Lecture 6

Energetic Particle Modes: solitons in active media and modulation of the radial envelope

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The GFLDR in the WKB limit

(Reminder Lecture 4)

- For localized mode structures, we can adopt the WKB representation

$$A_n(r) \sim \exp i \int nq'\theta_k(r)dr \quad ,$$

with $\theta_k(r) \equiv k_r(r)/(nq')$ the normalized radial wave vector.

- The GFLDR can be cast as [Zonca and Chen 2014]

$$\left[i\Lambda_n - \left(\delta\bar{W}_f + \delta\bar{W}_k \right)_n \right] A_n(r) = D_n(r, \theta_k, \omega) A_n(r) = 0 \quad ,$$

with $D_n(r, \theta_k, \omega)$ playing the role of a local dispersion function (Lecture 1)

  - Λ_n: generalized inertia response
  - $\delta\bar{W}_{fn}$: potential energy due to fluid plasma response
  - $\delta\bar{W}_{kn}$: potential energy due to kinetic plasma response

Nonlinear DAW field equations

(Lecture 1)

- Consistent with the WKB approach, the lowest order (local) dispersion relation must be satisfied in the linear limit

$$D_n^L(r, \theta_{k0}(r), \omega_0) = 0 \quad ;$$

- At the next order, $\omega = \omega_0 + i\partial_t$, $k_r = nq'\theta_{k0} - i\partial_r$; and the GFLDR theoretical framework describes the slow spatial and temporal evolution of the wave packet $A_n = \exp(-i\omega_0 t) A_{n0}(r,t)$ [Lu POP12; Chen RMP14].

$$\frac{\partial D_n^L}{\partial \omega_0}\left(i\frac{\partial}{\partial t}\right) A_{n0}(r,t) + \frac{\partial D_n^L}{\partial \theta_{k0}}\left(-\frac{i}{nq'}\frac{\partial}{\partial r} - \theta_{k0}\right) A_{n0}(r,t) + \frac{1}{2}\frac{\partial^2 D_n^L}{\partial \theta_{k0}^2}\left[\left(-\frac{i}{nq'}\frac{\partial}{\partial r} - \theta_{k0}\right)^2 - \frac{i}{nq'}\frac{\partial \theta_{k0}}{\partial r}\right] A_{n0}(r,t) = \left(\delta \bar{W}_f^{NL} + \delta \bar{W}_k^{NL}\right)_n - i\Lambda_n^{NL} + S_n^{ext}(r,t) \qquad k_r \equiv nq'\theta_{k0}$$

- Λ_n^{NL}: nonlinear response in the kinetic/singular layer; dominated by thermal plasma, wave-wave interactions, competition of Reynolds and Maxwell stress (Lectures by Prof. Liu Chen, following this course)
- $\delta\bar{W}_{fn}^{NL}$: nonlinear potential energy response, due to both thermal plasma and EPs. Non-resonant behaviors in the regular regions
- $\delta\bar{W}_{kn}^{NL}$: nonlinear potential energy due to resonant kinetic plasma response, due to both thermal plasma and EPs

□ Nonlinear dynamics of phase-space zonal structures driven by EPs is determined by $\delta\bar{W}_{kn}^{NL}$ and by nonlinear resonant EP behaviors. $\Rightarrow$ Here, we assume $\Lambda_n^{NL} = 0$ and $\delta\bar{W}_{fn}^{NL} = 0$.

□ In Lecture 5 (fishbones) and this Lecture (EPMs), we adopt a very simple model for Λ_n^L, $\delta\bar{W}_{fn}^L$ and $\delta\bar{W}_{kn}^L$.

□ In the following, use $\delta\bar{W}_{kn}^{NL}$ for simple case of precession resonance and vanishing Larmor radius and magnetic drift orbit width.

- From Lecture 4 (p. 15)

$$\delta \bar{W}_{nk} = \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm} \frac{\pi^2 q R_0}{c^2 k_\vartheta^2 |s|} \frac{e^2}{m} \left(\frac{\tau_b n^2 \bar{\omega}_{dn}^2}{\omega_0(\tau)} \right) \int_{-\infty}^{+\infty} \frac{\omega + \omega_0(\tau)}{n\bar{\omega}_{dn} - \omega_0(\tau) - \omega} e^{-i\omega t} Q_{k,\omega_0(\tau)} \hat{F}_0(\omega) d\omega \;;$$

- Definitions: $\mathcal{E} = v^2/2$, $\lambda = \mu B_0/\mathcal{E}$, $\tau_b = 2\pi/\omega_b$, $k_\vartheta = -nq/r$, $\omega_0(\tau) = \omega_{0r}(\tau) + i\gamma_0(\tau)$, $s = rq'/q$.

- Note that the wave-vector and frequency $(k, \omega_0(\tau))$ values are indicated on which the $Q_{k,\omega_0(\tau)}$ operator [acting on $\hat{F}_0(\omega)$] is to be computed.

- From Lecture 5 (p. 6), the Dyson Equation with $|\omega_{*EP}| \gg |\omega_0| \Rightarrow \omega_0(\tau)^{-1} Q_{k_0,\omega_0(\tau)} \bar{F}_0 \simeq (\omega_0 \Omega)^{-1} (-nq/r) \partial_r \bar{F}_0$ is (up to order $\sim \gamma_0/\omega_0$)

$$\hat{F}_0(\omega) = \frac{i}{\omega} \mathrm{St} \hat{F}_0(\omega) + \frac{i}{\omega} \hat{S}(\omega) + \frac{i}{2\pi\omega} \bar{F}_0(0) - 2\frac{i}{\omega} \left(\frac{n\bar{\omega}_{dk0}}{\omega_{0r}} \right) \frac{\partial}{\partial r}$$

$$\times \left[\frac{(\gamma_0 + i\omega) + (n\bar{\omega}_{dk0} - \omega_{0r})(\gamma_0/\omega_{0r})}{(n\bar{\omega}_{dk0} - \omega_{0r})^2 + (\gamma_0 + i\omega)^2} \left| \delta\bar{\xi}_{rk0}(r,\tau) \right|^2 \frac{\partial}{\partial r} \hat{F}_0 \left(\omega - 2i\gamma_0(\tau) \right) \right] \;.$$

Radial localization of linear EPM

- Expression for $\delta \bar{W}_f$: assume $2\kappa(s)\cos\theta_k \simeq 2\kappa(s) - \kappa(s)\theta_k^2$.

$$\delta \bar{W}_f = \frac{|s|\pi}{8}\left(1 + 2\kappa(s) - \frac{\alpha}{\alpha_c}\right) + \frac{|s|\pi}{8}\kappa(s)\frac{\partial^2}{\partial x^2} \quad ,$$

$$\alpha_c = \frac{s^2}{1+|s|} \quad ; \qquad \kappa(s) \simeq \frac{1}{2}\left(1 + \frac{1}{|s|}\right)e^{-1/|s|} \quad ; \qquad x \equiv |nq'|(r - r_0) \; .$$

- As reference case (Lecture 5), consider isotropic slowing-down $\bar{F}_0$

$$\bar{F}_0 = \frac{3P_{0E}}{4\pi E_F}\frac{\mathrm{H}(E_F/m_E - \mathcal{E})}{(2\mathcal{E})^{3/2} + (2E_c/m_E)^{3/2}} \quad ,$$

where H denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is $(3/2)P_{0E}$ for $E_F \gg E_c$, and EP energy is predominantly transferred to thermal electrons by collisional friction as it occurs for α-particles in fusion plasmas.

- Governing equations for EPM structure near the toroidal Alfvén eigenmode (TAE) gap:

$$
\begin{aligned}
i\Lambda_T A \;=\;& \frac{|s|\pi}{8}\left(1+2\kappa(s)-\frac{\alpha}{\alpha_c}\right)A+\frac{|s|\pi}{8}\kappa(s)\frac{\partial^2 A}{\partial x^2}\\
&+\frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E}\left(1-\frac{x^2/s^2}{k_\vartheta^2 L_{pE}^2}\right)\left\{1+\frac{\omega}{\bar\omega_{dF}}\left[\ln\left(\frac{\bar\omega_{dF}}{\omega}-1\right)+i\pi\right]\right\}A\ ,
\end{aligned}
$$

- Definitions: $\Lambda_T = (1/2)(\Gamma_+/\Gamma_-)^{1/2}$ [Zonca and Chen 2014], subscript T stands for toroidal, $\Gamma_\pm = \omega^2/\omega_A^2 - 1/4 \pm \epsilon_0\omega^2/\omega_A^2$, $\epsilon_0 = 2(r/R_0+\Delta')$.

- Note that the localized equilibrium EP radial profile has been assumed

$$
\alpha_E = \alpha_{0E}\exp\left[-(r-r_0)^2/L_{pE}^2\right] \simeq \alpha_{0E}\left(1-\frac{(r-r_0)^2}{L_{pE}^2}\right) = \alpha_{0E}\left(1-\frac{x^2}{k_\vartheta^2 s^2 L_{pE}^2}\right)\ .
$$

- Solutions for localized modes are readily found as $A(x,t) = A_0(x)e^{-i\omega t}$, with $\omega = \omega_0 + i\gamma_L$, $A_0(x) = \bar{A}_0 \exp(-x^2/2\Delta^2)$ and [Zonca and Chen 00]

$$\Delta^{-4} = \frac{3(r/R_0)^{1/2}}{\sqrt{2}\kappa(s)s^4k_\vartheta^2L_{pE}^2}\alpha_{0E}\left\{1 + \frac{\omega_0}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0} - 1\right) + i\pi\right]\right\} \quad ,$$

$$\frac{|s|\pi}{8}\left[1 + \left(2 - \mathbb{R}\mathrm{e}\left(\Delta^{-2}\right)\right)\kappa(s) - \frac{\alpha}{\alpha_c}\right] + \frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E}\left[1 + \frac{\omega_0}{\bar{\omega}_{dF}}\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0} - 1\right)\right] = 0 \quad ,$$

$$\frac{\gamma_L}{\omega_0} = \left[\frac{\omega_0/\bar{\omega}_{dF}}{1 - \omega_0/\bar{\omega}_{dF}} - \frac{\omega_0}{\bar{\omega}_{dF}}\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0} - 1\right)\right]^{-1}\left[\pi\frac{\omega_0}{\bar{\omega}_{dF}} - \left(\frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E}\right)^{-1}\right.$$
$$\left.\times\left(\Lambda_T(\omega_0) + \frac{|s|\pi}{8}\kappa(s)\mathbb{I}\mathrm{m}\left(\Delta^{-2}\right)\right)\right] \quad .$$

- EPM radial localization is determined by the energetic particle source and is effectively provided only in the presence of wave-particle resonant drive.

Nonlinear Dynamics of a single-n coherent EPM

- Linear properties: EPM radial localization is determined by the energetic particle source and is effectively provided only in the presence of wave-particle resonant drive.
- These properties have important consequences on the nonlinear dynamics as well. In the following, it is demonstrated that they also hold in the nonlinear case.
- Follow the same strategy as in the case of nonlinear fishbone (Lecture 5).
- Nonlinear wave-EP interactions are dominated by resonant particles. So, we must derive $\delta \bar{W}_k^{NL} \sim i\mathbb{I}\mathrm{m}\delta \bar{W}_k^{NL}$.

E: Discuss for the EPM case (as you did for fishbone) the reasons why the nonlinear response is dominated by resonant particles. Can you demonstrate that non-resonant particle response is $\sim |\gamma_0/\omega_0|$ smaller?

- The calculation of the nonlinear response is constructed from $\hat{F}_0(\omega)$, solution of the Dyson equation. It involves integral in velocity and frequency space with integrand (see p.5; and Lecture 5 p. 13)

$$\propto \left[\frac{(\gamma_0 + i\omega) + (n\bar{\omega}_{dk0} - \omega_{0r})(\gamma_0/\omega_{0r})}{(n\bar{\omega}_{dk0} - \omega_{0r})^2 + (\gamma_0 + i\omega)^2} \frac{k_\vartheta^2 c^2}{|\omega_0|^2 B_0^2} \left|\delta\bar{\phi}_{k0}(r,\tau)\right|^2 \frac{\partial}{\partial r} \hat{F}_0 \left(\omega - 2i\gamma_0(\tau)\right) \right]$$
$$\times \frac{ie^{-i\omega t}}{\omega} \frac{[\omega_{0r} + i(\gamma_0 - i\omega)] \left[(n\bar{\omega}_{dk0} - \omega_{0r}) + i(\gamma_0 - i\omega)\right]}{(n\bar{\omega}_{dk0} - \omega_{0r})^2 + (\gamma_0 - i\omega)^2}$$

E: Comment about similarities and differences with the fishbone case studied in Lecture 5. Can you discuss the role of mode structures?

- Except for the resonance (in velocity space), the frequency response of $\hat{F}_0(\omega)$ has a double peak at $\omega \sim 0$ and $\omega \sim 2i\gamma_0$ (E: Demonstrate this.). that can be used for the analytic calculation of $\delta\bar{W}_k^{NL}$ by Cauchy's residue theorem.

- For $|a|, |b| \ll 1$, assuming $\Re a > 0$ and $\Re b > 0$, recall from Lecture 5

$$\frac{1}{(x^2+a^2)(x^2+b^2)} \to \frac{\pi}{ab(a+b)}\delta(x) \quad .$$

- Shifting $\omega \to \omega + 2i\gamma_0$ in the frequency integral to $\left|\delta\bar{\phi}_{k0}(r,t)\right|^2 = \exp(2\gamma_0 t)\left|\delta\bar{\phi}_{k0}(r,\tau)\right|^2$; and

$$\simeq (-i/2)(2\gamma_0 - i\omega)^{-2}\left[\pi\delta(n\bar{\omega}_{dk0} - \omega_{0r})e^{-i\omega t}\frac{k_\vartheta^2 c^2}{|\omega_0|^2 B_0^2}\left|\delta\bar{\phi}_{k0}(r,t)\right|^2 \frac{\partial}{\partial r}\hat{F}_0(\omega)\right]$$

$$= (-i/2)\partial_t^{-2}\left[\pi\delta(n\bar{\omega}_{dk0} - \omega_{0r})e^{-i\omega t}\frac{k_\vartheta^2 c^2}{|\omega_0|^2 B_0^2}\left|\delta\bar{\phi}_{k0}(r,t)\right|^2 \frac{\partial}{\partial r}\hat{F}_0(\omega)\right] \quad .$$

E: Verify the above equation and use it for computing $\delta\bar{W}_k^{NL}$.

EPM solitons in active media: convective amplification

$$
\begin{aligned}
\delta\bar{W}_k^{NL} &\simeq \frac{3\pi(r/R_0)^{1/2}\alpha_E}{8\sqrt{2}|s|} i\pi \frac{\omega_0}{n\bar{\omega}_{dnF}} k_\vartheta^2 v_E^2 \rho_{LE}^2 \partial_t^{-2} \frac{\partial^2}{\partial r^2} |A_0|^2 \quad , \\
&= \frac{3\pi(r/R_0)^{1/2}|s|\alpha_E}{8\sqrt{2}} i\pi \frac{\omega_0}{n\bar{\omega}_{dnF}} k_\vartheta^2 v_E^2 k_\vartheta^2 \rho_{LE}^2 \partial_t^{-2} \frac{\partial^2}{\partial x^2} |A_0|^2
\end{aligned}
$$

- Definitions: $v_E^2 = E_F/m_E$, $\rho_{LE}^2 = v_E^2/\Omega_E^2$. The nonlinear EPM equation becomes (p. 7) (recalling $A = e^{-i\omega_{0r}t}A_0(r,t)$ and dropping subscript 0)

$$
\begin{aligned}
i\Lambda_T A &= \frac{|s|\pi}{8}\left(1 + 2\kappa(s) - \frac{\alpha}{\alpha_c}\right) A + \frac{|s|\pi}{8}\kappa(s)\frac{\partial^2 A}{\partial x^2} + \frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E} \\
&\times \left(1 - \frac{x^2/s^2}{k_\vartheta^2 L_{pE}^2}\right)\left\{1 + \frac{\omega}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega} - 1\right) + i\pi\left(1 + k_\vartheta^2 v_E^2 k_\vartheta^2 \rho_{LE}^2 \partial_t^{-2}\frac{\partial^2}{\partial x^2}|A|^2\right)\right]\right\} A \ .
\end{aligned}
$$

- Look for solutions that can be expressed as the convectively amplified propagating (self-similar) wave-packet

$$A(\xi,t) = U(\xi) e^{\int^t \gamma_0(t')dt'} \equiv W(\xi) e^{i\varphi(\xi)+\int_0^t \gamma_0(t')dt'} \quad ,$$

$$\xi - \xi_0 \equiv \frac{k_{n0}}{|sk_\vartheta|}(x - x_0) \equiv \frac{k_{n0}}{|sk_\vartheta|}\left(x - |sk_\vartheta| \int_0^t v_g(t')dt' \right) \quad ,$$

with k_{n0} denoting the nonlinear wave-vector and v_g the nonlinear group velocity.

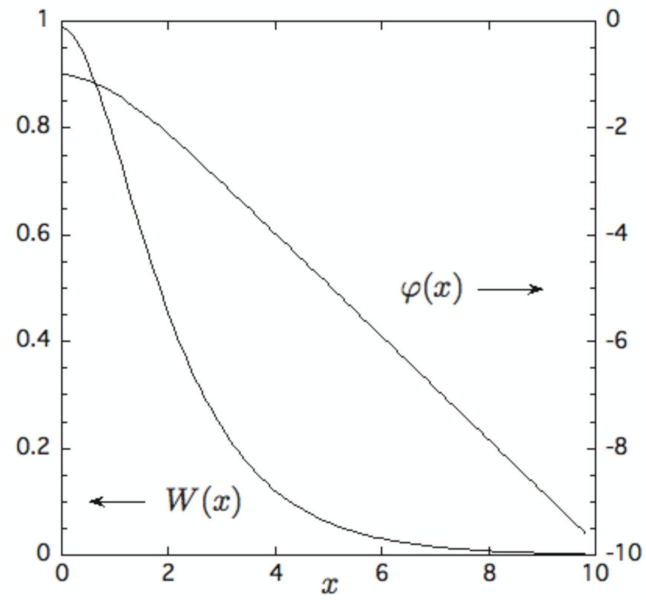
- Adopt the general procedure for isolating the behavior of the wave-packet soliton; i.e., balance the nonlinear term with the linear dispersiveness.

- This optimal balance fixes $k_{n0}v_g$ but the nonlinear group velocity

$$k_{n0}^2 v_g^2 = \frac{3\pi (r/R_0)^{1/2}\alpha_E}{2\sqrt{2}\kappa(s)} \frac{\omega_0}{\bar{\omega}_{dF}} k_\vartheta^4 v_E^2 \rho_{LE}^2 \exp\left(2\int_0^t \gamma_0(t')dt'\right) \quad ,$$

while $U(\xi)$ satisfies the nonlinear equation

$$\partial_\xi^2 U = \lambda_0 U - 2iU|U|^2 \quad .$$



- The value $\lambda_0 \simeq -0.47 + i1.32$ corresponds to the ground state of the corresponding complex nonlinear oscillator.

- For the time being, simply let

$$v_g^2 \simeq k_\vartheta^2 v_E^2 \rho_{LE}^2 \lambda_g^2 \exp\left(2\int_0^t \gamma_0(t')dt'\right) \quad ,$$

where $\lambda_g < 1$ is a coefficient, whose value is set by the maximization of the wave-particle power transfer in the "phase-locking" regime.

- Nonlinear EPM wave-packet dispersion relation

$$\frac{s|\pi}{8}\left[1+2\kappa(s)-\frac{\alpha}{\alpha_c}\right]+\frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E}\exp\left(-\frac{x_0^2/s^2}{k_\vartheta^2 L_{pE}^2}\right)\left\{1+\frac{\omega_0}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0}-1\right)+\frac{\pi}{2\lambda_g^2}\mathbb{R}\mathrm{e}\lambda_0\right]\right\}$$

$$\begin{aligned}\frac{\gamma_0}{\omega_0} &= \left[\frac{\omega_0/\bar{\omega}_{dF}}{1-\omega_0/\bar{\omega}_{dF}}-\frac{\omega_0}{\bar{\omega}_{dF}}\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0}-1\right)\right]^{-1}\left[\pi\frac{\omega_0}{\bar{\omega}_{dF}}\left(1+\frac{1}{2\lambda_g^2}\mathbb{I}\mathrm{m}\lambda_0\right)\right.\\ &\quad\left.-\left(\frac{3\pi(r/R_0)^{1/2}}{8\sqrt{2}|s|}\alpha_{0E}\right)^{-1}\exp\left(\frac{x_0^2/s^2}{k_\vartheta^2 L_{pE}^2}\right)\Lambda_T(\omega_0)\right] \quad .\end{aligned}$$

- The EPM wavepacket propagates as phase-locked amplified soliton as long as the nonlinear strengthened pressure gradient balances the decreasing drive and increasing continuum damping (radial nonuniformity).

- The value of λ_g^2, by maximization of the wave-particle power transfer in the phase-locking regime, is obtained from the nonlinear EPM dispersion relation and the condition

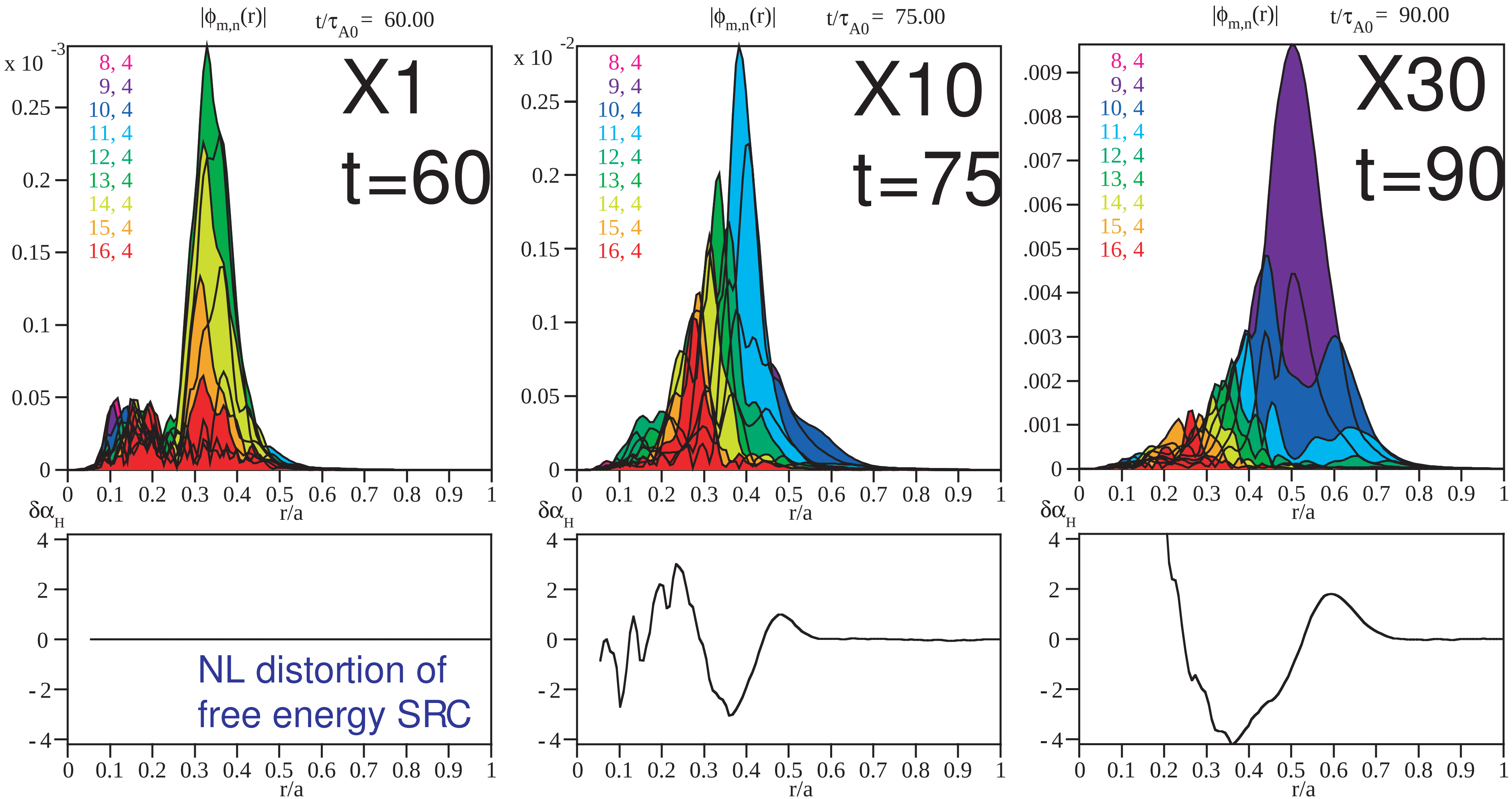
$$\frac{d\gamma_0}{d\lambda_g^2} = \frac{\partial\gamma_0}{\partial\lambda_g^2} + \frac{\partial\gamma_0}{\partial\omega_0}\frac{d\omega_0}{d\lambda_g^2} = 0 \quad .$$

- This equation has a solution $\lambda_g^2 \lesssim 1$ due to the optimal ordering in the nonlinear dispersion relation above and to the fact that $d\gamma_0/d\lambda_g^2 > 0$ for $\lambda_g^2 \to 0$, while $d\gamma_0/d\lambda_g^2 < 0$ for $\lambda_g^2 \to \infty$.

- For typical tokamak parameters, one obtains $\lambda_g \simeq 0.5 \div 0.6$, with a spread $\Delta\lambda_g \simeq \Delta\lambda_g^2 \simeq \gamma_0^{1/2}[-d^2\gamma_0/(d\lambda_g^2)^2]^{-1/2} \sim 0.1$.

Avalanches and NL EPM dynamics

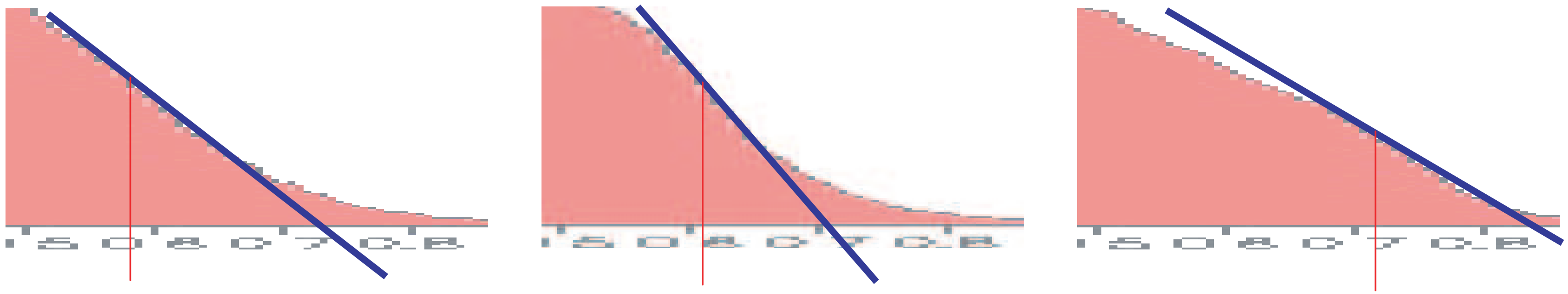
Zonca et al. IAEA, (2002)

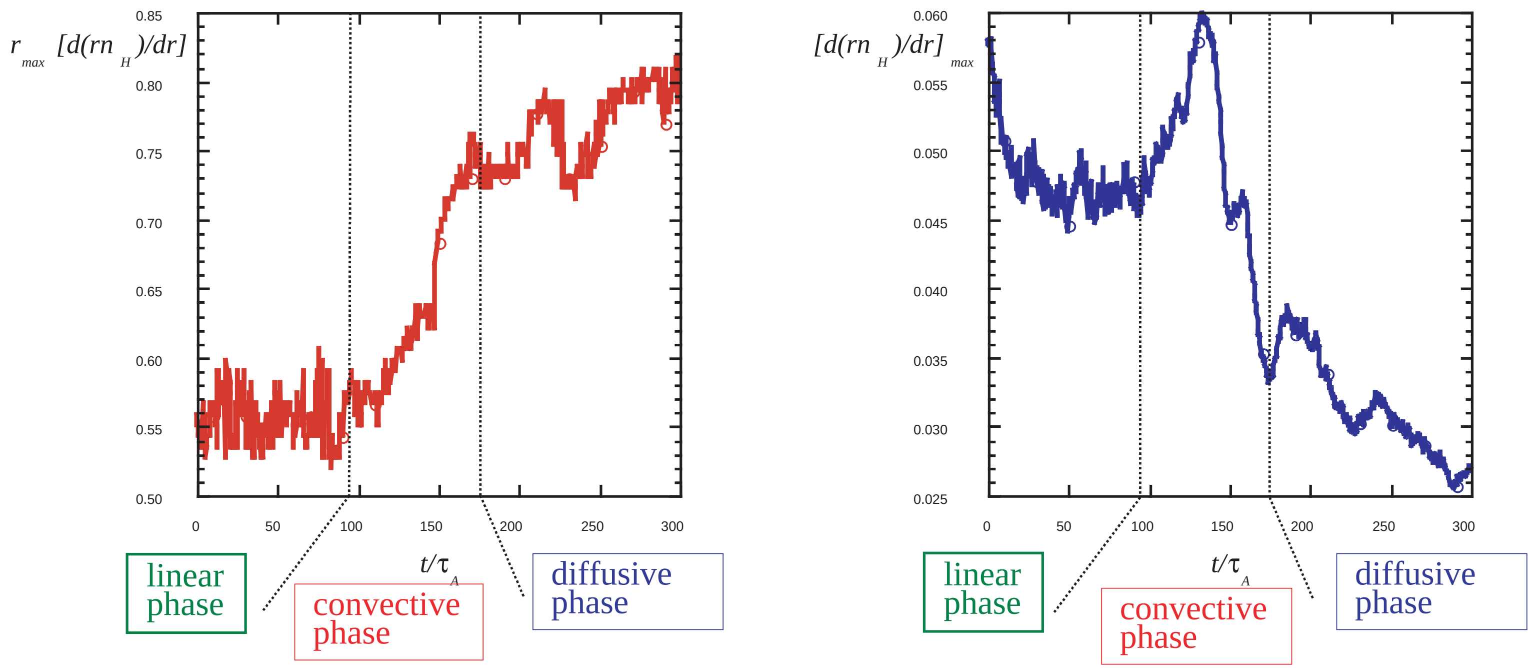


- ☐ Importance of toroidal geometry on wave-packet propagation and shape

Vlad et al. IAEA-TCM, (2003)

Propagation of the unstable front





- Gradient steepening and relaxation: spreading ... similar to turbulence

On the nonlinear Schrödinger equation

- The EPM soliton formation and convective wave-packet amplification yield

$$\partial_\xi^2 U = \lambda_0 U - 2iU|U|^2 \quad .$$

- Finite $\mathbb{I}\mathrm{m}\lambda_0 > 0$ is connected with enhanced nonlinear EP drive, in competition with continuum damping changing because or radial nonuniformity (see nonlinear EPM disp. rel. p. 15) and is due to the $-2iU|U|^2$ on the RHS. Thus, EPM are solitons in an active nonuniform medium.

- This is similar but not the same as the equation a nonlinear oscillator in the so-called "Sagdeev potential" $V = (-U^2 + U^4)/2$, which generates the equation of motion

$$\partial_\xi^2 U = U - 2U^3 \quad ,$$

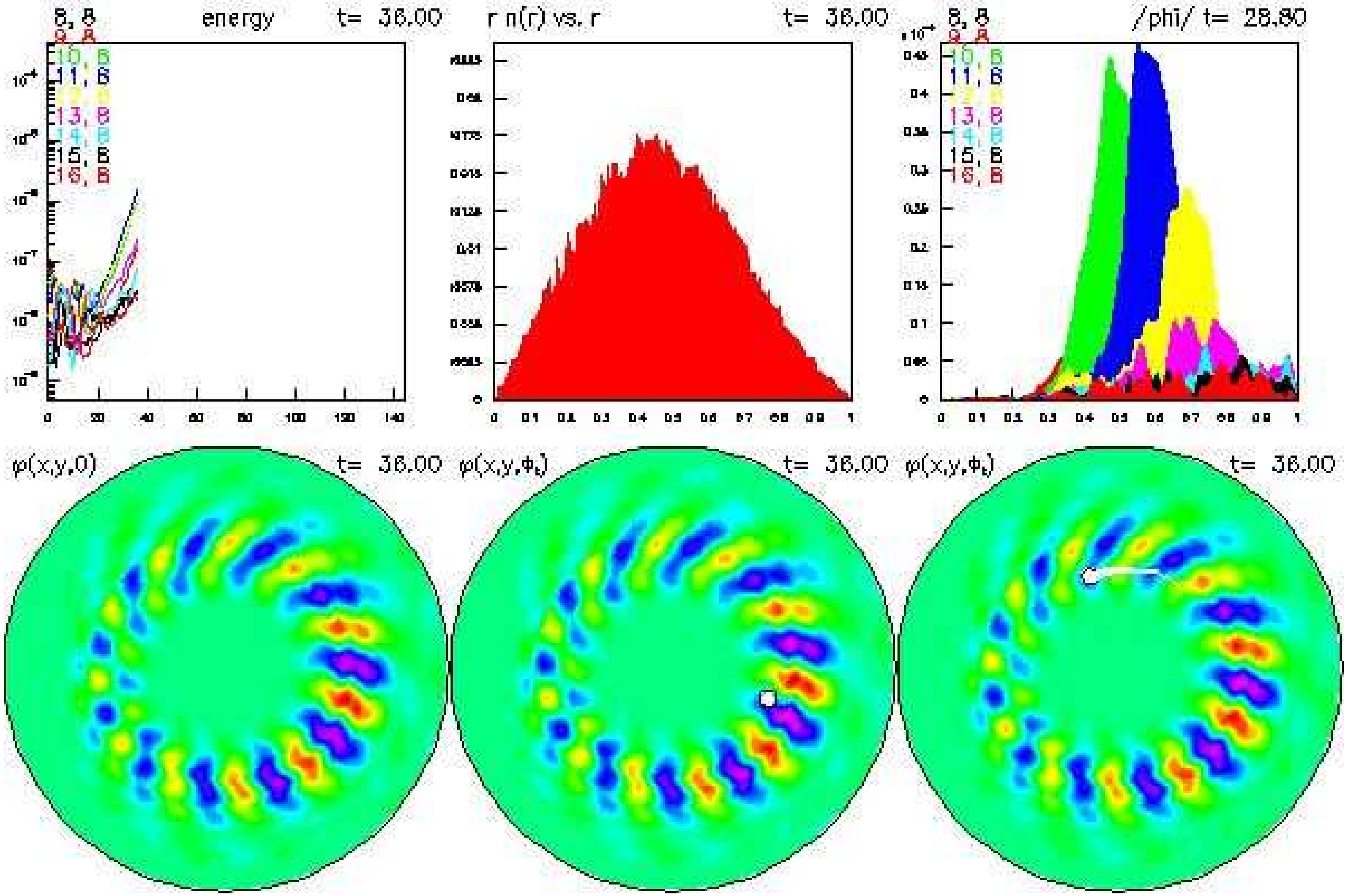
  and gives $U = \mathrm{sech}(\xi)$.

- This is the solution that appears in the problem of ITG turbulence spreading via soliton formation [Z. Guo et al PRL 09].

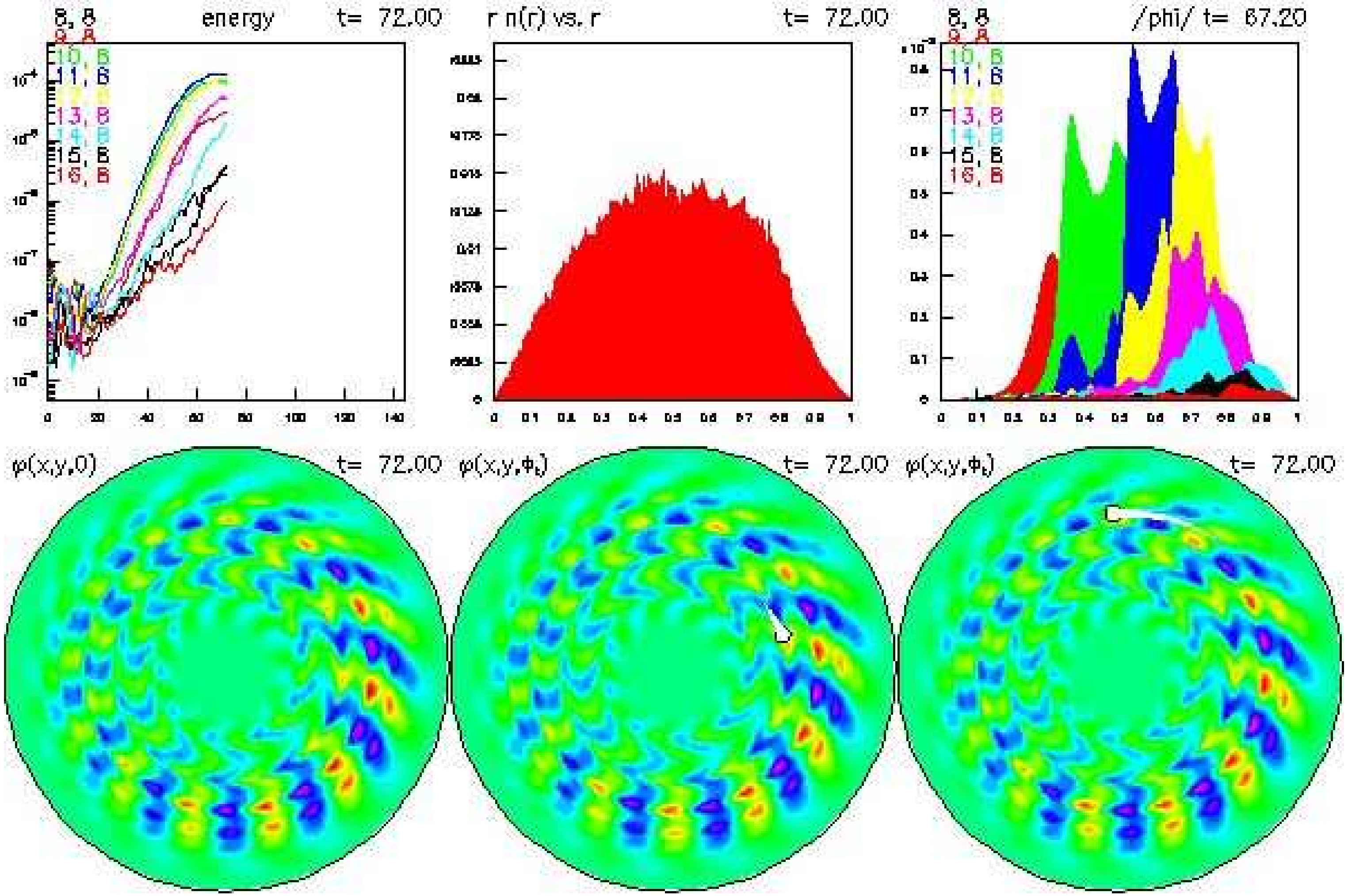
- More generally, this form appears in soliton-like solutions of the nonlinear Schrödinger equations; e.g.
    - the Gross-Pitaevsky equation describing the ground state of a quantum system of identical bosons [Gross 61; Pitaevsky 61]
    - the envelope of modulated water wave groups [Zakharov 68]
    - the propagation of the short optical pulse of a FEL in the superradiant regime [Bonifacio 90]

- The complex nature of EPM equation is novel and connected with the peculiar role of wave-particle resonances, which dominate the nonlinear dynamics of EPMs via resonant wave-particle power exchange, whose maximization yields two effects:
    - the mode radial localization, similar to the analogous mechanism discussed for the linear EPM mode structure
    - the strengthening of mode drive $\mathbb{I}\mathrm{m}\lambda_0 > 0$, connected with the steepening of pressure gradient, convectively propagating with the EPM wave-packet

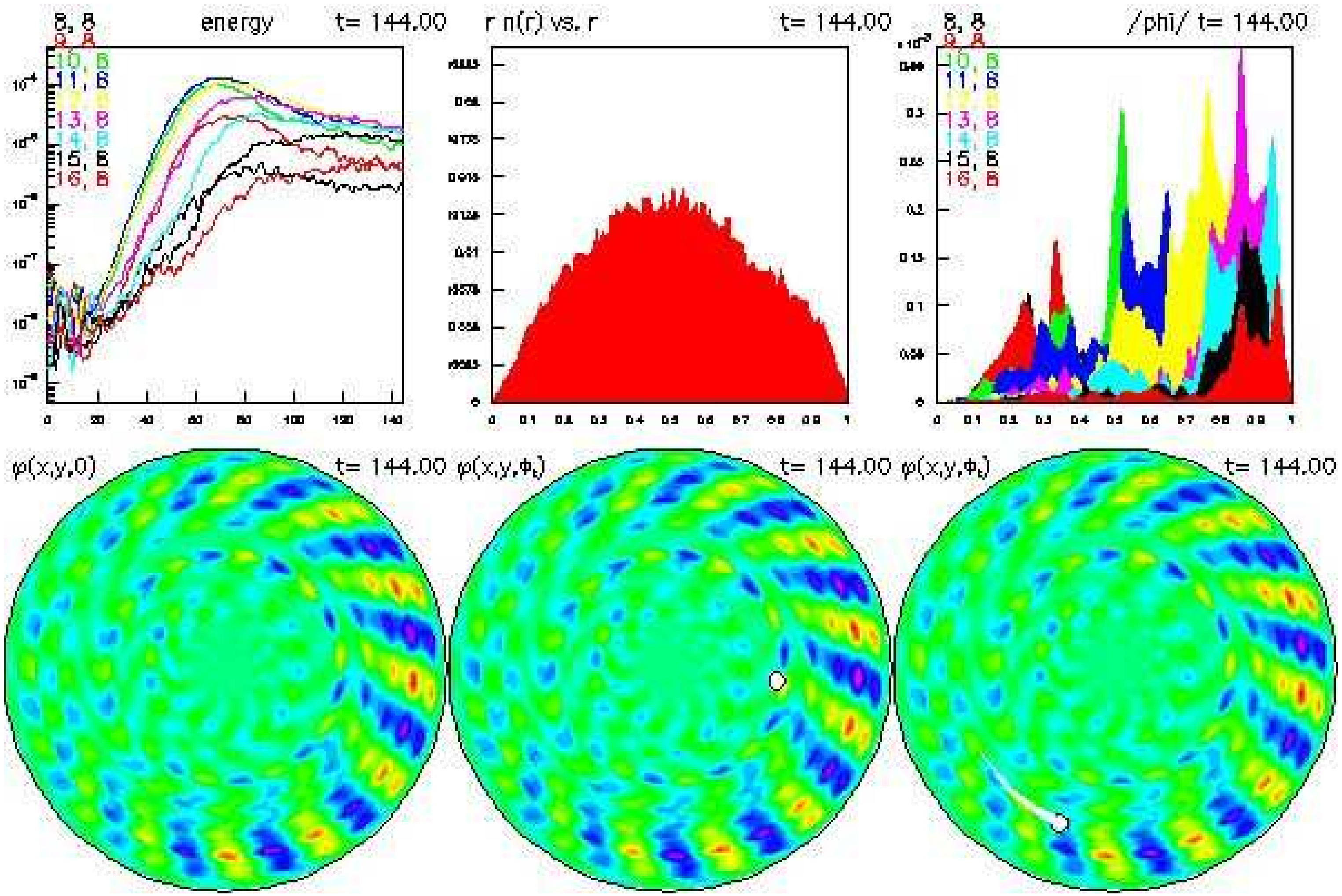
EPM modulational instability I (Varenna 00)



EPM modulational instability II (Varenna 00)



EPM modulational instability III (Varenna 00)



Theory on EPM modulational instability

- Recall (Lecture 4 p. 19) that nonlinear wave-wave and wave-EP interactions generate fines structures, shorter than $|nq'|^{-1}$.

- In the case of EPM, the fine structure modulations are reflected on $\hat{F}_0(\omega)$ and can become nonlinearly unstable by a process called modulational instability.

- The modulational instability of the EPM radial envelope function can be studies via the GFLDR (see p. 3) in the form given at p. 12.

- Assume that the EPM wave packet as a uniform pump, which may decay to upper/lower EPM sidebands by scattering off a corrugated suprathermal particle profile. Therefore, we let

$$
\begin{aligned}
\bar{A}_{n0} &\rightarrow \bar{A}_0 + \bar{A}_+ e^{i\int^r k_z dr - i\omega_z t} + \bar{A}_-^* e^{-i\int^r k_z dr + i\omega_z^* t} \quad , \\
\bar{A}_{-n0} &\rightarrow \bar{A}_0^* + \bar{A}_+^* e^{-i\int^r k_z dr + i\omega_z^* t} + \bar{A}_- e^{i\int^r k_z dr - i\omega_z t} \quad ,
\end{aligned}
$$

- Solve the problem in the local limit. The sideband dispersion functions on the LHS of the GFLDR on p.3 can be written as

$$D_+ \simeq \frac{\partial D_0}{\partial \omega_0}(\omega_z + i\gamma_d - \Delta_T) \quad ,$$
$$D_- \simeq -\frac{\partial D_0^*}{\partial \omega_0}(\omega_z + i\gamma_d + \Delta_T) \quad ,$$

with. ω_z the frequency of the zonal structure, γ_d being the sideband damping and Δ_T the frequency mismatch, which, for localized modes with $\partial D/\partial\theta_{kz} = 0$, $\theta_{kz} = k_z/(nq')$, is given by

$$\Delta_T = \omega(k_z) - \omega_0 = -\frac{1}{2}\frac{\partial^2 D_0/\partial\theta_{kz}^2}{\partial D_0/\partial\omega_0} \quad .$$

- On the RHS, the nonlinear contribution, due to NL EP response, is given by (E: Demonstrate the following relations)

$$
\begin{aligned}
\delta \bar{W}_k^{NL} &\simeq \frac{3\pi (r/R_0)^{1/2}\alpha_E}{8\sqrt{2}|s|} i\pi \frac{\omega_0}{n\bar{\omega}_{dnF}} k_\vartheta^2 v_E^2 \rho_{LE}^2 \partial_t^{-2} \frac{\partial^2}{\partial r^2} \left| A_{n0} \right|^2 A_{n0} \\
&= \frac{i\gamma_M^2}{(\omega_z + 2i\gamma_0)^2} \left| A_{n0} \right|^2 A_{n0} \\
&\rightarrow \frac{i\gamma_M^2}{(\omega_z + 2i\gamma_0)^2} \left[\left(2\bar{A}_+ + \frac{\bar{A}_0}{\bar{A}_0^*} \bar{A}_- \right) e^{i\int^r k_z dr - i\omega_z t} \right. \\
&\qquad \left. + \left(\frac{\bar{A}_0^*}{\bar{A}_0} \bar{A}_+ + 2\bar{A}_- \right) e^{-i\int^r k_z dr + i\omega_z^* t} \right] \\
\gamma_M^2 &= \frac{3\pi^2 (r/R_0)^{1/2}\alpha_E}{8\sqrt{2}|s|} \frac{\omega_0}{\bar{\omega}_{dF}} k_\vartheta^2 \rho_{LE}^2 k_z^2 v_E^2 |\bar{A}_0|^2 \ .
\end{aligned}
$$

- Note: $\gamma_M^2 \propto |\bar{A}_0|^2$ explains origin of terms $\propto \bar{A}_0/\bar{A}_0^*$ and $\propto \bar{A}_0^*/\bar{A}_0$.

- Collecting results, the GFLDR for the sidebands reads

$$
\begin{aligned}
\bar{A}_0^* D_+ \bar{A}_+ &= \frac{i\gamma_M^2}{(\omega_z + 2i\gamma_0)^2} \left(2\bar{A}_0^* \bar{A}_+ + \bar{A}_0 \bar{A}_- \right) \quad , \\
\bar{A}_0 D_- \bar{A}_- &= -\frac{i\gamma_M^2}{(\omega_z + 2i\gamma_0)^2} \left(\bar{A}_0^* \bar{A}_+ + 2\bar{A}_0 \bar{A}_- \right) \quad ,
\end{aligned}
$$

- These equations, yield the following dispersion relation for EPM modulational instability

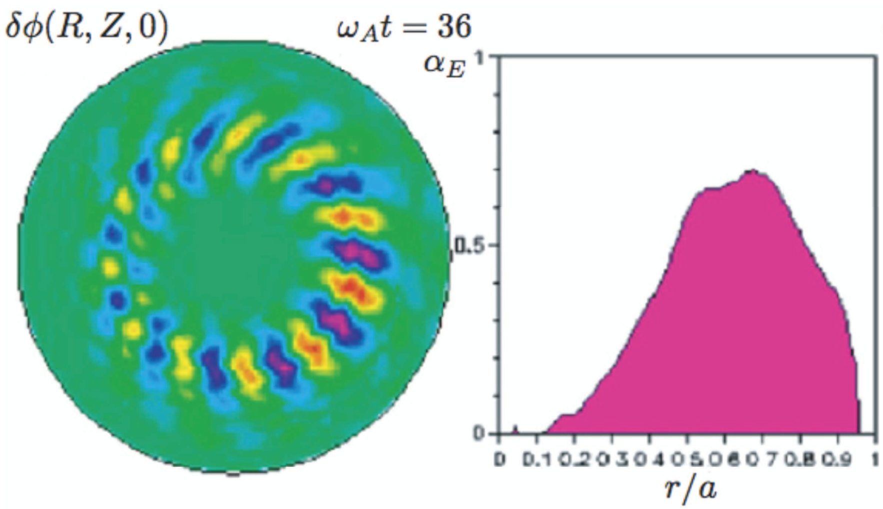
$$
\left| \frac{\partial D_0}{\partial \omega_0} \right|^2 \left(\Delta_T^2 - (\omega_z + i\gamma_d)^2 \right) + \frac{4i\gamma_M^2}{(\omega_z + 2i\gamma_0)^2} \left((\omega_z + i\gamma_d) \frac{\partial \mathbb{R}\mathrm{e} D_0}{\partial \omega_0} - i\Delta_T \frac{\partial \mathbb{I}\mathrm{m} D_0}{\partial \omega_0} \right) + \frac{3\gamma_M^4}{(\omega_z + 2i\gamma_0)^4} = 0
$$

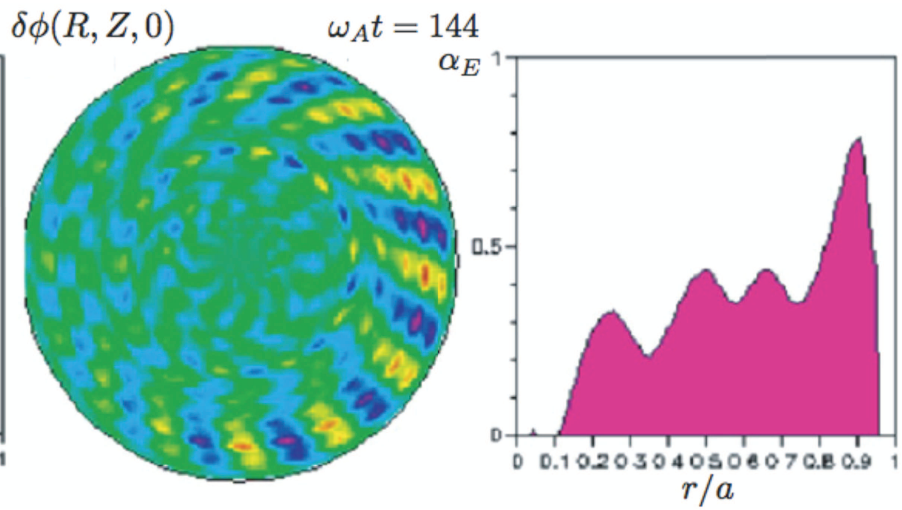
- At threshold, the onset condition for the EPM induced modulational instability gives $|\omega_z| \sim \epsilon_0 \omega_0 \sim \gamma_d \sim \gamma \sim \gamma_M / |\Delta_T/\omega_0|^{1/2}$, with $\Delta_T \sim \epsilon_0 \omega_0$.

- The modulational instability dispersion relation is easily evaluated above threshold, assuming that $|\partial \mathbb{I}\mathrm{m} D_0/\partial\omega_0| \gg |\partial \mathbb{R}\mathrm{e} D_0/\partial\omega_0|$ near the TAE gap. In this case, there is splitting of spectral lines together with the instability [Zonca et al 2006] (note scaling with amplitude, γ_M, is not the same as with wave-particle trapping)

$$|\Delta_T| \ll |\omega_z| \qquad\qquad |\Delta_T| \gg |\omega_z|$$

$$\omega_z = \pm \frac{3^{1/6} e^{\pm i\pi/3} \gamma_M^{2/3}}{|\Delta_T \partial \mathbb{I}\mathrm{m} D_0/\partial\omega_0|^{1/3}} \; . \qquad\qquad \omega_z = -2i\gamma_0 + \frac{[i\gamma_M, \sqrt{3} i\gamma_M]}{|\Delta_T \partial \mathbb{I}\mathrm{m} D_0/\partial\omega_0|^{1/2}} \; .$$





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