Lecture 5

Frequency sweeping, phase locking and supra-thermal particle transport:

heuristic models for understanding nonlinear behaviors in nonuniform plasmas

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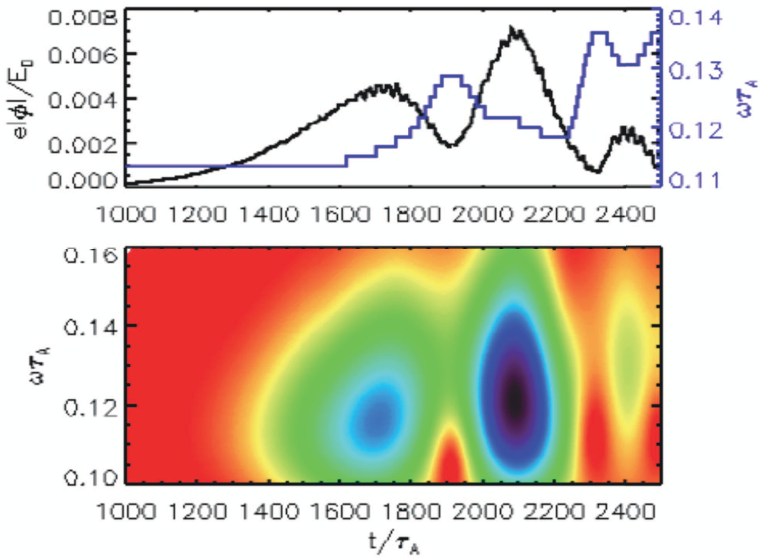
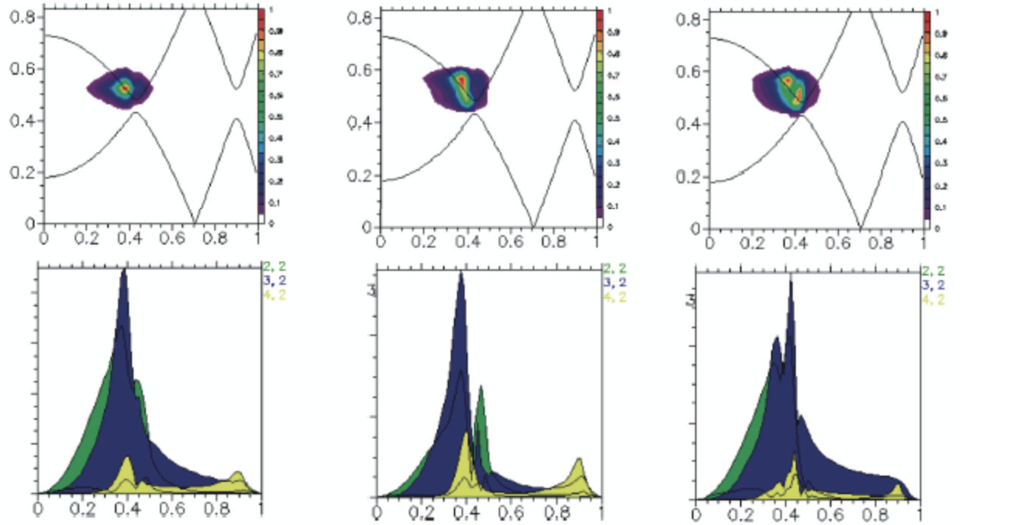
May 9.th, 2012

IFTS Intensive Course on Advanced Plasma Physics-Spring 2012,
Nonlinear charged particle dynamics in tokamaks: theory and applications
2–12 May 2012, IFTS – ZJU, Hangzhou

Some evidence from Lecture 4



Bursting BAE structures, resonance detuning by radial nonuniformity and frequency sweeping (Wang et al. submitted to PRL).



Nonlinear TAE evolutions, resonance detuning by radial nonuniformity and frequency (mode) splitting, accompanied by structure splitting and merging in phase space (Briguglio et al. to be submitted to PRL).

Nonuniform plasma behaviors

- In Lecture 1, Lecture 2 and Lecture 3, we have emphasized the roles of plasma nonuniformity and equilibrium geometry.
- The role of plasma nonuniformity is clear from the mode structure decomposition in action angle variables (see Lecture 2):

$$
\begin{aligned}
f(r,\theta,\xi) &= \sum_{m,n,\ell} \lambda_{n,m}\Lambda_{n,m} e^{i(n\bar{\omega}_d+\ell\omega_b)\tau+i\Theta_{NL\,n,m,\ell}} \mathcal{F}_{m,n,\ell}(\bar{r};\mu,J,P_\phi) \quad , \\
\Theta_{NL\,n,m,\ell} &= n\Delta\zeta - m\Delta\theta + n\left(\frac{\partial\bar{\omega}_d}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\bar{\omega}_d}{\partial J}\int_0^\tau \delta J d\tau'\right) \\
&\quad + \ell\left(\frac{\partial\omega_b}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\omega_b}{\partial J}\int_0^\tau \delta J d\tau'\right) \quad .
\end{aligned}
$$

$$
\Lambda_{n,m} = \exp\left[i\left(n\bar{q}(\bar{r})-m\right)\left(\frac{\partial\omega_b}{\partial P_\phi}\int_0^\tau \delta P_\phi d\tau' + \frac{\partial\omega_b}{\partial J}\int_0^\tau \delta J d\tau'\right) + in\omega_b\frac{d\bar{q}}{d\bar{r}}\int_0^\tau \delta r d\tau'\right] \quad .
$$

- Note that typically $\delta r/r \approx \delta P_\phi / P_\phi \approx (\omega_*/\omega)\delta H_0/H_0$ (see p.29 of Lecture 1): thus, for energetic particles the resonance-detuning is expected to be dominated by the radial displacement, since $|\omega_{*E}/\omega| \gg 1$.

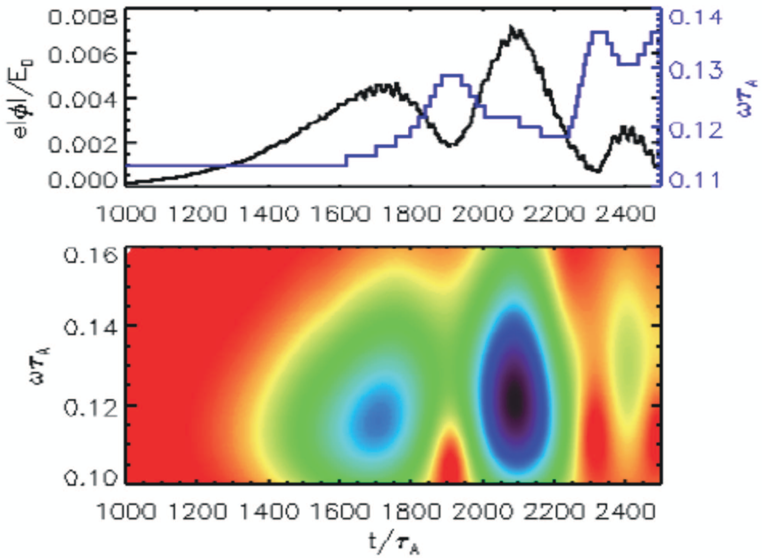
- From $\Theta_{NL\,n,m,\ell}$ and $\Lambda_{m,n}$ expressions on p.3, when orbit width and mode width are of the same order, a particle transported radially by NL processes by one mode/orbit width will have a modest $\mathcal{O}(1/n)$ resonance-detuning for the precession/bounce resonance of trapped particles, while the corresponding detuning for circulating particles will be important and $\mathcal{O}(1)$.

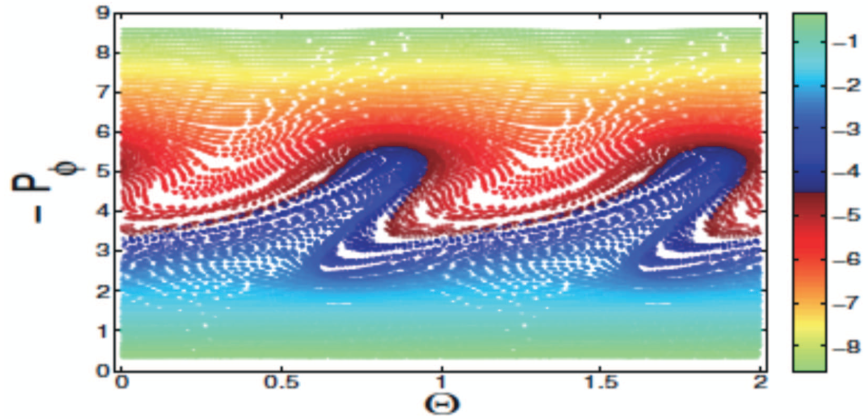
- Therefore, if $\gamma/\omega > \mathcal{O}(1/n)$ for fixed frequency modes, the finite radial structure will start to play a significant role for the precession/bounce resonance of trapped particles, while the same condition would be $\gamma/\omega > \mathcal{O}(\omega/n\omega_{*E})$ for strong frequency chirping modes (Zonca et al 05 and Spring 10 Lecture 5). These nonlinear processes will be characterized by secular radial motion on a time scale shorter than the wave-particle trapping time in the potential well of the wave (see p.23 of Lecture 2 and also the work by Zhichen Feng and Xiao and Lin 11).

- On the contrary, for circulating particle resonances these results imply that the saturation should be local or nearly local, except for the very strong drive and chirping frequency $\gamma/\omega > \mathcal{O}(\omega/\omega_{*E})$, although finite radial mode structures start playing a role at $\gamma/\omega \lesssim \mathcal{O}(\omega/\omega_{*E})$ (Levin et al. 72).
- This is confirmed by recent numerical simulations (Wang et al 12 and Briguglio et al 12) with the hybrid MHD Gyrokinetic Code HMGC (Briguglio et al 95,98).
- This discussion clearly shows the particular role of precession resonance in the trapped particle transport.
- Further details in Lecture 6.

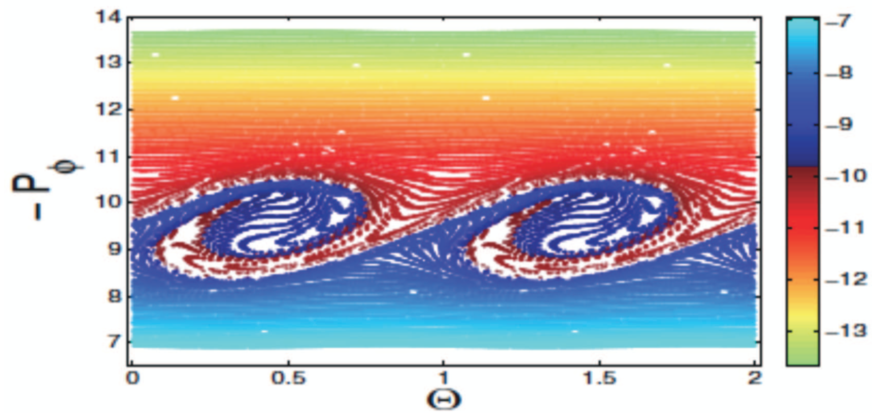
Frequency sweeping and phase locking

- Evidence from Lecture 4 (X. Wang et al, submitted to PRL)





$v_\parallel > 0$



$v_\parallel < 0$

Bursting BAE structures, resonance detuning by radial nonuniformity and frequency sweeping yield $v_\parallel$ symmetry breaking (Wang et al. submitted to PRL.)

- Different spatiotemporal scales, connected with plasma nonuniformity and equilibrium geometry, determine resonance detuning (see p.15 Lecture 2), i.e. they determine the finite interaction length and finite interaction time for the wave-particle resonance.

- Consider resonance detuning for the nonlinear BAE case studied in Lecture 4. Linear resonance $\omega = \omega_0$

$$\omega = \ell\omega_t + (n\bar{q} - m)\omega_t \qquad \text{negligible } \bar{\omega}_d$$

$$\Delta\dot{\Theta} = \left(n\partial_r\bar{q} + \frac{\omega_0}{\omega_t^2}\partial_r\omega_t \right) \omega_t\Delta r - (\omega - \omega_0)$$

- This expression for resonance detuning explains the $v_\parallel$ symmetry breaking in the presence of frequency sweeping.

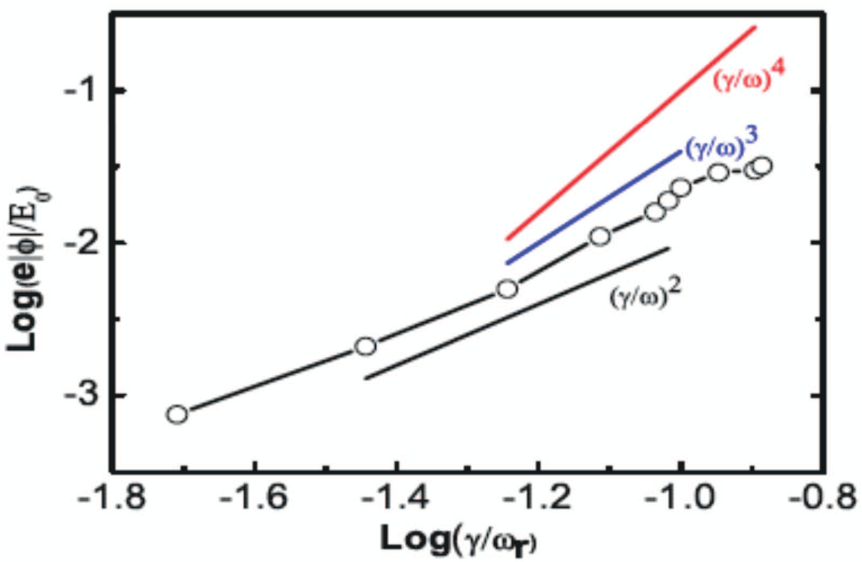
- It further demonstrates that finite interaction length and finite interaction time may be very different for frequency sweeping modes with respect to fixed frequency modes: this brings consequences on mode nonlinear dynamics and on resonant particle transport.

- Prerequisite for this to occur is that spatiotemporal scales of frequency sweeping are shorter than those of wave-particle trapping in the absence of sweeping. Otherwise, existence of adiabatic invariance would imply that nonlinear dynamics is similar to that of a 1D uniform plasma system (see Spring 10 Lecture 3) $\Longrightarrow$ non-adiabatic frequency sweeping and mode particle pumping (White et al 83).

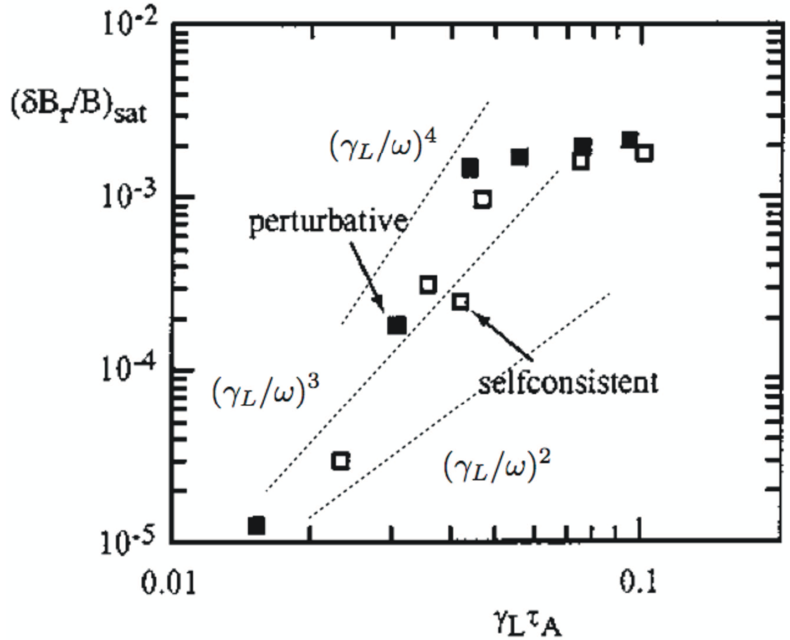
- Anticipation of Lecture 6: when a given mode structure (Energetic Particle Mode, Chen 94) can maintain the resonance condition by frequency sweeping (phase locking), macroscopic secular particle transport takes place.

- This is a condition for maximizing energy extraction from energetic particles (mode growth) and related transport processes.

- Further evidence of resonance detuning and importance or radial nonuniformity is given by scaling of the saturation amplitude with the growth rate.



BAE saturation amplitude scaling vs. γ/ω (p.17 Lecture 4) (Wang et al. submitted to PRL).



TAE/EPM saturation amplitude scaling vs. γ/ω_A (Briguglio et al. 98).

The nonlinear pendulum model

- Assume that we are dealing with a mode structure with few harmonics effectively coupled together: localized mode structures due to BAE and/or TAE, as in Lecture 4.

- As model Ansatz, assume that the radial envelope of the modes (in this case the mode structure of the single harmonics) can be represented by a Lorentzian function

$$\bar{A} = \frac{\bar{A}_0}{1 + (X - X_0)^2}$$

- Definitions: $X = k_r \Delta r$, X_0 is the radial position of the Lorentzian function peak (thus, allowing that the peak does not coincide with the surface of unperturbed particle motion), $\bar{A} = (n^2/\omega)\bar{q}\bar{q}'\mathcal{A} \approx (k_\perp \rho_L)(k_\perp R)s(e\delta\phi/T) \approx (k_\perp R)s(\delta B_r/B)$, $\bar{A}_0$ is the value of $\bar{A}$ at X_0.

- Assume that also for the TAE case there is one dominant resonance. Then, one can obtain the following equations of motion near resonance, including frequency sweeping

$$\frac{dY}{d\tau} = X - \Omega \ ,$$

$$\frac{dX}{d\tau} = \frac{\bar{A}_0(\tau)}{1+(X-X_0)^2}\left(\frac{k_r}{n\bar{q}'}\sin\Theta_0 + Y\cos\Theta_0\right) \ .$$

- Additional definitions: $\tau = \omega_t t$, $Y = (k_r/n\bar{q}')(\Theta-\Theta_0)$, $\Omega(\tau) = (k_r/n\bar{q}')(\omega - \omega_0)/\omega_t$. For $\Omega = 0$, the equations above clearly admit a stable fixed point at $\Theta_0 = \pi$ and an unstable fixed point at $\Theta_0 = 0$ (see Spring 11 Lecture 4).

- In the derivation of the above system of equations, we have expanded $\sin\Theta$ about Θ_0. This is because we are interested in either one of these cases:
  - Weak or moderate chirping: the effect of finite radial mode structure can be investigated assuming small oscillations
  - Strong chirping: phase locking allows us to assume weak phase variations, so small oscillations

- Solution can be found in the form $X = \Omega + \delta X$

$$
\begin{aligned}
\frac{dY}{d\tau} &= \delta X \quad , \\
\frac{d\delta X}{d\tau} &= -\dot{\Omega} + \frac{\bar{A}_0(\tau)}{1 + (\delta X + \Omega - X_0)^2} \left(\frac{k_r}{n\bar{q}'} \sin \Theta_0 + Y \cos \Theta_0 \right) \quad .
\end{aligned}
$$

- For $|\Omega|, |X_0| \ll |\delta X|$, i.e. if the mode structure undergoes a weak read-justment in the dynamic saturation process, we can investigate the effect of nonuniformity looking at solutions of the system about $\Theta_0 = \pi$

$$
\begin{aligned}
\frac{dY}{d\tau} &= \delta X \quad , \\
\frac{d\delta X}{d\tau} &= - \frac{\bar{A}_0(\tau) Y}{1 + \delta X^2} \quad .
\end{aligned}
$$

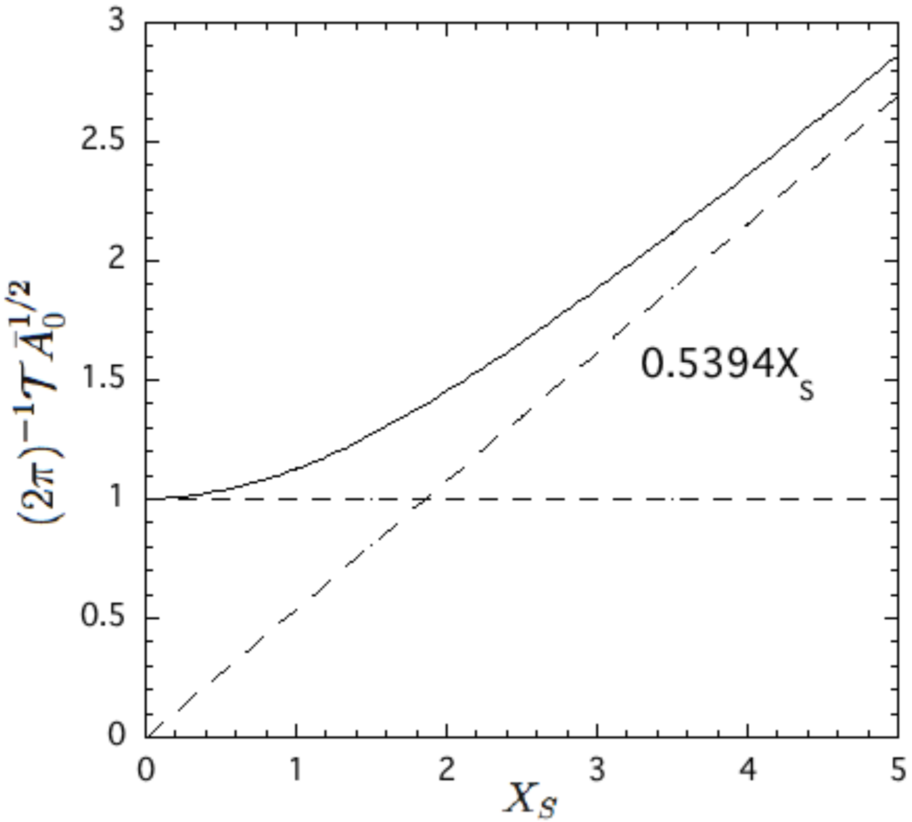
- This system which is separable and can be trivially integrated by quadratures, yielding

$$\delta X^2(2+\delta X^2)+2\bar{A}_0 Y^2 = X_S^2(2+X_S^2) \ ,$$

with X_S representing the value of δX at $Y=0$, i.e. the maximum excursion of the periodic orbit described by the considered equations.

- The period of the motion is given by

$$\begin{aligned}
\mathcal{T}(\bar{A}_0, X_S) &= \frac{4}{\bar{A}_0^{1/2}} \int_0^{X_S} \frac{(1+X^2)dX}{\sqrt{X_S^2(1+X_S^2/2)-X^2(1+X^2/2)}} \\
&= \frac{2\pi}{\bar{A}_0^{1/2}} \int_0^1 \frac{(2/\pi)(1+t^2X_S^2)dt}{\sqrt{(1+X_S^2/2)-t^2(1+t^2X_S^2/2)}} \ .
\end{aligned}$$



The integral in the expression for $\mathcal{T}(\bar{A}_0, X_S)$ is a function of X_0, which is easily computed by asymptotic expansions and is 1 for $X_0 \to 0$ and $\propto X_S$ for $X_S \gg 1$.

- The characteristic rate for a particle to undergo a radial excursion Δr_S, corresponding to $X_S = |k_r|\Delta r_S$, is given by $\gamma_{NL}/\omega_t = \bar{A}_0^{1/2}\Gamma(X_S)$, where $\Gamma(X_S)$ is the inverse of the normalized period shown in the figure; i.e., $\Gamma(X_S) = 1$ for small X_S, while $\Gamma(X_S) \propto X_S^{-1}$ for large X_S. Note, also, that $X_S \approx \bar{A}_0^{1/2}\Gamma(X_S)|k_r|/(n\bar{q}')$, having accounted for $Y = k_r(\Theta - \Theta_0)/(n\bar{q}') \approx |k_r|/(n\bar{q}')$.

- If $X_S = |k_r|\Delta r_S << 1$, as assumed in the the original work by Berk and Breizman on the nonlinear saturation of TAE modes (Berk and Breizman 90), the nonlinear dynamics is the same as in a uniform system and wave saturation can occur only when the particle distribution function in the resonance region is flattened, so that the nonlinear drive is significantly reduced and brought back to threshold.

- This occurs for $\gamma_{NL}/\omega_t = \bar{A}_0^{1/2} \simeq \gamma_L/\omega_t$, with γ_L the linear drive, yielding the well known estimate

$$\bar{A}_0 \propto (\gamma_L/\omega)^2$$

- When $X_S = |k_r|\Delta r_S \gtrsim 1$, saturation occurs because particle get off resonance by radial detuning, as observed first by (Briguglio et al. 98). This process is different from that due to wave-particle trapping. In fact it takes place because of a significant radial transport of particles combined with the radial structure of the mode.

- Thus, it is strictly connected with the nonuniform nature of the system and its geometry (the process is different for circulating particles, as in the case of transit resonance we are analyzing, and magnetically trapped particles).

- Similar to the uniform case, saturation is expected when the radial resonance detuning rate is of the order of the linear growth rate, i.e. when $\bar{A}_0^{1/2}\Gamma(X_S) \propto \bar{A}_0^{1/2}/X_S \approx \gamma_L/\omega_t$. However, we also have $X_S \approx \bar{A}_0^{1/2}\Gamma(X_S)|k_r|/(n\bar{q}') \propto (\bar{A}_0^{1/2}/X_S)|k_r|/(n\bar{q}')$, as shown above, yielding

$$X_S \propto \bar{A}_0^{1/4}|k_r|^{1/2}/(n\bar{q}')^{1/2} \quad \Rightarrow \quad \gamma_{NL}/\omega_t = \bar{A}_0^{1/2}\Gamma(X_S) \propto \bar{A}_0^{1/4}(n\bar{q}')^{1/2}/|k_r|^{1/2}.$$

- The saturation condition then becomes

$$\bar{A}_0 \approx (k_\perp R)s\frac{\delta B_r}{B} \propto (\gamma_L/\omega)^4 \frac{k_r^2}{(n\bar{q}')^2} \quad .$$

- For BAE and TAE modes, the radial structure has two characteristic scale lengths set by the distance of mode rational surfaces $1/(n\bar{q}')$ (the longest one) and by the linear mode frequency distance $\delta\omega$ from the accumulation points in the shear Alfvén continuous spectrum gap (the shortest one).

- For BAE modes this scale is $(\delta\omega/\omega)^{1/2}/(n\bar{q}')$, while for TAE it is $(r/R_0)^{1/2}(\delta\omega/\omega)^{1/2}/(n\bar{q}')$.

- Near the accumulation point, the short scale radial structure of the mode is dictated by the drive rather than by the mode frequency and $\gamma_L/\omega \approx (\delta\omega/\omega)$. Thus, the saturation condition scaling, corresponding to particle radial displacement of the order of the short scale mode structure is

$$\bar{A}_0 \approx (k_\perp R) s \frac{\delta B_r}{B} \propto (\gamma_L/\omega)^3 \quad . \qquad (\times \frac{R_0}{r} \text{ for TAE})$$

- Meanwhile, the saturation condition scaling, corresponding to particle radial displacement of the order of the long scale mode structure is

$$\bar{A}_0 \approx (k_\perp R) s \frac{\delta B_r}{B} \propto (\gamma_L/\omega)^4 \quad .$$

- When the mode drive become even stronger, the saturation amplitude will eventually become independent on the drive. This can be understood as follows:
  - For very weak drive, flattening of the distribution function within the wave-particle trapping separatrix width in particle phase space yields $\bar{A}_0 \propto (\gamma_L/\omega)^2$.
  - For increasing drive, plasma nonuniformity plays an important role and mode saturation take place when radial particle displacement is in the order of the mode characteristic scales, giving either $\bar{A}_0 \propto (\gamma_L/\omega)^3$ or $\bar{A}_0 \propto (\gamma_L/\omega)^4$ depending on whether the relevant spatial scale is the short or the long scale of the mode structure.

- This condition can be translated into the corresponding condition on the rate at which particles are displaced the relevant mode radial scale: on this time scale, the linear drive is expected to change significantly and therefore to yield saturation when it is of the order of the growth rate.

- For strong enough drive, particles will be displaced by

$$X_S \propto \bar{A}_0^{1/4} |k_r|^{1/2} / (n\bar{q}')^{1/2}$$

$$\text{at the rate :} \quad \gamma_{NL}/\omega_t = \bar{A}_0^{1/2} \Gamma(X_S) \propto \bar{A}_0^{1/4} (n\bar{q}')^{1/2} / |k_r|^{1/2}.$$

- This is the estimate used before; however, there exists a (equilibrium dependent) $X_{S\mathrm{crit}}$ above which particle do not effectively interact with the mode. So, for $X_S \gtrsim X_{S\mathrm{crit}}$, for increasing drive saturation will occur at a faster rate but will have a saturation amplitude independent on the mode drive

$$\bar{A}_0 \propto X_{S\mathrm{crit}}^4 (k_r/n\bar{q}')^2$$

- Original observations of these scaling regimes in (Briguglio et al 98), recently emphasized for NL BAE saturation excited by supra-thermal slowing down ions (W. Xin et al. submitted to PRL).

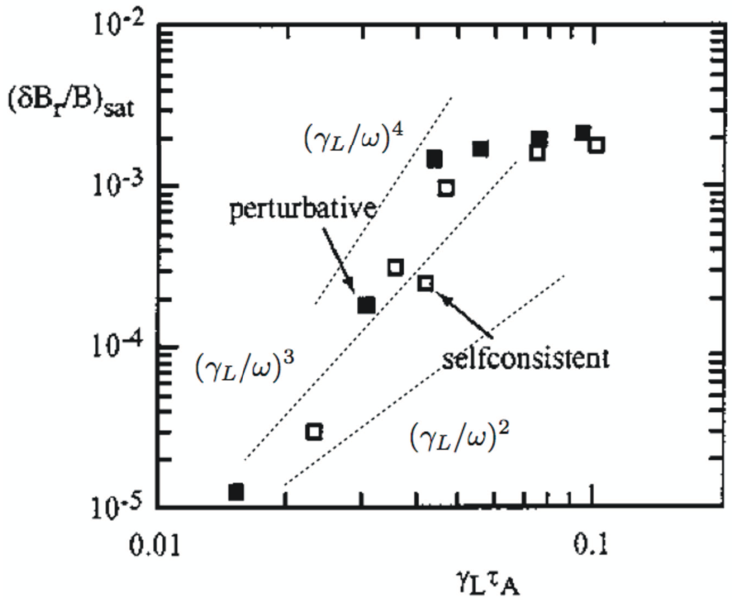


Figure 9 of (Briguglio et al 98), which had the following caption: *The maximum amplitude of the fluctuating radial magnetic field versus linear growth rate for perturbative (full boxes) and self-consistent (empty boxes) $n = 1$ simulations, performed at different values of β_H. The other parameters are the same ones considered in the rest of the paper. The departure, at moderately high γ_L values, from the very-low-γ_L scaling of the saturated amplitude can be traced back to the finite mode radial width compared to that of the resonance region.*

E: Discuss the underlying physics assumptions, which allow to adopt the nonlinear pendulum model for the description of the saturated Alfvén Eigenmode scaling with the normalized growth rate.

E: The existence various scaling regimes is connected with the existence of multiple scale structures of the Alfvén Eigenmodes and, as a consequence of this, of multiple time scales in the wave-particle decorrelation (resonance detuning). Can you justify this statement?

E: In the light of the previous two exercises, discuss the conditions such that the different scalings are actually observed in numerical simulations. Do you expect that all scalings are always visible? If not, give reasons why this is the case and identify which of the provided scalings are "robust", i.e. they should always be observed.

The relay-runner model

- For the NL BAE simulations by (W. Xin et al. submitted to PRL), we have $|\Omega|, |X_0| \lesssim |\delta X|$, so that the frequency sweeping (and phase locking) yield a small but appreciable change in the mode structure.

- Similar conditions, $|\Omega|, |X_0| \lesssim |\delta X|$, are verified in NL TAE simulations (Briguglio et al. to be submitted to PRL; see p.2 and Lecture 4) yielding to phase space, frequency and mode structures splitting and merging.

- It is instructive to explore another limiting solution of the equations reported on p.11, corresponding to an EPM avalanche propagating radially, yielding secular radial supra-thermal particle transport as the EPM wave-packet is convectively amplified (see Spring 10 Lecture 5 and this year's Lecture 6).

$$\left.\begin{aligned} \frac{dY}{d\tau} &= X - \Omega \;, \\ \frac{dX}{d\tau} &= \frac{\bar{A}_0(\tau)}{1+(X-X_0)^2}\left(\frac{k_r}{n\bar{q}'}\sin\Theta_0 + Y\cos\Theta_0\right) \;. \end{aligned}\right\} \Rightarrow \left\{ \begin{aligned} & X = X_0 = \Omega \;\; ; \;\; Y = 0 \\ & \frac{dX}{d\tau} = \frac{k_r}{n\bar{q}'}\bar{A}_0(\tau)\sin\Theta_0 \end{aligned}\right.$$

- Phase locking and self-consistent frequency chirping

$$\dot{\Omega} = \frac{dX}{d\tau} = \frac{k_r}{n\bar{q}'}\bar{A}_0(\tau)\sin\Theta_0$$

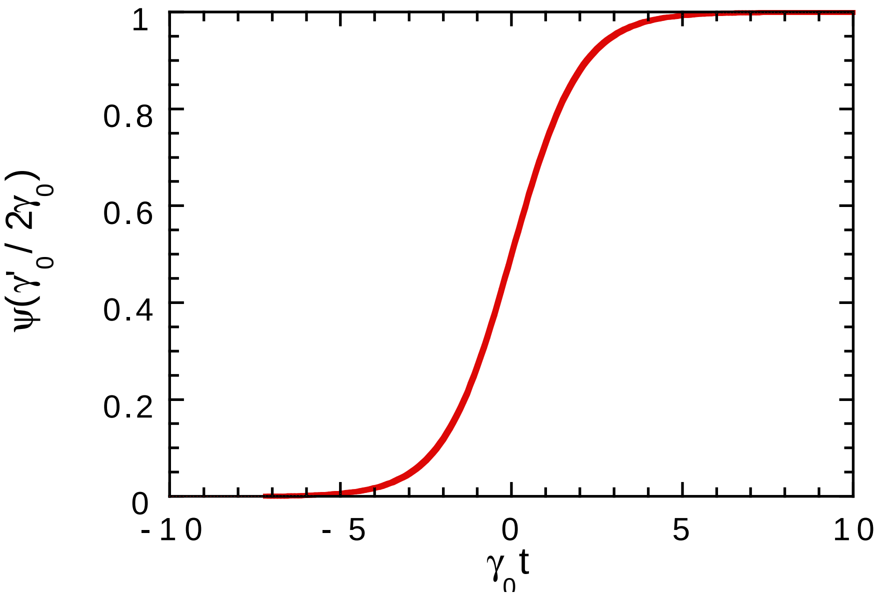
  are essential ingredients of the EPM avalanche, which requires a self-consistent solution of a NL PDE (see Spring 10 Lecture 5 and this year's Lecture 6).

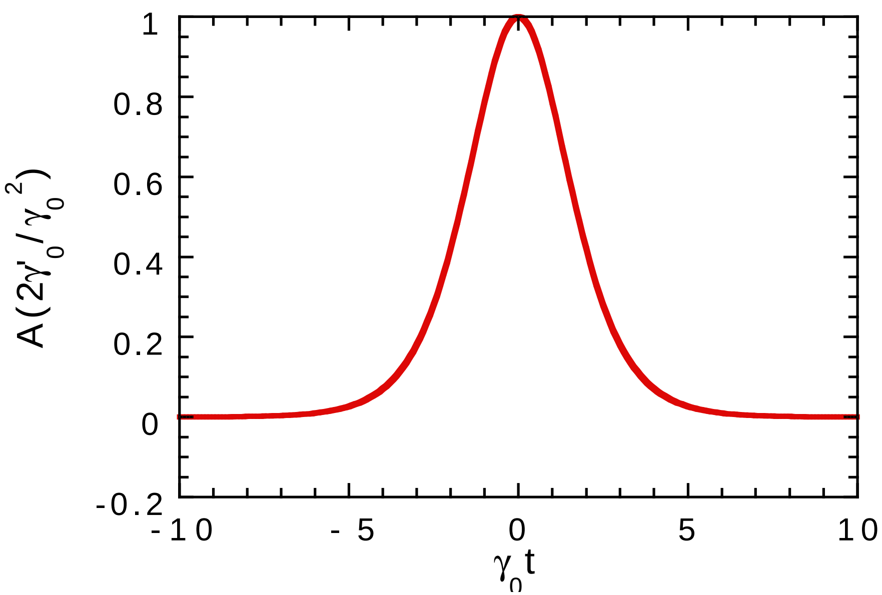
- However, one can develop a heuristic simplified model to describe the EPM convective amplification and frequency chirping.

- After suitable rescaling of time, previous equations give $\Omega = \dot{X} = \bar{A}_0$; so, by definition (first equation) and by heuristic model Ansatz (second equation) we have (Zonca and Chen 99)

$$\begin{aligned} \ddot{X} &= \dot{\bar{A}}_0 = \gamma(X)\bar{A}_0 \\ \gamma(X) &= \gamma_0 - \gamma_0' X \ . \end{aligned}$$

- This system was proposed for explaining EPM NL dynamics and frequency chirping connected with the mode particle pumping (White et al 83). Model parameters are γ_0 (normalized initial drive) and γ_0' (normalized drive detuning/damping).





$$X = \frac{2\gamma_0}{\gamma_0'}\frac{\exp(\gamma_0 t)}{1+\exp(\gamma_0 t)} \qquad \bar{A}_0 = \frac{\gamma_0^2}{2\gamma_0'}\frac{4\exp(-\gamma_0 t)}{(1+\exp(-\gamma_0 t))^2}$$

- Due to the change in resonance condition, the outward secular motion of the wave-packet corresponds to a downward frequency chirping ($X = \Omega$): original results apply to precession resonance (Zonca and Chen 99).

The peak displacement corresponds to

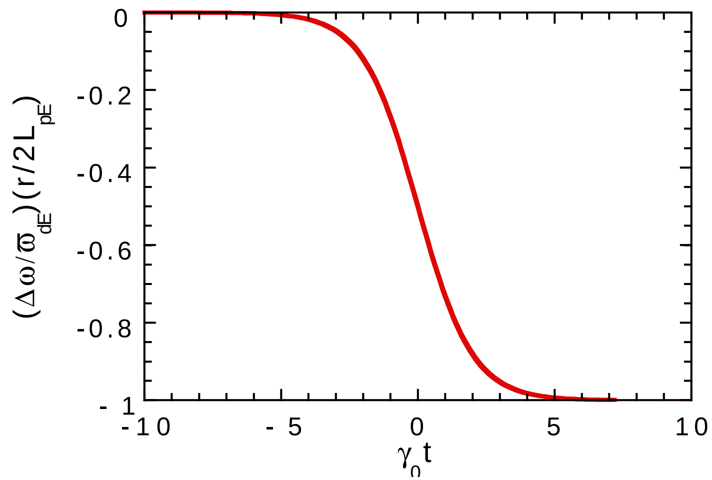
$$\frac{\Delta r}{r} \approx \frac{2L_{pE}}{r}$$

The peak radial amplitude corresponds to a max. radial velocity

$$v_r \approx 0.5\gamma_L L_{pE}$$

. . . and to a fluctuation level of

$$\frac{\delta B_r}{B} \approx \left(\frac{L_{pE}}{R_0}\right)\left(\frac{\gamma_L}{\bar{\omega}_{dE}}\right)$$



Assuming $\omega \simeq \omega_{dE}$, we have $(\Delta\omega/\omega_{dE}) \approx (\Delta r/r)$, i.e.,

$$\frac{\Delta\omega}{\omega_{dE}} \approx -\frac{2L_{pE}}{r}\frac{\exp(\gamma_0 t)}{1+\exp(\gamma_0 t)}$$

References and reading material

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