to

$$J_{n+1} = J_n \;\; ; \;\; \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) \;\; ; \;\; \alpha = \omega_1/\omega_2 \equiv \text{rotation number}$$

- The mapping is called the *twist mapping* and the use of $\alpha(J_{n+1})$ is for convenience.
- The twist mapping , as two-dimensional mapping, must be area preserving

$$\frac{\partial(J_{n+1}, \theta_{n+1})}{\partial(J_n, \theta_n)} \equiv [\theta_{n+1}, J_{n+1}] = 1$$

Near-integrable systems

- Area preserving mapping are of particular interest and are connected with canonical mappings.

E: Demonstrate that canonical mappings, i.e. mappings that are generated by Hamilton's equations, must be area preserving.

- Consider the near-integrable Hamiltonian obtained as perturbation of a two-dimensional integrable system

$$H(\boldsymbol{J}, \boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta})$$

- For this system, one expects that the twist mapping is modified into the *perturbed twist mapping*

$$J_{n+1} = J_n + \epsilon f(J_{n+1}, \theta_n) \;\; ; \;\; \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) + \epsilon g(J_{n+1}, \theta_n)$$

E: Show that the functions f and g are periodic in θ. Show that, since they are functions of J_{n+1} rather than J_n, the area-preserving form of the mapping is particularly simple.

E: Is there any loss of generality having chosen f and g as functions of J_{n+1} rather than J_n? Why?

- Note that the perturbed twist mapping can be obtained from the generating function

$$F_2 = J_{n+1}\theta_n + 2\pi\mathcal{A}(J_{n+1}) + \epsilon\mathcal{G}(J_{n+1},\theta_n) \;\; ; \;\; J_n = \frac{\partial F_2}{\partial \theta_n} \;\; ; \;\; \theta_{n+1} = \frac{\partial F_2}{\partial J_{n+1}}$$

- Thanks to these simple rules, the perturbed twist mapping is obtained as

$$\alpha = \frac{d\mathcal{A}}{dJ_{n+1}} \;\; ; \;\; f = -\frac{\partial \mathcal{G}}{\partial \theta_n} \;\; ; \;\; g = \frac{\partial \mathcal{G}}{\partial J_{n+1}}$$

- From these relations, one easily sees that the condition for area preservation is

$$\frac{\partial g}{\partial \theta_n} + \frac{\partial f}{\partial J_{n+1}} = 0$$

E: Show that this condition is actually giving area preservation from the definition of the perturbed twist mapping.

- For many interesting mappings, f is independent of J and $g \equiv 0$. The the perturbed twist mapping takes the form of a *radial twist mapping*

$$J_{n+1} = J_n + \epsilon f(\theta_n) \;\; ; \;\; \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1})$$

Hamiltonian forms and mappings

- In two dimensions, the functions f and g in the perturbed twist mapping are determined from integration of the Hamilton's equations

$$\frac{dJ_1}{dt} = -\epsilon \frac{\partial H_1}{\partial \theta_1}$$

- The jump in action over one period in the θ_2 motion is $\Delta J_1 = \epsilon f(J_{n+1}, \theta_n)$, using the unperturbed motion for the integration

$$\Delta J_1 = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \theta_1}(J_{n+1}, J_2, \theta_n + \omega_1 t, \theta_{20} + \omega_2 t) dt$$

- The jump is phase can be obtained from the area preserving condition

$$g(J,\theta) = -\int^{\theta} \frac{\partial f}{\partial J} d\theta'$$

- For computation of ΔJ_1, the second degree of freedom can be expressed in generic (not necessarily action-angle) canonical coordinates.

$$\Delta J_1 = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \theta_1}(J_{n+1}, \theta_n + \omega_1 t, p_2(t), q_2(t))dt$$

- This method can be generalized to N degrees of freedom

$$\Delta \boldsymbol{J} = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \boldsymbol{\theta}}(\boldsymbol{J}_{n+1}, J_N, \boldsymbol{\theta}_n + \boldsymbol{\omega} t, \theta_{N0} + \omega_N t)dt$$

- One can then derive the jumps of $N-1$ actions $\epsilon \boldsymbol{f} = \Delta \boldsymbol{J}$. One then finds $\boldsymbol{g} = \partial \mathcal{G}/\partial J_{n+1}$, knowing $\mathcal{G}$ from direct integration of $\boldsymbol{f} = -\partial \mathcal{G}/\partial \boldsymbol{\theta}_n$.

- One can construct the inverse procedure, e.g., construct the equations in Hamiltonian form, given the radial twist mapping.
- This can be done considering that the iteration number plays the role of time and using the function

$$\delta_1(n) = \sum_{m=-\infty}^{\infty} \delta(n-m) = 1 + 2\sum_{q=1}^{\infty} \cos 2\pi nq$$

- The radial twist mapping can then be written as

$$(dJ/dn) = \epsilon f(\theta)\delta_1(n) \;\; ; \;\; (d\theta/dn) = 2\pi\alpha(J)$$

which are expressed in Hamiltonian form with

$$H(J,\theta,n) = 2\pi \int^J \alpha(J')dJ' - \epsilon\delta_1(n) \int^\theta f(\theta')d\theta'$$

References and reading material

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