Lecture 5

Nonlinear description of fishbones: nonlinear evolution (mode-particle pumping) and reduced (predator-prey) equations

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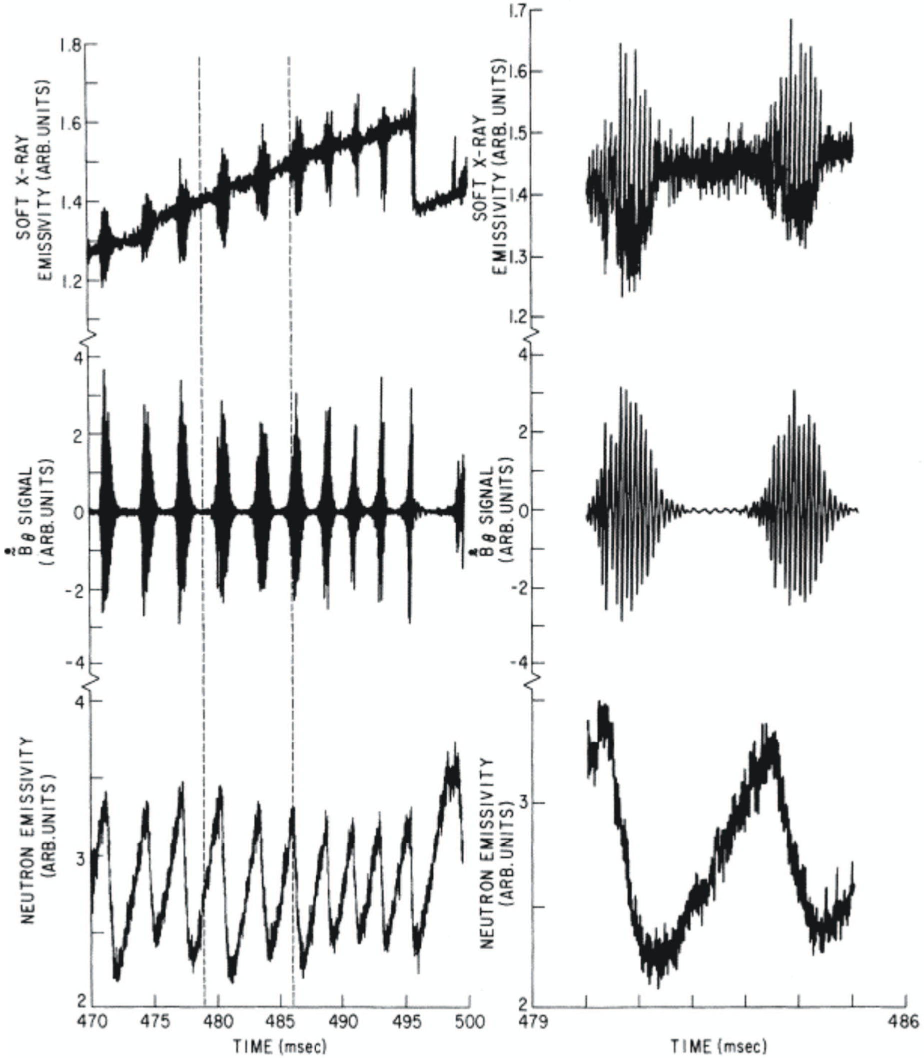
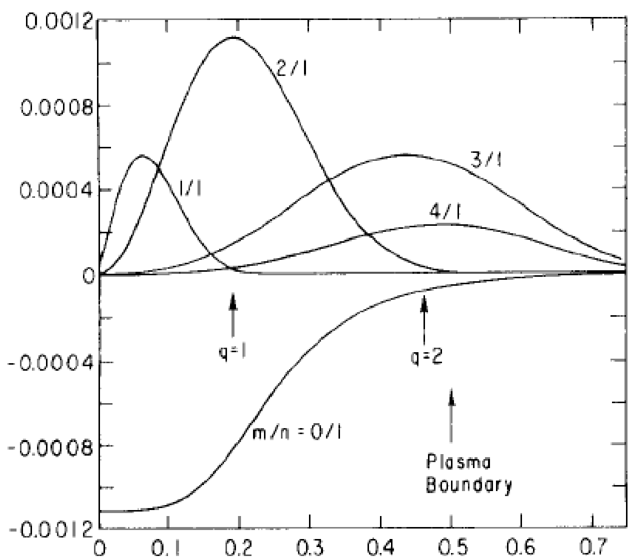
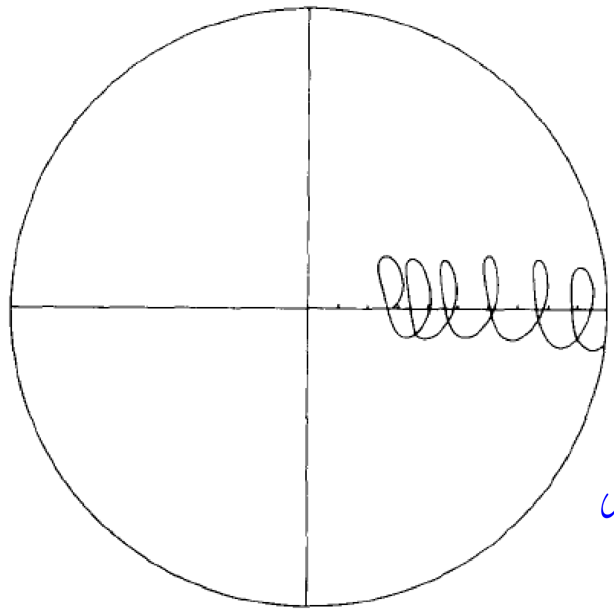
Observation of fishbone oscillations

- Experimental observation of fishbones in PDX [McGuire et al. 83] with macroscopic losses of ⊥ injected fast ions ...



- Followed by numerical simulation of the mode-particle pumping (secular) loss mechanism [White et al 83] ...





$\dot{\omega} \approx \omega_B^2$

- ... and the theoretical explanation of the resonant internal kink excitation by energetic particles and the (model) dynamic description of the fishbone cycle [Chen, White, Rosenbluth 84]

The fishbone dispersion relation

- The problem was solved by [Chen, White, Rosenbluth 84]: first example of the General Fishbone Like Dispersion Relation (GFLDR; $n = 1$ Lecture 4)

$$i|s|\Lambda_n = \delta\hat{W}_{nf} + \delta\hat{W}_{nk} \quad .$$

- Definitions (see Lecture 4): drop subscript n for simplicity, $\Delta q_0 = 1 - q_0$,

$$\Lambda^2 = (\omega/\omega_A^2)(\omega - \omega_{*pi}) \left(1 + 0.5q^2 + 1.6q^2(R_0/r)^{1/2}\right)$$

[Graves et al. 2000]

$$\delta\hat{W}_f = 3\pi\Delta q_0 \left(13/144 - \beta_{ps}^2\right)\left(r_s^2/R_0^2\right) \quad ; \qquad \beta_{ps} = -(R_0/r_s^2)^2 \int_0^{r_s} r^2 (d\beta/dr) dr$$

[Bussac et al. 1975]

$$\delta\hat{W}_k = \frac{2}{r_s^2}\int_0^{r_s} r^3 dr \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm} \frac{\pi^2 R_0}{c^2 q}\frac{e^2}{m}\left(\frac{\tau_b \bar{\omega}_d^2}{\omega_0(\tau)}\right)\int_{-\infty}^{+\infty} \frac{\omega + \omega_0(\tau)}{\bar{\omega}_d - \omega_0(\tau) - \omega} e^{-i\omega t} Q_{k,\omega_0(\tau)} \hat{F}_0(\omega) d\omega$$

(see Lecture 4)

- Nonlinear wave-EP interactions are accounted for via $(n = 1)$ Dyson equation

$$\hat{F}_0(\omega) = \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}\bar{F}_0(0) + \frac{e}{m}\frac{nc}{\omega(d\psi/dr)}\frac{\partial}{\partial r}\left\{\left[\frac{Q^*_{k_0,\omega_0(\tau)}}{\omega_0^*(\tau)}\right.\right.$$

$$\left.\left.\times\frac{\hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega - \omega_0(\tau) + n\bar{\omega}_{dk0}} + \frac{Q_{k_0,\omega_0(\tau)}}{\omega_0(\tau)}\frac{\hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega + \omega_0^*(\tau) - n\bar{\omega}_{dk0}}\right]\hat{\omega}_{dk0}\left|\delta\bar{\phi}_{k0}(r,\tau)\right|^2\right\} .$$

- Solution strategy of the nonlinear problem follows that outlined in Lecture 4 p. 13:
  - At the lowest order, the problem is solved satisfying the linear dispersion relation
  - At next order the nonlinear dynamics describes the mode amplitude and frequency evolution; and EP transport

- Introduce the radial displacement

$$\delta\bar{\xi}_{rk0}(r,\tau) = \frac{k_\vartheta c}{\omega_0 B_0}\delta\bar{\phi}_{k0}(r,\tau) = -\frac{nqc}{\omega_0 r B_0}\delta\bar{\phi}_{k0}(r,\tau) \ .$$

- From the Dyson Equation, assume $|\omega_{*EP}| \gg |\omega_0| \Rightarrow \omega_0(\tau)^{-1}Q_{k_0,\omega_0(\tau)}\bar{F}_0 \simeq (\omega_0\Omega)^{-1}(-nq/r)\partial_r\bar{F}_0$. Furthermore, $d\psi/dr = B_0(r/q)$.Then, by direct substitution, at the two lowest orders in γ_0/ω_0

$$\hat{F}_0(\omega) = \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}\bar{F}_0(0) - 2\frac{i}{\omega}\left(\frac{n\bar{\omega}_{dk0}}{\omega_{0r}}\right)\frac{\partial}{\partial r}$$
$$\times\left[\frac{(\gamma_0+i\omega)+(n\bar{\omega}_{dk0}-\omega_{0r})(\gamma_0/\omega_{0r})}{(n\bar{\omega}_{dk0}-\omega_{0r})^2+(\gamma_0+i\omega)^2}\left|\delta\bar{\xi}_{rk0}(r,\tau)\right|^2\frac{\partial}{\partial r}\hat{F}_0\left(\omega-2i\gamma_0(\tau)\right)\right] \ .$$

- This expression has to be employed within the $\delta\hat{W}_k$ expression on p. 4.

- Write the expression of $\delta\hat{W}_k$ in more compact form, noting, for deeply trapped particles, $\bar{\omega}_d \equiv \tilde{\omega}_d \mathcal{E}$ and $\omega_{0r} \equiv \tilde{\omega}_d \mathcal{E}_0$. Furthermore $\tau_b = 2\pi q R_0 \mathcal{E}^{-1/2}(R_0/r)^{1/2}$, as well as $k_\vartheta \propto -(nq/r)$, $\bar{\omega}_d^2 \propto \tilde{\omega}_d^2 \propto (nq/r)^2$. Finally

$$(\bar{\omega}_d - \omega_0(\tau) - \omega)^{-1} \simeq \tilde{\omega}_d^{-1}\left(\mathcal{E} - \mathcal{E}_0 - i(\gamma_0 - i\omega)/\tilde{\omega}_d\right)^{-1} \quad ,$$

- Calculation of $\delta\hat{W}_k^{NL}$ involves a velocity space and frequency integral with integrand

$$\propto \frac{\pi^2 R_0}{c^2 q}\frac{e^2}{m}\left(\frac{\tau_b \bar{\omega}_d^2}{\omega_0(\tau)}\right)\frac{\omega + \omega_0(\tau)}{\bar{\omega}_d - \omega_0(\tau) - \omega} Q_{k,\omega_0(\tau)} \hat{F}_0(\omega)$$

$$\simeq \frac{\pi^2 m R_0}{B_0^2}\left(-\frac{n\Omega}{r}\right)\left(\tau_b \bar{\omega}_d^2\right)\frac{(n\bar{\omega}_{dk0} - \omega_{0r}) + i(\gamma_0 - i\omega)}{(n\bar{\omega}_{dk0} - \omega_{0r})^2 + (\gamma_0 - i\omega)^2}\frac{\partial}{\partial r}\hat{F}_0(\omega)$$

- Thanks to these expressions we can define resonant particle β_E as

$$\beta_{Er}(r;\omega_0(\tau)) = 2\frac{\pi^2}{B_0^2} m|\Omega| \frac{r}{q^2} \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \tau_b \bar{\omega}_d^2$$
$$\times \int_{-\infty}^{\infty} \frac{(\gamma_0 - i\omega)}{(\bar{\omega}_d - \omega_{0r})^2 + (\gamma_0 - i\omega)^2} e^{-i\omega t} \hat{F}_0(\omega) d\omega \quad .$$

- With this expression,

$$\mathbb{I}\mathrm{m}\delta\hat{W}_k = -\frac{R_0}{r_s}\int_0^{r_s} q^2 \frac{r}{r_s}\left(\frac{R_0}{r}\right)^{1/2} \frac{\partial}{\partial r}\left[\left(\frac{r}{R_0}\right)^{1/2} \beta_{Er}(r;\omega_0(\tau))\right] dr$$
$$= \frac{R_0}{r_s}\int_0^{r_s}\left[-rq^2\frac{\partial \beta_{Er}}{\partial r} - q^2\frac{\beta_{Er}}{2}\right]\frac{dr}{r_s} \quad ,$$

E: Demonstrate that the main contribution to $\beta_{Er}(r;\omega_0(\tau))$ comes from a narrow region near the maximum of the wave-EP power transfer.

E: Derive the representation of $\mathbb{I}\mathrm{m}\delta\hat{W}_k$ step by step.

- In the same way, we can define non-resonant particle β_E as

$$\hat{\beta}_E(r;\omega_0(\tau)) = 2\frac{\pi^2}{B_0^2} m\left|\Omega\right| \frac{r}{q^2}\int \mathcal{E}d\mathcal{E}d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \tau_b \bar{\omega}_d^2$$
$$\times \int_{-\infty}^{\infty} \frac{(\bar{\omega}_d-\omega_{0r})}{(\bar{\omega}_d-\omega_{0r})^2+(\gamma_0-i\omega)^2} e^{-i\omega t}\hat{F}_0(\omega)d\omega \ .$$

- With this expression,

$$\mathbb{R}\mathrm{e}\delta\hat{W}_k \simeq \mathbb{R}\mathrm{e}\delta\hat{W}_k^L = -\frac{R_0}{r_s}\int_0^{r_s} q^2\frac{r}{r_s}\left(\frac{R_0}{r}\right)^{1/2}\frac{\partial}{\partial r}\left[\left(\frac{r}{R_0}\right)^{1/2}\hat{\beta}_E(r;\omega_0(\tau))\right]dr \ ,$$

E: Discuss similarities and differences of $\hat{\beta}_E(r;\omega_0(\tau))$ expression (non-resonant) with $\beta_{Er}(r;\omega_0(\tau))$ (resonant). Can you provide a qualitative reason why the former is due to non-resonant and the latter to resonant particles?

Nonlinear fishbone dynamics

- As reference case, consider a simple isotropic slowing down distribution,

$$\bar{F}_0 = \frac{3P_{0E}}{4\pi E_F} \frac{\mathrm{H}(E_F/m_E - \mathcal{E})}{(2\mathcal{E})^{3/2} + (2E_c/m_E)^{3/2}} \quad ,$$

  where H denotes the Heaviside step function and the normalization condition is chosen such that the EP energy density is $(3/2)P_{0E}$ for $E_F \gg E_c$, and EP energy is predominantly transferred to thermal electrons by collisional friction as it occurs for α-particles in fusion plasmas.

- Reconsider the expressions of $\mathbb{R}\mathrm{e}\delta\hat{W}_k$ and $\hat{\beta}_E(r;\omega_0(\tau))$, given on p. 9; and

$$\delta\hat{W}_f + \mathbb{R}\mathrm{e}\delta\hat{W}_k^L \simeq 0 \quad .$$

- For $\mathbb{R}\mathrm{e}\Lambda^2 > 0$ (see p. 4), this equation (real part of GFLDR) determines the real frequency of the fishbone mode for $|\gamma_0/\omega_0| \ll 1$ [Chen 1984].

- The fishbone frequency is set by the condition $\omega_0/\bar{\omega}_{dF} \simeq$ const, to be computed at the position of the radial shell where the most significant EP contribution is localized.

E: Demonstrate that the statement above is correct.

E: Demonstrate that $\mathbb{R}\mathrm{e}\delta\hat{W}_k \simeq \mathbb{R}\mathrm{e}\delta\hat{W}_k^L$ and that $|\mathbb{R}\mathrm{e}\delta\hat{W}_k^{NL}| \sim |\gamma_0/\omega_0||\mathbb{I}\mathrm{m}\delta\hat{W}_k^{NL}|$. [Hint: see the following discussion].

E: Given $\omega_0/\bar{\omega}_{dF} \simeq$ const (at the radial position of the most significant EP contribution), can you explain how the fishbone frequency can change during nonlinear dynamic evolution? How is this related with phase-locking?

- The GFLDR (imaginary part; for $|\gamma_0/\omega_0| \ll 1$), gives the evolution of the fishbone amplitude (Lecture 4)

$$\gamma_0 \left(-\partial \mathbb{R}\mathrm{e}\delta\hat{W}_k^L / \partial \omega_{0r} \right) = \mathbb{I}\mathrm{m}\delta\hat{W}_k - |s|\Lambda(\omega_{0r}) \quad .$$

- This equation expresses the competition between EP drive and continuum damping; and can be cast as

$$\frac{\partial}{\partial t}\ln|\delta\xi_{r0}|^2 = \frac{2(R_0/r_s)}{\left(-\partial\mathbb{R}\mathrm{e}\delta\hat{W}_k^L/\partial\omega_{0r}\right)}\left\{-\int_0^{r_s} q^2\frac{r}{r_s}\left(\frac{R_0}{r}\right)^{1/2}\right.$$
$$\left.\times\frac{\partial}{\partial r}\left[\left(\frac{r}{R_0}\right)^{1/2}\beta_{Er}(r;\omega_0(\tau))\right]dr - \left(\frac{r_s}{R_0}|s|\Lambda(\omega_{0r})\right)\right\} .$$

E: Derive this equation step by step. Can you show that to properly describe the fishbone cycle you now need an equation for $\beta_{Er}(r;\omega_0(\tau))$ which can be derived from the Dyson equation?

- The simplest way to close the (GFLDR) evolution for the fishbone amplitude is to obtain the evolution equation for $\beta_{Er}(r;\omega_0(\tau))$ directly from the Dyson equation for $\hat{F}_0(\omega)$ (p. 6 and Lecture 4)).

- Calculation of $\delta\hat{W}_k$ and $\beta_{Er}(r;\omega_0(\tau))$ involves a vel. space and freq. integral with integrand

$$\propto \left[\frac{(\gamma_0+i\omega)+(n\bar{\omega}_{dk0}-\omega_{0r})(\gamma_0/\omega_{0r})}{(n\bar{\omega}_{dk0}-\omega_{0r})^2+(\gamma_0+i\omega)^2}\left|\delta\bar{\xi}_{rk0}(r,\tau)\right|^2\frac{\partial}{\partial r}\hat{F}_0\left(\omega-2i\gamma_0(\tau)\right)\right]$$

$$\times\frac{ie^{-i\omega t}}{\omega}\frac{\left[\omega_{0r}+i(\gamma_0-i\omega)\right]\left[(n\bar{\omega}_{dk0}-\omega_{0r})+i(\gamma_0-i\omega)\right]}{(n\bar{\omega}_{dk0}-\omega_{0r})^2+(\gamma_0-i\omega)^2}$$

E: Discuss the spatial and temporal scales involved in this expression. Which contribution diverges for $\gamma_0\pm i\omega\to 0$? Which one is regular? Can you identify which particles, resonant and/or non-resonant, are responsible for these behaviors?

E: Reconsider now the (E) on p. 11: can you demonstrate, with help of the expression above, that $|\mathbb{R}\mathrm{e}\delta\hat{W}_k^{NL}|\sim|\gamma_0/\omega_0||\mathbb{I}\mathrm{m}\delta\hat{W}_k^{NL}|$?. And that, therefore, $\mathbb{R}\mathrm{e}\delta\hat{W}_k\simeq\mathbb{R}\mathrm{e}\delta\hat{W}_k^L$?

- Recall that $\left|\delta\bar{\xi}_{rk0}(r,t)\right|^2 = \exp(2\gamma_0 t)\left|\delta\bar{\xi}_{rk0}(r,\tau)\right|^2$. For strongly growing modes, $\gamma_0 \sim |\omega| \Rightarrow \gamma_0 \sim \tau_{NL}^{-1}$, as for SAW excited by EP in tokamaks, which still have $|\gamma_0/\omega_0| \ll 1$.

E: In the light of this last statement, discuss how you would be able to simplify the calculation of $\delta\hat{W}_k^{NL}$.

- Calculations can be done considering that

$$\int \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b} \; ; \qquad \int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)} \; .$$

- Therefore, for $|a|, |b| \ll 1$, assuming $\mathbb{R}\mathrm{e}a > 0$ and $\mathbb{R}\mathrm{e}b > 0$,

$$\frac{x^2}{(x^2+a^2)(x^2+b^2)} \to \frac{\pi}{a+b}\delta(x) \qquad \frac{1}{(x^2+a^2)(x^2+b^2)} \to \frac{\pi}{ab(a+b)}\delta(x) \; .$$

- Considering strongly growing modes; *i.e.*, $\gamma_0 > |\omega|$, the condition $\mathbb{R}\mathrm{e}a > 0$ and $\mathbb{R}\mathrm{e}b > 0$ above is crucial for the correct causality constraint to be computed.

- Shifting $\omega \rightarrow \omega + 2i\gamma_0$ in the frequency integral to $\left|\delta\bar{\xi}_{rk0}(r,t)\right|^2 = \exp(2\gamma_0 t)\left|\delta\bar{\xi}_{rk0}(r,\tau)\right|^2$; the integrand on p. 13 becomes

$$\simeq (-i/2)(2\gamma_0 - i\omega)^{-2}\left[\pi\delta(n\bar{\omega}_{dk0} - \omega_{0r})e^{-i\omega t}\left|\delta\bar{\xi}_{rk0}(r,t)\right|^2 \partial_r \hat{F}_0(\omega)\right]$$

$$= (-i/2)\partial_t^{-2}\left[\pi\delta(n\bar{\omega}_{dk0} - \omega_{0r})e^{-i\omega t}\left|\delta\bar{\xi}_{rk0}(r,t)\right|^2 \partial_r \hat{F}_0(\omega)\right] \quad .$$

E: Discuss the symbolic meaning of the operator ∂_t^{-2} in the equation above.

Nonlinear fishbone equations

- With the previous result, the nonlinear equation for $\beta_{Er}(r;\omega_0(\tau))$ is [Chen RMP14]

$$\partial_t \beta_{Er} = \left(\dot{\beta}_{ErS} - \nu_{ext}\beta_{Er}\right) + \partial_t^{-1}\left(\frac{R_0}{r}\right)^{1/2}\left\{\frac{q}{r}\frac{\partial}{\partial r}\left[\frac{r}{q}|\omega_0|^2|\delta\xi_{r0}|^2\frac{\partial}{\partial r}\left(\left(\frac{r}{R_0}\right)^{1/2}\beta_{Er}\right)\right]\right\} .$$

- From external source and collision operator, plus the definition of $\beta_{Er}(r;\omega_0(\tau))$, we have

$$\dot{\beta}_{ErS} \equiv 2\frac{\pi^2}{B_0^2}m\,|\Omega|\,\frac{r}{q^2}\int \mathcal{E}d\mathcal{E}d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1}\tau_b\bar{\omega}_d^2\frac{\gamma_0}{(\bar{\omega}_d-\omega_{r0})^2+\gamma_0^2}S(t) \ ,$$

$$\nu_{ext}\beta_{Er} \equiv -2\frac{\pi^2}{B_0^2}m\,|\Omega|\,\frac{r}{q^2}\int \mathcal{E}d\mathcal{E}d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1}\tau_b\bar{\omega}_d^2\frac{\gamma_0}{(\bar{\omega}_d-\omega_{r0})^2+\gamma_0^2}\mathrm{St}F_0(t) \ ,$$

- These equations are closed by the fishbone amplitude evolution equation (p. 12)

$$\frac{\partial}{\partial t}\ln|\delta\xi_{r0}|^2 = \frac{2(R_0/r_s)}{\left(-\partial\mathbb{R}\mathrm{e}\delta\hat{W}_k^L/\partial\omega_0\right)}\left\{-\int_0^{r_s} q^2\frac{r}{r_s}\left(\frac{R_0}{r}\right)^{1/2}\right.$$
$$\left.\times\frac{\partial}{\partial r}\left[\left(\frac{r}{R_0}\right)^{1/2}\beta_{Er}(r;\omega_0(\tau))\right]dr - \left(\frac{r_s}{R_0}|s|\Lambda(\omega_0)\right)\right\} \quad .$$

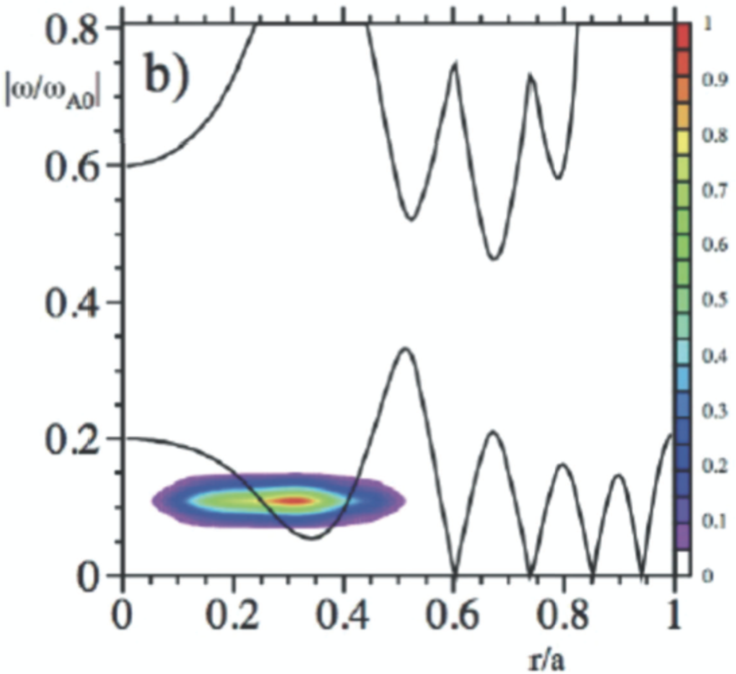
E: Go back to pp. 10, 11 and consider the real part of the GFLDR. Can you explain the fishbone frequency chirping by phase locking?

- Nonlinear equation for $\beta_{Er}(r;\omega_0(\tau))$ demonstrates that fishbone saturation should occur for $|\delta\xi_{r0}| \sim r_s|\gamma_L/\omega_0|$, consistent with numerical simulation results [Vlad et al. 2013]. While EP transport is secular, accompanied by phase locking (mode particle pumping) [White et al 1983].

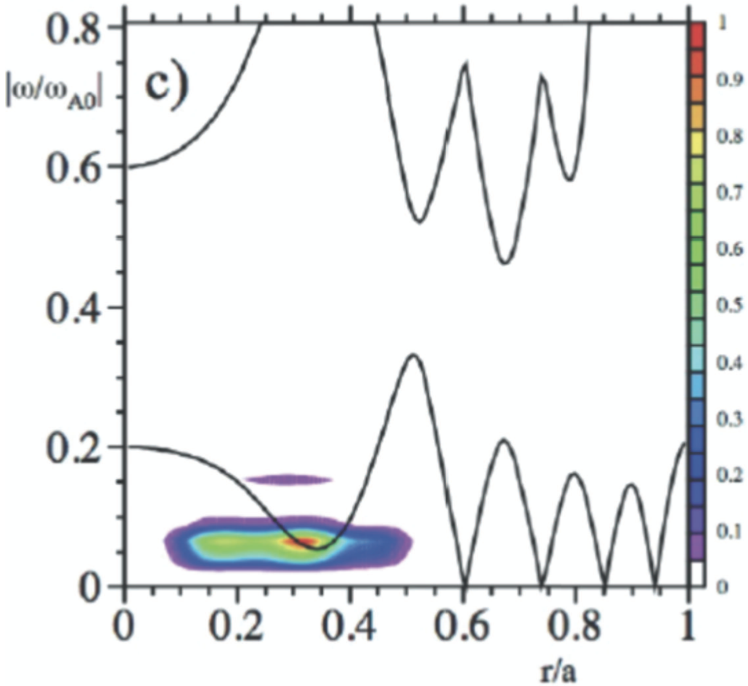
Electron-Fishbone Hybrid MHD-Gyrokinetic simulations

(Lecture 1)

- Frequency chirping and EP radial redistribution [Vlad et al., 2013]

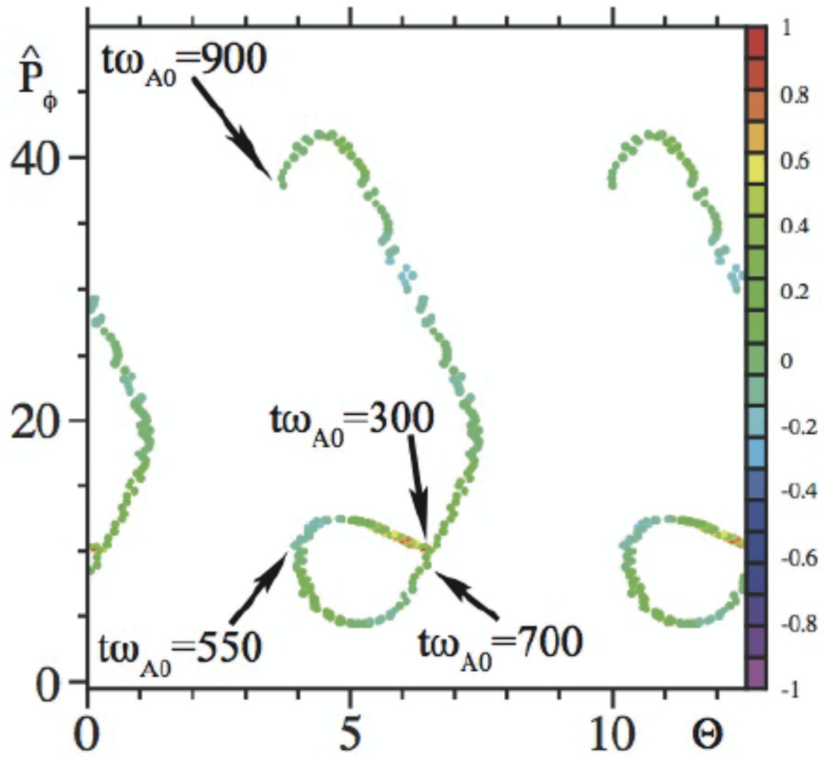


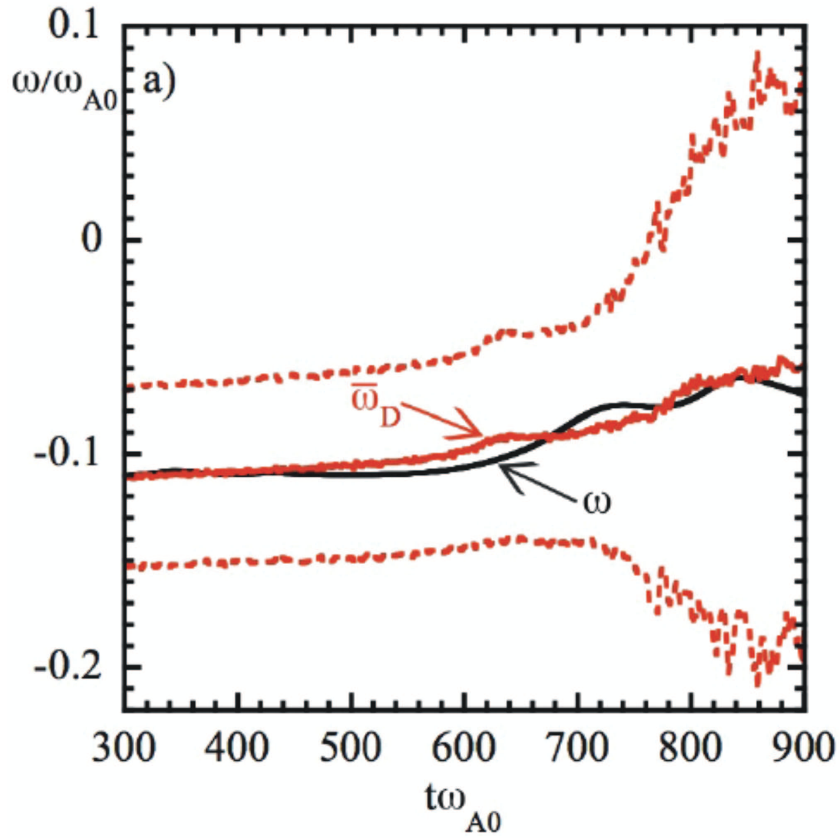
$T = 300$ (linear)



$T = 900$ (saturation)

- Secular radial motion and phase locking [Vlad et al., 2013]

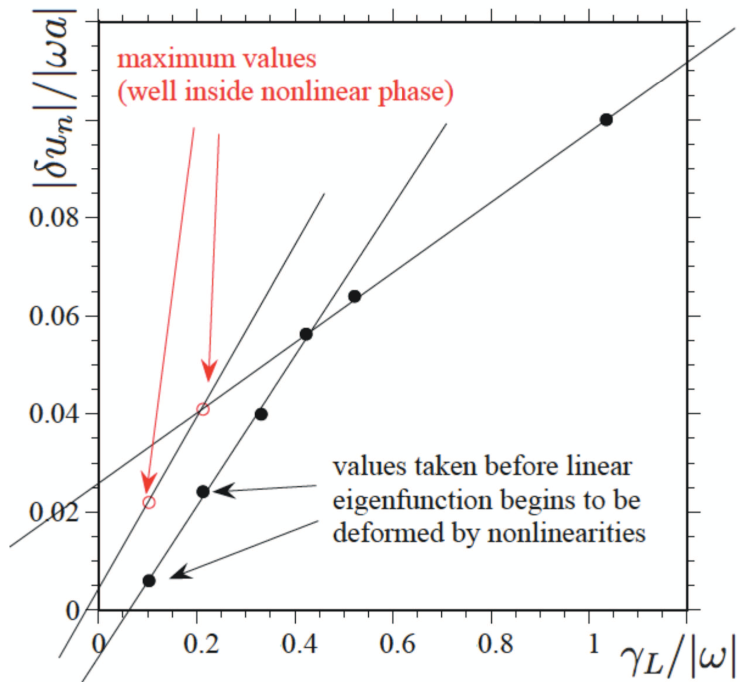




- Resonant EPs convect outward with radial speed $|\delta u_n|$ $\Rightarrow$ Nonlinear saturation occurs when $|\delta u_n|/\gamma_L \sim r_s$

  [$r_s \sim$ mode structure width $\to$ Wave-EP interaction domain]

- [Vlad et al., 2013] simulation results



Reduced nonliear equations: predator-prey model

- Original work on fishbone proposed a qualitative model for fishbone cycle, based on the understanding of the mode particle pumping $\Leftrightarrow$ phase locking $\Rightarrow$ secular EP loss mechanism.
- This understanding implies $|\delta\xi_{r0}| \sim r_s|\gamma_L/\omega_0|$
- From [Chen 1984], fishbone cycle can be described as

$$\begin{aligned} d\beta/d\tau &= S - A\beta_c \ , \\ dA/d\tau &= \gamma_0 \left(\beta/\beta_c - 1\right) A \ . \end{aligned}$$

- This system of equations can be obtained from our NL fishbone equations.
  - the first one from $\beta_{Er}(r;\omega_0(\tau)) \to \beta$ equation
  - the second one from the fishbone amplitude evolution equation

Here, τ is a normalized time, $A = |\delta\boldsymbol{\xi}_{r0}|/r_s$ is the normalized fishbone amplitude and γ_0 is a measure of the linear growth rate.

E: Derive this reduced description of the fishbone cycle from the nonlinear fishbone equations. [Hint: $\partial_t^{-1} \sim \tau_{NL} \sim r_s/(|\omega_0||\delta\boldsymbol{\xi}_{r0}|)$ in the $\beta_{Er}(r;\omega_0(\tau))$ evolution equation. Furthermore $\partial_r^2 \sim -1/r_s^2$. (Chen RMP 2014)].

- The solution of the predator-prey model is cyclic; *i.e.*, it can be generally written as $F(A,\beta) = \text{const}$, where $F(A,\beta)$ has a maximum at the fixed point position $\beta = \beta_c$, $A = S/\beta_c$. [E: demonstrate this.].
- A crucial feature of the model is the linear dependence on A of the loss term in the β evolution equation. This is consequence of the ∂_t^{-2} operator acting on the nonlinear response, which is the manifestation of secular resonant EP losses by mode particle pumping [White et al 1983].
- This term was proposed in the original model by [Chen 1984] on the basis of intuitive representation of the underlying physics of the fishbone burst cycle, and indicates the fundamental difference of that approach with respect to the predator-prey model discussed by [Coppi 1986], which adopts a loss term $\propto A^2$.

References and reading material

K. McGuire et al., Phys. Rev. Lett. **50**, 891 (1983).

R.B. White, R.J. Goldston, K. McGuire et al., Phys. Fluids **26**, 2958, (1983).

L. Chen, R.B. White and M.N. Rosenbluth, Phys. Rev. Lett. **52**, 1122, (1984).

F. Zonca, L. Chen, S. Briguglio, G. Fogaccia, G. Vlad and X. Wang, *Nonlinear dynamics of phase-space zonal structures and energetic particle physics*, Proceedings of the 6th IAEA Technical Meeting on "Theory of Plasmas Instabilities", Vienna, Austria, May 27 - 29, 2013.

F. Zonca, L. Chen, S. Briguglio, G. Fogaccia, G. Vlad and X. Wang, *Nonlinear dynamics of phase-space zonal structures and energetic particle physics in fusion plasmas*, submitted to New J. Phys. (2014).

M. N. Bussac, R. Pellat, D. Edery et al., Phys. Rev. Lett. **35**, 1638 (1975).

J. P. Graves, R. J. Hastie and K. I Hopcraft, Plasma Phys. Control. Fusion **42**, 1049 (2000).

B. Coppi and F. Porcelli, Phys. Rev. Lett. **57**, 2272 (1986).