Lecture 4

Nonlinear dynamics of phase-space zonal structures: governing equations, renormalized particle response and Dyson equation

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Phase-space zonal structures in toroidal system

- In Lecture 3 we derived self-consistent equations for nonlinear evolution of phase-space structures in a beam-plasma system:
  - Dyson Equation: phase-space structure evolution from Vlasov equation
    - Broad spectrum: quasilinear diffusion equation
    - Narrow spectrum: wave-particle trapping
  - Poisson Equation: evolution of (nearly resonant) fluctuations with $\omega \simeq \omega_p$

- In this Lecture, we derive the self-consistent equations for evolution of phase-space zonal structures in the presence of energetic particle (EP) driven shear Alfvén waves (SAW).
  - Dyson Equation: phase-space zonal structure evolution from NL Gyrokinetic equation (Lecture 4 of Spring 2010 and Lecture 6 of Spring 2013 Intensive Courses)
  - SAW excited by EP: general fishbone like dispersion relation (GFLDR) for the evolution of fluctuations (Lecture 4 of Spring 2009 and Lecture 3 of Spring 2013 Intensive Courses)
  - New twist: system equilibrium geometry and nonuniformity (Lecture 1)

Review of Nonlinear Gyrokinetic Equation

- The fluctuating particle distribution functions are decomposed in adiabatic and nonadiabatic responses as [Frieman and Chen 1982] (see Lecture 1 of Spring 2013 Intensive Course and Lectures by B.D. Scott, Hangzhou 2013)

$$\delta f = e^{-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}\left[\delta g - \frac{e}{m}\frac{1}{B_0}\frac{\partial \bar{F}_0}{\partial \mu}\langle \delta L_g \rangle\right] + \frac{e}{m}\left[\frac{\partial \bar{F}_0}{\partial \mathcal{E}}\delta\phi + \frac{1}{B_0}\frac{\partial \bar{F}_0}{\partial \mu}\delta L\right] \quad ,$$

$$\delta L_g = \delta\phi_g - \frac{v_\parallel}{c}\delta A_{\parallel g} = e^{\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}\delta L = e^{\boldsymbol{\rho}\cdot\boldsymbol{\nabla}}\left(\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel\right) \quad .$$

E: Show that $\exp(-\boldsymbol{\rho}\cdot\boldsymbol{\nabla}) \equiv \exp\left(-i\mathbf{k}_\perp\cdot\mathbf{v}\times\mathbf{b}/\omega_c\right)$ is the generator of coordinate transformation from guiding center to particle position

E: Consider $\mathbf{k}_\perp\cdot\mathbf{v}\times\mathbf{b}/\omega_c = (k_\perp v_\perp/\omega_c)\cos\alpha \equiv \lambda\cos\alpha$, with α the gyrophase, and the identity $e^{-i\lambda\cos\alpha} = \sum_{\ell=-\infty}^{\infty}(-i)^\ell J_\ell(\lambda)e^{i\ell\alpha}$ (J_ℓ are Bessel functions). Show that gyrophase average yields

$$\overline{\exp\left(-i\mathbf{k}_\perp\cdot\mathbf{v}\times\mathbf{b}/\omega_c\right)\delta\phi} = J_0(\lambda)\delta\phi$$

E: What is the physical/mathematical meaning of the operator $J_0(\lambda)$?

- The nonadiabatic response of the particle distribution function, δg_k, is obtained from the nonlinear gyrokinetic equation [Frieman and Chen 1982]:

$$\left(\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp\right) \delta g = -\left(\frac{e}{m}\frac{\partial}{\partial t}\left\langle \delta L_g \right\rangle \frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{c}{B_0}\boldsymbol{b} \times \boldsymbol{\nabla}\left\langle \delta L_g \right\rangle \cdot \boldsymbol{\nabla} \bar{F}_0\right) - \frac{c}{B_0}\boldsymbol{b} \times \boldsymbol{\nabla}\left\langle \delta L_g \right\rangle \cdot \boldsymbol{\nabla} \delta g\,.$$

- Here, the magnetic drift velocity $\boldsymbol{v}_d$ is

$$\boldsymbol{v}_d = \frac{\boldsymbol{b}}{\Omega} \times \left(\mu \boldsymbol{\nabla} B_0 + \boldsymbol{\kappa} v_\parallel^2\right) \simeq \frac{\left(\mu B_0 + v_\parallel^2\right)}{\Omega} \boldsymbol{b} \times \boldsymbol{\kappa} \quad ,$$

- Furthermore, $\boldsymbol{\kappa} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{b}$ is the magnetic field curvature and $\boldsymbol{\nabla} B_0 \simeq \boldsymbol{\kappa} B_0$ in the low-β limit, consistent with well-known cancellations in the linear vorticity equation, arising from the perpendicular pressure balance and plasma equilibrium condition [Hasegawa and Sato 1989]

$$\boldsymbol{\nabla}_\perp \left(B_0 \delta B_\parallel + 4\pi \delta P_\perp \right) \simeq 0 \quad .$$

- Introduce the notation $\boldsymbol{b} \cdot \boldsymbol{\nabla} \delta\psi = -(1/c)\partial_t \delta A_\parallel$, so that $\delta\phi = \delta\psi$ corresponds to $\delta E_\parallel = 0$. The further decomposition to separate the convective response is often used

$$\delta g \equiv \delta K + i \frac{e}{m} Q \bar{F}_0 \partial_t^{-1} \langle \delta\psi_g \rangle \quad .$$

$$iQ\bar{F}_0 = -\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \frac{\partial}{\partial t} + \frac{\boldsymbol{b} \times \boldsymbol{\nabla} \bar{F}_0}{\Omega} \cdot \boldsymbol{\nabla} \quad .$$

$$\omega_{*p} = \omega_{*n} + \omega_{*T} \quad , \qquad \omega_{*n} = \left(\frac{T_0 c}{e n_0 B_0} \right) (\boldsymbol{b} \times \boldsymbol{\nabla} n_0) \cdot \boldsymbol{k}_\perp \quad , \qquad \omega_{*T} = \left(\frac{c}{e B_0} \right) (\boldsymbol{b} \times \boldsymbol{\nabla} T_0) \cdot \boldsymbol{k}_\perp \quad .$$

- Field equations are generally provided by the quasineutrality condition

$$\sum \left\langle \frac{e^2}{m}\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \right\rangle_v \delta\phi + \boldsymbol{\nabla}\cdot \sum \left\langle \frac{e^2}{m}\frac{2\mu}{\Omega^2}\frac{\partial \bar{F}_0}{\partial \mu}\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v \boldsymbol{\nabla}_\perp \delta\phi + \sum \langle e J_0(\lambda)\delta g\rangle_v = 0 \ .$$

- ... and the vorticity equation

$$B_0\left(\nabla_\parallel + \frac{\delta \boldsymbol{B}_\perp}{B_0}\cdot\boldsymbol{\nabla}\right)\left(\frac{\delta j_\parallel}{B_0}\right) - \boldsymbol{\nabla}\cdot\sum\left\langle \frac{e^2}{m}\frac{2\mu}{\Omega^2}\left(B_0\frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{\partial \bar{F}_0}{\partial \mu}\right)\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v \boldsymbol{\nabla}_\perp \frac{\partial}{\partial t}\delta\phi$$

$$-\sum ec\boldsymbol{b}\times\boldsymbol{\nabla}\left\langle \frac{2\mu}{\Omega^2}\bar{F}_0\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v \cdot \boldsymbol{\nabla}\nabla_\perp^2\delta\phi + \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\sum\left\langle m\left(\mu B_0 + v_\parallel^2\right)J_0\delta g\right\rangle_v$$

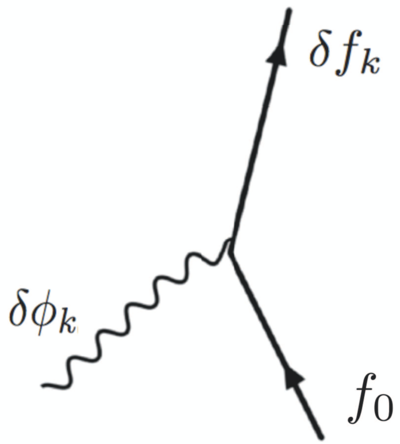
$$+\delta\boldsymbol{B}_\perp\cdot\boldsymbol{\nabla}\left(\frac{j_{\parallel 0}}{B_0}\right) + \sum e\left\langle J_0\left[\frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\left(J_0\delta\phi\right)\cdot\boldsymbol{\nabla}\delta g\right] - \frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\left(J_0\delta g\right)\right\rangle_v$$

$$+\frac{c}{B_0}\boldsymbol{b}\times\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\sum\left\langle \frac{e^2}{m}\frac{2\mu}{\Omega^2}\frac{\partial \bar{F}_0}{\partial \mu}\left(\frac{1-J_0^2}{\lambda^2}\right)\right\rangle_v \boldsymbol{\nabla}_\perp\delta\phi\right] = 0 \ .$$

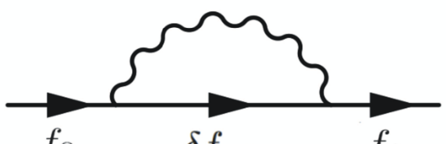
RP: Discuss the physics origin of various terms in quasineutrality condition and vorticity equation. Discuss how general these equations are and find out whether numerical codes exist that solve them in this form. Can you derive the small Larmor radius limit of these equations?

- These equations are very general and will not be used in their general form in this Lecture Series. Are given here as reference [Chen RMP14].

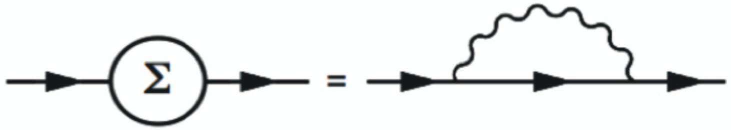
- This Lecture: follow by analogy with the nonlinear beam-plasma system
  - Investigate only wave-EP nonlinear interactions via the fastest processes which modify the "equilibrium" distribution function
  - Assume that the plasma responds to a nearly periodic perturbation $(\omega_0, \boldsymbol{k}_0)$
  - Novel twist: the plasma does not behave as linear dielectric medium (unlike the thermal plasma in the beam-plasma problem) but can respond non-adiabatically to EP transports

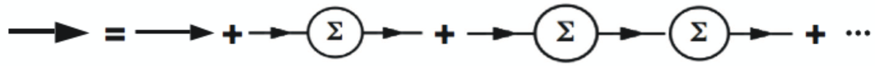


Generation of the distribution δf_k due to the interaction of f_0 with $\delta\phi_k$, corresponding to the solution of the GFLDR (Poisson Eq. in Lecture 1).



Nonlinear distortion of f_0 due to emission and absorption of the field $\delta\phi_k$.





The diagram of the process is defined in the top frame, while the solution of the "Dyson" equation corresponds to the summation of all terms in the Dyson series (bottom).

[From Lecture 3]

The General Fishbone Like Dispersion Relation

- The General Fishbone Like Dispersion Relation (GFLDR) is a general framework for the nonlinear description of SAW excited by EPs using nonlinear quasineutrality condition and vorticity equation (p.7).
- Detailed derivations of the GFLDR are given by [Zonca and Chen 2014] and Lecture 4 of Spring 2009 and Lecture 3 of Spring 2013 Intensive Courses.
- This Lecture: synthetic summary of essential results.

- From Lecture 2, summarize the generic representation of a generic fluctuation $f(r,\theta,\xi\equiv\zeta-q(r)\theta)$ in Clebsch coordinates [Z.X. Lu etal 12]

$$f(r,\theta,\xi)=\sum_n e^{in\xi}F_n(r,\theta) \quad ,$$

$$\begin{aligned}
F_n(r,\theta) &= 2\pi\sum_\ell e^{2\pi i\ell nq}\hat{F}_n(r,\theta-2\pi\ell) \\
&= \sum_m e^{i(nq-m)\theta}\int e^{i(m-nq)\vartheta}\hat{F}_n(r,\vartheta)d\vartheta \quad . \\
f(r,\theta,\xi) &= \sum_{m,n} e^{in\xi}e^{i(nq-m)\theta}F_{m,n}(r) \quad , \\
F_{m,n}(r) &= \int e^{i(m-nq)\vartheta}\hat{F}_n(r,\vartheta)d\vartheta \quad .
\end{aligned}$$

- For localized mode structures, we can adopt the WKB representation

$$\begin{aligned} \hat{F}_n(r,\vartheta) &= A_n(r)\hat{F}_{n0}(r,\vartheta) \simeq A_n(r)\hat{F}_{n0}(\vartheta) \\ A_n(r) &\sim \exp i \int nq'\theta_k(r)dr \ , \end{aligned}$$

with $\theta_k(r) \equiv k_r(r)/(nq')$ the normalized radial wave vector.

- The GFLDR can be cast as [Zonca and Chen 2014]

$$\left[i\Lambda_n - \left(\delta\bar{W}_f + \delta\bar{W}_k \right)_n \right] A_n(r) = D_n(r,\theta_k,\omega)A_n(r) = 0 \ ,$$

with $D_n(r,\theta_k,\omega)$ playing the role of a local dispersion function (Lecture 1)

  - Λ_n: generalized inertia response
  - $\delta\bar{W}_{fn}$: potential energy due to fluid plasma response
  - $\delta\bar{W}_{kn}$: potential energy due to kinetic plasma response

Nonlinear DAW field equations

(Lecture 1)

- Consistent with the WKB approach, the lowest order (local) dispersion relation must be satisfied in the linear limit

$$D_n^L(r, \theta_{k0}(r), \omega_0) = 0 \quad ;$$

- At the next order, $\omega = \omega_0 + i\partial_t$, $k_r = nq'\theta_{k0} - i\partial_r$; and the GFLDR theoretical framework describes the slow spatial and temporal evolution of the wave packet $A_n = \exp(-i\omega_0 t) A_{n0}(r,t)$ [Lu POP12; Chen RMP14].

$$\frac{\partial D_n^L}{\partial \omega_0}\left(i\frac{\partial}{\partial t}\right) A_{n0}(r,t) + \frac{\partial D_n^L}{\partial \theta_{k0}}\left(-\frac{i}{nq'}\frac{\partial}{\partial r} - \theta_{k0}\right) A_{n0}(r,t) + \frac{1}{2}\frac{\partial^2 D_n^L}{\partial \theta_{k0}^2}\left[\left(-\frac{i}{nq'}\frac{\partial}{\partial r} - \theta_{k0}\right)^2 - \frac{i}{nq'}\frac{\partial \theta_{k0}}{\partial r}\right] A_{n0}(r,t) = \left(\delta \bar{W}_f^{NL} + \delta \bar{W}_k^{NL}\right)_n - i\Lambda_n^{NL} + S_n^{ext}(r,t) \qquad k_r \equiv nq'\theta_{k0}$$

- Λ_n^{NL}: nonlinear response in the kinetic/singular layer; dominated by thermal plasma, wave-wave interactions, competition of Reynolds and Maxwell stress (Lectures by Prof. Liu Chen, following this course)
- $\delta\bar{W}_{fn}^{NL}$: nonlinear potential energy response, due to both thermal plasma and EPs. Non-resonant behaviors in the regular regions
- $\delta\bar{W}_{kn}^{NL}$: nonlinear potential energy due to resonant kinetic plasma response, due to both thermal plasma and EPs

□ Nonlinear dynamics of phase-space zonal structures driven by EPs is determined by $\delta\bar{W}_{kn}^{NL}$ and by nonlinear resonant EP behaviors. $\Rightarrow$ Here, we assume $\Lambda_n^{NL} = 0$ and $\delta\bar{W}_{fn}^{NL} = 0$.

□ In Lecture 5 (fishbones) and Lecture 6 (EPMs), we are going to adopt a very simple model for Λ_n^L, $\delta\bar{W}_{fn}^L$ and $\delta\bar{W}_{kn}^L$.

□ In the following, concentrate on the derivation of $\delta\bar{W}_{kn}^{NL}$ for simple case of precession resonance and vanishing Larmor radius and magnetic drift orbit width.

- Remember diagrams on p.9 and definition of $\delta\bar{W}_{kn}^{NL}$ [Chen 1984] in Lecture 4 and Lecture 5 of Spring 2010 Intensive Lectures

$$\delta\bar{W}_{nk} = \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm} \frac{\pi^2 q R_0}{c^2 k_\vartheta^2 |s|} \frac{e^2}{m} \left(\frac{\tau_b n^2 \bar{\omega}_{dn}^2}{\omega_0(\tau)} \right) \int_{-\infty}^{+\infty} \frac{\omega + \omega_0(\tau)}{n\bar{\omega}_{dn} - \omega_0(\tau) - \omega} e^{-i\omega t} Q_{k,\omega_0(\tau)} \hat{F}_0(\omega) d\omega$$

- Definitions: $\mathcal{E} = v^2/2$, $\lambda = \mu B_0/\mathcal{E}$, $\tau_b = 2\pi/\omega_b$, $k_\vartheta = -nq/r$, $\omega_0(\tau) = \omega_{0r}(\tau) + i\gamma_0(\tau)$, $s = rq'/q$.

- Note that the wave-vector and frequency $(k, \omega_0(\tau))$ values are indicated on which the $Q_{k,\omega_0(\tau)}$ operator [acting on $\hat{F}_0(\omega)$] is to be computed.

E: Go back to p.9 and reconsider diagrams illustrating the processes involved with the nonlinear excitation of SAW by EPs. Discuss where the nonlinearity due to wave-particle interactions enters in the expression of $\delta\bar{W}_{nk}$.

- Strictly speaking, the expression of $\delta \bar{W}_{nk}$ applies for the WKB form of the GFLDR (p. 12), derived for localized. However, a very similar expression holds for the $n = 1$ fishbone mode, which has a typical internal kink mode structure.

E: Describe the macroscopic mode structure of an internal kink mode. What makes local (WKB) and global expressions of $\delta \bar{W}_{nk}$ so similar?

E: Introducing $s(r)$ as (radial dependent) magnetic shear, while s indicates shear at the $q(r_s) = 1$ surface, located at r_s. With help of Spring 2010 lecture notes, demonstrate that one can write the fishbone $\delta \bar{W}_{nk}$ as

$$\delta \bar{W}_{nk} = \frac{2}{|s| r_s^2} \int_0^{r_s} |s(r)| \, \delta \bar{W}_{nk}(r) \big|_{\text{Eq. at p.15}} \, r dr$$

Bounce averaged mode structures

- Recall Lecture 2 and the general result on mode structure representation, lifted to the particle phase-space; *i.e.*, sampled by particles interacting with the mode.

$$
\begin{aligned}
f(r,\theta,\xi) &= \sum_{m,n,\ell} e^{i(n\bar{\omega}_d+\ell\omega_b)\tau+i\Theta_{m,n,\ell}}\mathcal{P}_{m,n,\ell}\circ F_{m,n}(\bar{r}+\Delta r) \quad , \\
\Theta_{m,n,\ell} &= n\Delta\zeta - m\Delta\theta + n\left(\frac{\partial\bar{\omega}_d}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\bar{\omega}_d}{\partial J}\int_0^\tau \delta J d\tau'\right) \\
&\quad +\ell\left(\frac{\partial\omega_b}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\omega_b}{\partial J}\int_0^\tau \delta J d\tau'\right) - i\ln\Lambda_{m,n} \quad .
\end{aligned}
$$

$$
\Lambda_{m,n} = \exp\left[i\left(n\bar{q}(\bar{r})-m\right)\left(\frac{\partial\omega_b}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\omega_b}{\partial J}\int_0^\tau \delta J d\tau'\right) + in\omega_b\frac{d\bar{q}}{d\bar{r}}\int_0^\tau \delta r d\tau'\right] \quad .
$$

- Furthermore, with $\lambda_{m,n} = \exp\left[i\left(n\bar{q}(\bar{r}) - m\right)\omega_b\tau\right]$, the projection operators $\mathcal{P}_{m,n,\ell}$ are defined as

$$\mathcal{P}_{m,n,\ell} \circ F_{m,n} = \frac{\lambda_{m,n}}{2\pi} \oint e^{\{in\Xi(\eta)+i[n\bar{q}(\bar{r})-m]\Theta(\eta)-i\ell\eta\}} F_{m,n}(\bar{r} + \rho(\eta))d\eta$$

- Definitions introduced for compactness: $\Theta_{m,n,\ell} \equiv \Theta_{NL\,m,n,\ell} - i\ln\Lambda_{m,n}$ (recall $\Lambda_{m,n} = \lambda_{m,n} \equiv 1$ for magnetically trapped particles.

- Note the following:

    - The bounce averaged response is obviously obtained for the $\ell = 0$ bounce harmonic

    - For negligible Larmor radius and magnetic drift orbit width $\lambda_{m,n} \exp in\bar{\omega}_d\tau \simeq \exp in(\zeta - q\theta)$

- Introducing $\overline{(\ldots)} = \tau_b^{-1} \oint (\ldots) d\theta/\dot{\theta}$ as magnetic drift bounce averaging [Zonca et al 13, Chen RMP14]

$$\delta\bar{\phi}_n = e^{in(\zeta - q\theta)} \sum_m \mathcal{P}_{m,n,0} \circ \delta\phi_{m,n} = e^{-inq\theta} \overline{e^{inq\theta} \delta\phi_n} \quad ,$$

- Recalling that

$$\delta\phi_n(r,\theta) = 2\pi \sum_\ell e^{2\pi i \ell n q} \delta\hat{\phi}_n(r, \theta - 2\pi\ell) = \sum_m e^{i(nq-m)\theta} \int e^{i(m-nq)\vartheta} \delta\hat{\phi}_n(r,\vartheta) d\vartheta$$

$$\delta\phi_{m,n}(r) = \int e^{i(m-nq)\vartheta} \delta\hat{\phi}_n(r,\vartheta) d\vartheta = A_n(r) \int e^{i(m-nq)\vartheta} \hat{\delta}\hat{\phi}_{n0}(r,\vartheta) d\vartheta$$

Iis possible to see that the fluctuation intensity varies faster than $\sim |nq'|^{-1}$

$$\left|\delta\bar{\phi}_n(r,t)\right|^2 = (2\pi)^2 \left|A_n(r,t)\right|^2 \sum_{\ell,\ell'} e^{-2\pi i n q \ell'} \overline{\delta\hat{\phi}^\dagger_{-n0}\Big|_{\vartheta = 2\pi(\ell-\ell')}} \; \overline{\delta\hat{\phi}_{n0}\Big|_{\vartheta = 2\pi\ell}} \quad .$$

E: Go back to p. 11,12 and to Lecture 2. Explain with your own words why the radial structures $\sim |nq'|^{-1}$ are intrinsic to fluctuations in toroidal system. What is the role of magnetic shear?

E: Convince yourself that structures shorter than $|nq'|^{-1}$ are common to nonlinear structures. These can be due to both wave-wave interactions [see lectures by Prof. Liu Chen following this course] and/or modulations via wave-particle interactions of the EP radial profiles. What is the relevant mechanism in our case? Can you explain why?

Bounce averaged phase-space zonal structures

- Derivation here follows that for the bounce averaged mode structures, applied to EP phase-space.
- Again, assume negligible Larmor radius and magnetic drift orbit width and consider EP precessional motion/resonance only for magnetically trapped particles.
- Using the same decomposition as for mode structures [Zonca et al. 13, 14], and separating adiabatic and non-adiabatic responses (p.4), the representation of phase-space zonal structures is $(n = \ell = 0)$

$$\delta f_z = \sum_m \left\{ \mathcal{P}_{m,0,0} \circ \left[J_0(\lambda) \delta g \right]_{m,0} \right\} - \left[J_0(\lambda) \left(\frac{e}{m} \frac{1}{B_0} \frac{\partial \bar{F}_0}{\partial \mu} \langle \delta L_g \rangle \right) \right]_{0,0} + \frac{e}{m} \left[\frac{\partial \bar{F}_0}{\partial \mathcal{E}} \delta\phi + \frac{1}{B_0} \frac{\partial \bar{F}_0}{\partial \mu} \delta L \right]_{0,0} .$$

- Using the nonlinear gyrokinetic equation [Frieman and Chen 1982] (p.5); and assuming that $|k_\parallel| \ll |\boldsymbol{k}_\perp|$ [Zonca et al 13, Chen RMP14]

$$\frac{\partial \delta g_z}{\partial t} = -\mathcal{P}_{0,0,0}\circ\left(\frac{e}{m}\frac{\partial}{\partial t}\left\langle\delta L_g\right\rangle_z\frac{\partial \bar{F}_0}{\partial \mathcal{E}}\right)_{0,0} + i\sum_m \mathcal{P}_{m,0,0}\circ\frac{c}{d\psi/dr}\frac{\partial}{\partial r}\sum_n n\left(\delta g_n\left\langle\delta L_g\right\rangle_{-n}\right)_{m,0} \quad ,$$

where $d\psi/dr$ is the derivative of the equilibrium magnetic flux.

- This equation contains both zonal flows and fields (the first term on the RHS) as well as the nonlinear effect of EP on equilibrium via wave-EP interactions, dominated by wave-particle resonances.

- In turn, the feedback of phase space zonal structures onto δg_n $(n \neq 0)$ is

$$\left(\frac{\partial}{\partial t} - \frac{inc}{d\psi/dr}\left\langle\delta L_g\right\rangle_z\frac{\partial}{\partial r} + v_\parallel\nabla_\parallel + \boldsymbol{v}_d\cdot\boldsymbol{\nabla}_\perp\right)\delta g_n = i\frac{e}{m}\left(Q\bar{F}_0 - \frac{nB_0}{\Omega d\psi/dr}\mathcal{P}_{0,0,0}\circ\frac{\partial \delta g_z}{\partial r}\right)\left\langle\delta L_g\right\rangle_n \quad .$$

- Accounts for zonal flows/fields as well as corrugation of radial profiles. (Lecture 1)

(Lecture 1)

- These equations for δg_z and δg_n are closed by the $(\delta\phi_n, \delta A_{\parallel n})$ DAW field equations for the dynamic evolution of Alfvénic fluctuations and by the equations for the zonal flows/fields and $(\delta\phi_z, \delta A_{\parallel z})$.

- Studied so far in simplified limits:

  - Neglecting wave-particle resonances $\Rightarrow$ dominant zonal flows/fields [Chen POP00; Chen NF01; Guo PRL09; Kosuga POP12]

  - Neglecting effect of zonal flows/fields $\Rightarrow$ dominant EP wave-particle resonances [Zonca Th.Fus.Pl.00; NF05; PPCF06]

$\Rightarrow$(Lecture 5 and Lecture 6)

- Further simplification noting:

  - for SAW excited by EP $|\omega_{*EP}| \gg |\omega_0| \Rightarrow Q\bar{F}_0 \simeq \Omega^{-1}\boldsymbol{b} \times \boldsymbol{\nabla}\bar{F}_0 \cdot (-i\boldsymbol{\nabla})$

  - again, consider precession resonance only and negligible Larmor radius and magnetic drift orbit width

- Defining $F_0 \equiv \bar{F}_0 + \mathcal{P}_{0,0,0} \circ \delta g_z$, simplified equations become, because of $|\omega_{*EP}| \gg |\omega_0|$

$$\frac{\partial F_0}{\partial t} = i\mathcal{P}_{0,0,0} \circ \sum_m \mathcal{P}_{m,0,0} \circ \frac{c}{d\psi/dr}\frac{\partial}{\partial r}\sum_n n\left(\delta g_n \left\langle \delta L_g \right\rangle_{-n}\right)_{m,0} \quad ,$$

$$\left(\frac{\partial}{\partial t} + v_\parallel \nabla_\parallel + \boldsymbol{v}_d \cdot \boldsymbol{\nabla}_\perp\right)\delta g_n = i\frac{e}{m} Q F_0 \left\langle \delta L_g \right\rangle_n \quad .$$

- Neglecting Larmor radius and magnetic drift orbit width

$$\frac{\partial F_0}{\partial t} = i\sum_n \frac{nc}{d\psi/dr}\frac{\partial}{\partial r}\left(\delta \bar{g}_n \delta\bar{\phi}_{-n} - \delta\bar{g}_{-n}\delta\bar{\phi}_n\right) = i\sum_n \frac{nc}{d\psi/dr}\frac{\partial}{\partial r}\left(\delta\bar{K}_n\delta\bar{\phi}_{-n} - \delta\bar{K}_{-n}\delta\bar{\phi}_n\right)$$

$$+\sum_n \frac{1}{|\omega_n|^2}\frac{nc}{d\psi/dr}\frac{\partial}{\partial r}\left(\frac{nc}{d\psi/dr}\frac{\partial F_0}{\partial r}\frac{\partial}{\partial t}\left|\delta\bar{\phi}_n\right|^2\right) \quad .$$

E: Demonstrate that the term on 2nd line is time reversible and that it corresponds to interactions with non-resonant EPs.

E: Can you explain why we can neglect it, at the lowest order?

- Keeping only the $\delta\bar{K}_n$ terms; and using Laplace transform to express solutions (p. 11 Lecture 3)

$$
\begin{aligned}
\delta\hat{\bar{K}}_k(\omega) &= \frac{e}{m}\int_{-\infty}^{+\infty}\frac{\hat{\omega}_{dk}}{y}\frac{Q_{k,y}\hat{F}_0(\omega-y)}{n\bar{\omega}_{dk}-\omega}\delta\hat{\bar{\phi}}_k(y)dy \;, \\
\hat{\omega}_{dk}\delta\hat{\bar{\phi}}_k &\equiv e^{-inq\theta}\overline{e^{inq\theta}\omega_d\delta\phi_n} \simeq n\bar{\omega}_{dk}\delta\hat{\bar{\phi}}_k \;.
\end{aligned}
$$

(for deeply magnetically trapped particles)

- The Dyson equation for $\hat{F}_0(\omega)$ becomes

$$
\begin{aligned}
\hat{F}_0(\omega) &= \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega)+\frac{i}{\omega}\hat{S}_0(\omega)+\frac{i}{2\pi\omega}\bar{F}_0(0) \\
&+\frac{nc}{\omega(d\psi/dr)}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\left[\delta\hat{\bar{\phi}}_k(y)\delta\hat{\bar{K}}_{-k}(\omega-y)-\delta\hat{\bar{\phi}}_{-k}(y)\delta\hat{\bar{K}}_k(\omega-y)\right]dy \;.
\end{aligned}
$$

E: Compare this expression with that on p. 12 of Lecture 3, which applies for the beam-plasma problem in a 1D uniform system. Comment about similarities and differences.

- Here, the effect of collisions, formally denoted by $\mathrm{St}\hat{F}_0(\omega)$, and of an external source term, $\hat{S}_0(\omega)$, are included; while $\bar{F}_0(0)$ denotes the initial value of F_0 at $t = 0$.
- Final step: substitute the $\delta\hat{\bar{K}}_k(\omega)$ solution into Dyson Equation and use the Laplace transform representation for weakly varying fluctuation fields (p. 15 Lecture 3)

$$\delta\hat{\bar{\phi}}_{k_0}(\omega) = \frac{i}{2\pi}\frac{\delta\bar{\phi}_{k_0}}{\omega - \omega_0} \quad , \qquad \delta\hat{\bar{\phi}}_{-k_0}(\omega) = \frac{i}{2\pi}\frac{\delta\bar{\phi}^*_{k_0}}{\omega + \omega^*_0}$$

□ The Dyson Equation for nearly periodic fluctuations becomes

$$\hat{F}_0(\omega) = \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}\bar{F}_0(0) + \frac{e}{m}\frac{nc}{\omega(d\psi/dr)}\frac{\partial}{\partial r}\left\{\left[\frac{Q^*_{k_0,\omega_0(\tau)}}{\omega^*_0(\tau)}\right.\right.$$
$$\left.\left.\times\frac{\hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega - \omega_0(\tau) + n\bar{\omega}_{dk0}} + \frac{Q_{k_0,\omega_0(\tau)}}{\omega_0(\tau)}\frac{\hat{F}_0\left(\omega - 2i\gamma_0(\tau)\right)}{\omega + \omega^*_0(\tau) - n\bar{\omega}_{dk0}}\right]\hat{\omega}_{dk0}\left|\delta\bar{\phi}_{k0}(r,\tau)\right|^2\right\}$$

E: Compare this expression with that on p. 17 of Lecture 3, which applies for the beam-plasma problem in a 1D uniform system. Comment about similarities and differences.

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