Lecture 1

Guiding center particle motion in toroidal systems and integral invariants

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Lecture Plan and Teaching Material

Abstract: This Intensive Course is articulated in lectures, which are the natural continuation of the IFTS Intensive Course on Advanced Plasma Physics-Spring 2011 on Non-linear charged particle dynamics. Most of the relevant introductory material can be found in the aforementioned Lecture Notes and in the Lectures Notes of the second part of that course, given by Prof. Liu Chen, focusing on non-linear plasma theory. Useful source of information is also the book by A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, Second Edition, Springer (2010).

Main areas that will be explored are:

i) Guiding center particle motion in toroidal systems and integral invariants. (Lecture 1)

ii) Hamiltonian mappings for toroidal systems: numerical implementation for synthetic diagnostics and analytic forms of Hamiltonian mappings for theoretical studies. (Lecture 2)

iii) Applications and examples of Hamiltonian mappings from previous courses. (Lecture 3)

iv) Cases of practical interest: the nonlinear BAE problem and the nonlinear TAE problem from recent hybrid simulation results. (Lecture 4)

v) Frequency sweeping, phase locking and supra-thermal particle transport: heuristic models for understanding nonlinear behaviors in nonuniform plasmas. (Lecture 5)

vi) From local to global nonlinear dynamic descriptions. (Lecture 6)

Lecture Notes: Available in electronic form. At the end of each lecture and during the Q&A time, a list of specific reading material is given explicitly. Should you have difficulty in finding literatures and papers, please contact me at fulvio.zonca@enea.it.

Exercises: In the lecture notes, Exercises (E) of various difficulty levels are suggested. These are meant to be important part of the lectures themselves.

Nonlinear gyrokinetic equations of motions

- For reference, we use the *Hamiltonian* formulation of nonlinear gyrokinetic equations given in Brizard and Hahm 07, without giving a detailed derivation of those equations. Please refer to the given reference.
- The Brizard and Hahm 07 approach is based on the introduction of two *near-identity* phase-space coordinate transformations (refer to Spring 11 Lecture Notes):
  - from particle to guiding center: remove the fast gyroangle dependence in equilibrium fields
  - from guiding center to gyrocenter: remove the fast gyroangle dependence in fluctuating fields

E: Read the derivation of nonlinear gyrokinetic equations of motion in Brizard and Hahm 07 and discuss/comment the equations reported in the following.

- The transformation from particle $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{p})$ to gyrocenter phase space coordinates $\bar{\mathcal{Z}} = (\bar{\boldsymbol{X}}, \bar{p}_\parallel, \bar{\mu}, \bar{\zeta})$ has Jacobian $m^2 B^*_\parallel$. With $\boldsymbol{b} \equiv \boldsymbol{B}_0 / B_0$

$$B^*_\parallel = B_0 + (c/e)\boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})\bar{p}_\parallel$$

- Define $\boldsymbol{B}^* = \boldsymbol{b} B^*_\parallel + (c/e)(\boldsymbol{b} \times \boldsymbol{\kappa})\bar{p}_\parallel$. Then

$$\bar{\mu} = v_\perp^2 / (2B_0) + \ldots$$

and the parallel canonical moment is

$$\bar{p}_\parallel = m v_\parallel + \frac{e}{c} \langle \delta A_{\parallel gc} \rangle + \ldots \qquad \langle \ldots \rangle \equiv \frac{1}{2\pi} \int (\ldots)\, d\bar{\zeta} \ ,$$

$$\delta A_{\parallel gc}(\boldsymbol{X}, \mu, \zeta, t) = \delta A_\parallel(\boldsymbol{X} + \boldsymbol{\rho}, t)$$

- The nonlinear gyrocenter Hamiltonian equations are

$$\dot{\bar{\boldsymbol{X}}} = \frac{c\boldsymbol{b}}{eB^*_\parallel} \times \bar{\boldsymbol{\nabla}} \bar{H} + \frac{\boldsymbol{B}^*}{B^*_\parallel} \frac{\partial \bar{H}}{\partial \bar{p}_\parallel} \quad ,$$

$$\dot{\bar{p}}_\parallel = -\frac{\boldsymbol{B}^*}{B^*_\parallel} \cdot \bar{\boldsymbol{\nabla}} \bar{H}$$

- The nonlinear gyrokinetic Hamiltonian is expanded as a perturbation series (Brizard and Hahm 07) with $\bar{H} = \bar{H}_0 + \epsilon_\delta \bar{H}_1 + \epsilon_\delta^2 \bar{H}_2 + \ldots$ (Spring 11 Lectures) and $\Omega = eB_0/(mc)$:

$$\bar{H}_0 = \frac{\bar{p}_\parallel^2}{2m} + m\bar{\mu}B_0 \quad ,$$

$$\bar{H}_1 = e\left\langle \delta\psi_{gc} \right\rangle = e\left\langle \delta\phi_{gc} - \frac{\boldsymbol{v}}{c} \cdot \delta\boldsymbol{A}_{gc} \right\rangle \quad ,$$

$$\bar{H}_2 = -\frac{e^2}{2\Omega}\left\langle \left\{ \delta\tilde{\Psi}_{gc}, \delta\tilde{\psi}_{gc} \right\}_0 \right\rangle + \frac{e^2}{2mc^2}\left\langle |\delta\boldsymbol{A}_{gc}|^2 \right\rangle \quad . \tag{1}$$

- In the notation of (Brizard and Hahm 07)

$$\delta\tilde{\psi}_{gc} \equiv \delta\psi_{gc} - \langle\delta\psi_{gc}\rangle \qquad \text{and} \qquad \delta\tilde{\Psi}_{\rm gc} \equiv \int {\rm d}\bar{\zeta}\delta\tilde{\psi}_{\rm gc}$$

while $\{,\}_0$ denotes the lowest order gyrocenter Poisson brackets (see p. 25 and p. 29 of Brizard and Hahm 07)

$$\{F, G\}_0 = \frac{\Omega}{mB_0}\left(\frac{\partial F}{\partial\bar{\zeta}}\frac{\partial G}{\partial\bar{\mu}} - \frac{\partial F}{\partial\bar{\mu}}\frac{\partial G}{\partial\bar{\zeta}}\right)$$

- For this system, the Liouville theorem holds as follows

$$\bar{\nabla}\cdot\left[B_\parallel^*\dot{\bar{\boldsymbol{X}}}\right] + \frac{\partial}{\partial\bar{p}_\parallel}\left(B_\parallel^*\dot{\bar{p}}_\parallel\right) = 0 \quad ,$$

while the nonlinear gyrokinetic equation can be written as

$$\frac{\partial\bar{F}}{\partial t} + \left\{\bar{F}, \bar{H}\right\} = \frac{\partial\bar{F}}{\partial t} + \dot{\bar{\boldsymbol{X}}}\cdot\frac{\partial\bar{F}}{\partial\bar{\boldsymbol{X}}} + \dot{\bar{p}}_\parallel\frac{\partial\bar{F}}{\partial\bar{p}_\parallel} = 0 \quad .$$

- The drift approximation for the particle motions.

E: Introduce $\epsilon \sim \epsilon_B \sim \epsilon_\delta$, with ϵ_B a smallness parameter to describe the weak equilibrium magnetic field nonuniformity. Using the above expressions for the equations of motion, show that one can obtain the following expressions

$$\begin{aligned}
\dot{\boldsymbol{X}}_\perp &= \frac{B_0}{B_\parallel^*}\frac{\boldsymbol{b}}{\Omega} \times \left\{ \bar{\boldsymbol{\nabla}} \left(\frac{\bar{H}}{m} \right) + \boldsymbol{\kappa} v_\parallel \left(v_\parallel + \frac{e}{mc} \langle \delta A_{\parallel gc} \rangle \right) \right\} + o(\epsilon^3) v_t \\
\boldsymbol{b} \cdot \dot{\boldsymbol{X}} &= v_\parallel + o(\epsilon^3) v_t = \frac{\bar{p}_\parallel}{m} - \frac{e}{mc} \langle \delta A_{\parallel gc} \rangle + o(\epsilon^3) v_t \\
\dot{\bar{p}}_\parallel &= -m \bar{\mu} \nabla_\parallel B_0 - \frac{B_0}{\Omega B_\parallel^*} \bar{p}_\parallel \bar{\mu} (\boldsymbol{b} \times \boldsymbol{\kappa}) \cdot \boldsymbol{\nabla} B_0 \\
& \quad - \left(\nabla_\parallel + \frac{B_0}{\Omega B_\parallel^*} \frac{\bar{p}_\parallel}{m} (\boldsymbol{b} \times \boldsymbol{\kappa}) \cdot \boldsymbol{\nabla} \right) (\bar{H}_1 + \bar{H}_2 + \dots) \ .
\end{aligned}$$

- The expressions obtained so far are consistent with the intuitive expressions one may obtain from the general expression for the drift velocity of a particle in a magnetic field under the action of a generic force $\boldsymbol{F}$

$$\boldsymbol{v}_{DF} = \frac{1}{m\Omega}\boldsymbol{b} \times \boldsymbol{F}$$

E: Derive the above expression for $\boldsymbol{v}_{DF}$.

- The gyroaveraged forces are calculated as

$$\boldsymbol{F} = e\left\langle \delta \boldsymbol{E}_{gc} \right\rangle + \frac{e}{c}\left\langle \boldsymbol{v} \times \delta \boldsymbol{B}_{gc} \right\rangle$$

- Calculate groaveraged forces using the explicit definitions given on p.5

$$e\left\langle \delta \boldsymbol{E}_{gc} \right\rangle = -e\bar{\boldsymbol{\nabla}} J_0(\lambda)\delta\phi - \frac{e}{c}\frac{\partial}{\partial t} J_0(\lambda)\delta \boldsymbol{A}$$

having introduced Bessel functions of argument $\lambda = k_\perp (2\bar{\mu} B_0)^{1/2}/\Omega$.

$$
\begin{aligned}
\frac{e}{c}\langle \boldsymbol{v} \times \delta \boldsymbol{B}_{gc} \rangle \quad = \quad & \frac{e}{c} v_\parallel \boldsymbol{b} \times \left(\bar{\boldsymbol{\nabla}} \times J_0(\lambda) \delta \boldsymbol{A} \right) \\
& - \bar{\boldsymbol{\nabla}} \left(\frac{2J_1(\lambda)}{\lambda} m \bar{\mu} \delta B_\parallel \right) + m \bar{\mu} \delta \boldsymbol{B} \cdot \bar{\boldsymbol{\nabla}} \left(\frac{2J_1(\lambda)}{\lambda} \boldsymbol{b} \right)
\end{aligned}
$$

E: Derive the above result providing the detailed steps of the calculation.

E: Compare the expressions one would obtain from the gyroaveraged forces with those obtained from the nonlinear gyrokinetic equations of motions. Is there a one on one correspondence? Give reasons for any discrepancy you may find and clarify whether such discrepancy is important or not.

Guiding center motion

- Following Brizard and Hahm 07, the guiding center motion is described in the non-uniform magnetic field at equilibrium $\Rightarrow$ No equilibrium $\boldsymbol{E} \times \boldsymbol{B}$ (strong) flow is considered in this lecture.

- From equations of motion (see p.6), one readily derives

$$\dot{\boldsymbol{X}} = \frac{B_0}{B_\parallel^*}\left(v_\parallel \boldsymbol{b} + \frac{v_\parallel}{\Omega}\boldsymbol{\nabla} \times (\boldsymbol{b} v_\parallel)\right) \quad ,$$

$$\dot{v}_\parallel = -\frac{\boldsymbol{B}^*}{B_\parallel^*} \cdot \bar{\mu} \boldsymbol{\nabla} B_0 = -\bar{\mu}\left(\boldsymbol{b} \cdot \boldsymbol{\nabla} B_0 + \frac{mc}{eB_\parallel^*} v_\parallel \boldsymbol{b} \times \boldsymbol{\kappa} \cdot \boldsymbol{\nabla} B_0\right) \quad .$$

- Note that, if $\boldsymbol{B}^* \to \boldsymbol{B}_0$, equations above (Littlejohn 83) reduce to those given by (Northrop 63). From now on, use δ-notation in front of physical quantities involving fluctuating fields.

- For carrying out actual calculations, let us introduce the straight field aligned flux coordinates (ψ, θ, ζ)

  - ψ is the poloidal flux, that can be replaced by a radial like flux coordinate r

  - $\zeta = \phi - \nu(\psi, \theta)$, with ϕ the geometric toroidal angle and the periodic function $\nu(\psi, \theta)$ chosen such that the safety factor $\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\zeta / \boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\theta = q(\psi)$; thus, for the magnetic field $\boldsymbol{B}_0 = F(\psi)\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi$, and Jacobian $\mathcal{J} = (\boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\zeta)^{-1}$

$$\nu(\psi, \theta) = \int_0^\theta \frac{F(\psi)\mathcal{J}(\psi, \theta')}{R^2(\psi, \theta')} d\theta' - q(\psi)\theta \quad .$$

  - the choice of θ then specifies the Jacobian $\mathcal{J}$

E: Give examples of well-known straight field aligned flux coordinate systems specifying for each of them the choice of the Jacobian. Hint: see (Spring 09 Lecture Notes).

- For a recent and detailed description of the mapping between generic toroidal coordinates and (ψ, θ, ζ) see (Z.X. Lu et al 12).
- Often, it is also useful introducing the Clebsch coordinates (ψ, θ, ξ), with $\xi = \zeta - q(\psi)\theta$, unchanged Jacobian $\mathcal{J}$ and equilibrium magnetic field representation
$$\boldsymbol{B}_0 = \boldsymbol{\nabla}\xi \times \boldsymbol{\nabla}\psi$$

E: Explain why the Jacobian is unchanged when changing coordinates from (ψ, θ, ζ) to (ψ, θ, ξ).

E: Give a geometric interpretation of the equilibrium magnetic field lines and their connection with the surfaces $\psi = \psi_0$ and $\xi = \xi_0$.

- Using these coordinates, we can explicitly compute the guiding center motion as: E: fill in the details of the derivation!

$$
\begin{aligned}
\dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla} &= \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\left[\frac{\partial}{\partial\theta}+\frac{mc}{e}\frac{\partial}{\partial\psi}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\psi\right)\frac{\partial}{\partial\theta}+\frac{mc}{e}\frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\psi}\right.\\
&\left.+\frac{mc}{e}\frac{\partial}{\partial\psi}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\xi\cdot\boldsymbol{\nabla}\psi\right)\frac{\partial}{\partial\xi}+\frac{mc}{e}\frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\xi\cdot\boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\xi}\right]\\
&= \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\left[\frac{\partial}{\partial\theta}+\frac{mc}{e}\frac{\partial}{\partial\psi}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\psi\right)\frac{\partial}{\partial\theta}+\frac{mc}{e}\frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\psi}\right.\\
&+\frac{mc}{e}\frac{\partial}{\partial\psi}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\psi\right)\frac{\partial}{\partial\xi}+\frac{mc}{e}\frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\theta\right)\frac{\partial}{\partial\xi}\\
&\left.-\frac{mc}{e}\frac{\partial}{\partial\psi}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\theta\cdot\boldsymbol{\nabla}\psi\right)q\frac{\partial}{\partial\xi}-\frac{mc}{e}\frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel\boldsymbol{b}\times\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta\right)q_\psi\theta\frac{\partial}{\partial\xi}\right]\quad .
\end{aligned}
$$

- This expression is further simplified noting

$$\boldsymbol{b} \times \boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\psi = -\frac{F(\psi)}{\mathcal{J} B_0} \quad ,$$

from which we obtain easily,

$$\begin{aligned} \dot{\psi} &= -\frac{v_\parallel}{\mathcal{J} B_\parallel^*} \frac{\partial G}{\partial \theta} \quad , \\ \dot{\theta} &= \frac{v_\parallel}{\mathcal{J} B_\parallel^*} \frac{\partial G}{\partial \psi} \quad . \end{aligned}$$

$$G = \psi - F(\psi) v_\parallel / \Omega \simeq -(c/e) P_\phi$$

which is connected with the lowest order expression (in the drift parameter expansion) for the toroidal canonical angular momentum.

- In order to obtain the last equation for $\dot{\zeta}$ and/or $\dot{\xi}$, consider

$$\partial_\psi|_{\theta,\xi} = \partial_\psi|_{\theta,\zeta} + q_\psi \theta \partial_\zeta|_{\psi,\theta} \quad , \quad \partial_\theta|_{\psi,\xi} = \partial_\theta|_{\psi,\zeta} + q\partial_\zeta|_{\psi,\theta} \quad , \quad \partial_\xi|_{\psi,\theta} = \partial_\zeta|_{\psi,\theta}$$

and that

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\psi = B_0 - \frac{q(\psi)F(\psi)}{\mathcal{J}B_0} \quad ,$$

$$\boldsymbol{b} \times \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\theta = \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F}{\mathcal{J}B_0}\frac{\partial \nu(\psi,\theta)}{\partial \psi} \quad .$$

In this way, one obtains

$$\dot{\zeta} = \frac{q v_\parallel}{\mathcal{J}B_\parallel^*} + \frac{v_\parallel}{\mathcal{J}B_\parallel^*}\frac{mc}{e}\left[\frac{\partial}{\partial\psi}\left(\mathcal{J}B_0 v_\parallel - q\frac{F v_\parallel}{B_0}\right) + \frac{\partial}{\partial\theta}\left(\mathcal{J}v_\parallel \frac{\boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\theta}{B_0 R^2} - \frac{F v_\parallel}{B_0}\frac{\partial\nu(\psi,\theta)}{\partial\psi}\right)\right] \quad ,$$

$$\dot{\xi} = \dot{\zeta} - q\dot{\theta} - q_\psi \theta \dot{\psi} = \frac{v_\parallel}{\mathcal{J} B_\parallel^*} \frac{mc}{e} \left\{ \frac{\partial}{\partial \psi} \left(\mathcal{J} B_0 v_\parallel \right) + \frac{\partial}{\partial \theta} \left[\mathcal{J} v_\parallel \frac{\nabla \psi \cdot \nabla \theta}{B_0 R^2} - \frac{F v_\parallel}{B_0} \left(\frac{\partial \nu(\psi, \theta)}{\partial \psi} + q_\psi \theta \right) \right] \right\} .$$

- It is worthwhile noting that, since the dominant contribution of $\dot{\theta}$ comes from $\dot{\theta} \simeq v_\parallel / (\mathcal{J} B_\parallel^*)$, the expression of $\dot{\xi}$ contains only slow magnetic drift motions, as it happens for $\dot{\psi}$.
- Thus, ξ is the physically relevant variable for expressing the slow toroidal precession, as it was expected since $\boldsymbol{B}_0 = \nabla \xi \times \nabla \psi$.

□ The precession frequency is defined then as

$$\bar{\omega}_d = \frac{\omega_b}{2\pi} \oint \left(\dot{\xi} + \theta \dot{q} \right) \frac{d\theta}{\dot{\theta}} \quad ,$$

where ω_b is the bounce or transit frequency, respectively for trapped and circulating particles, i.e.

$$\omega_b = \frac{2\pi}{\oint d\theta/\dot{\theta}} \quad .$$

E: Justify these definitions of precession and bounce/transit frequency. (Refer to Lecture 2 for more details.)

□ Here, the integration is defined along the complete particle orbit, defined by Eqs. $\dot{\psi}$ and $\dot{\theta}$ on p.15. Obviously, ω_d and ω_b are functions of $H_0, \bar{\mu}, P_\phi$ or, equivalently, on $\bar{\mu}, J, P_\phi$, with J the second invariant defined below.

Integral invariants and action angle coordinates

- The quantities $(\bar{\mu}, \bar{\zeta})$ and (P_ϕ, ϕ) are action-angle coordinates for the unperturbed particle motion. For fully characterizing the equilibrium particle motion in action-angle coordinates, we are still missing (J, η), with

$$J = m \oint v_\parallel d\ell = m \oint \left| \dot{\boldsymbol{X}} \right|^2 dt = m \oint v_\parallel^2 dt$$

being the *second invariant* and η the respective conjugate canonical angle, defined as

$$\eta = \omega_b \int_0^\theta \frac{d\theta'}{\dot{\theta}'} \quad .$$

E: Justify the definition of η as the canonical angle conjugate to J and discuss the implications of interpreting $v_\parallel$ in the definition of J as parallel velocity along $\boldsymbol{b}$ or along the guiding center trajectory arc-length.

- With the definitions above, J can be computed once for all given the reference magnetic equilibrium, i.e.

$$J = J(H_0, \bar{\mu}, P_\phi) \quad \leftrightarrow \quad H_0 = H_0(\bar{\mu}, J, P_\phi) \quad .$$

- Meanwhile, from the definition of J and ω_b

$$\frac{\partial J}{\partial H_0} = \frac{2\pi}{\omega_b} \quad \leftrightarrow \quad \omega_b = 2\pi \frac{\partial H_0}{\partial J} \quad .$$

E: Demonstrate this last statement. Hint: refer to the solution of the take-home exam of the Spring 11 Lectures.

Note: The distinction between μ and $\bar{\mu}$ (guiding center and gyrocenter magnetic moment) is unneeded in the absence of fluctuations. We maintain it here for consistency of notation.

Integral invariants in the presence of fluctuations

- For our present scope, we need to discuss particle motion up to first order in the field amplitude. So, given that $\bar{\mu}$ is constant, we need to describe $\mathcal{O}(\epsilon_\delta)$ corrections of the guiding center equations of motion and determine $\dot{H}_0$ and $\dot{P}_\phi$ at $\mathcal{O}(\epsilon_\delta)$.
- In this Lecture, we focus on general properties of integral invariants of motion in the presence of fluctuations. A more detailed analysis is the subject of Lecture 2.
- For simplicity, consider only fluctuations belonging to the shear Alfvén branch: neglect $\delta B_\parallel$ and $\delta \boldsymbol{A}_\perp$.
- Next, we rewrite the equations of motions on p.8 to obtain explicit expressions of $\delta \dot{\boldsymbol{X}}$ and $\delta \dot{v}_\parallel$.

- The leading order equation for $\delta\dot{\boldsymbol{X}}$ is (dropping the subscript gc in $\langle\ldots\rangle$ for brevity)

$$
\begin{aligned}
\delta\dot{\boldsymbol{X}} &= \frac{c}{B_\parallel^*}\boldsymbol{b}\times\boldsymbol{\nabla}\langle\delta\phi\rangle + v_\parallel\frac{\langle\delta\boldsymbol{B}_\perp\rangle}{B_\parallel^*} \\
&= \frac{c}{B_\parallel^*}\boldsymbol{b}\times\left(\boldsymbol{\nabla}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\boldsymbol{\nabla}\langle\delta A_\parallel\rangle\right) + v_\parallel\frac{\langle\delta A_\parallel\rangle}{B_\parallel^*}\boldsymbol{\nabla}\times\boldsymbol{b} \\
&= \frac{c}{B_\parallel^*}\boldsymbol{b}\times\left(\boldsymbol{\nabla}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\boldsymbol{\nabla}\langle\delta A_\parallel\rangle\right) + \mathcal{O}\left(\frac{r}{nR_0}\right) \quad .
\end{aligned}
$$

- Note that, here, terms of $\mathcal{O}(r/nR_0)$ are typically negligible within the present gyrokinetic ordering. However, they should be properly taken into account to demonstrate conservation laws related with P_ϕ (see below).

- Keeping only leading order terms, explicit expressions for $\delta\dot{\psi}$, $\delta\dot{\theta}$ and $\delta\dot{\xi}$ can be obtained from the expression of $\delta\dot{\boldsymbol{X}}$

$$
\begin{aligned}
\delta\dot{\psi} &= \frac{cB_0}{B_\parallel^*}\left(\frac{\partial}{\partial\xi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\xi}\langle\delta A_\parallel\rangle\right) - \frac{c}{B_\parallel^*}\frac{F}{\mathcal{J}B_0}\left(\frac{\partial}{\partial\theta}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\theta}\langle\delta A_\parallel\rangle\right) \quad , \\
\delta\dot{\theta} &= \frac{c}{B_\parallel^*}\frac{F}{\mathcal{J}B_0}\left(\frac{\partial}{\partial\psi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\psi}\langle\delta A_\parallel\rangle\right) \\
&\quad + \frac{c}{B_\parallel^*}\left[\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta}{B_0R^2} - \frac{F}{\mathcal{J}B_0}\left(\frac{\partial\nu}{\partial\psi} + q_\psi\theta\right)\right]\left(\frac{\partial}{\partial\xi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\xi}\langle\delta A_\parallel\rangle\right) \quad , \\
\delta\dot{\xi} &= -\frac{cB_0}{B_\parallel^*}\left(\frac{\partial}{\partial\psi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\psi}\langle\delta A_\parallel\rangle\right) \\
&\quad - \frac{c}{B_\parallel^*}\left[\frac{\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\theta}{B_0R^2} - \frac{F}{\mathcal{J}B_0}\left(\frac{\partial\nu}{\partial\psi} + q_\psi\theta\right)\right]\left(\frac{\partial}{\partial\theta}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\theta}\langle\delta A_\parallel\rangle\right) \quad .
\end{aligned}
$$

- Similarly, it is possible to derive the expression for $\delta\dot{v}_\parallel$

$$
\begin{aligned}
m\delta\dot{v}_\parallel &= -e\left(\frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta}\langle\delta\phi\rangle + \frac{1}{c}\frac{\partial}{\partial t}\langle\delta A_\parallel\rangle\right) \\
&\quad -\frac{c}{B_\parallel^*}mv_\parallel\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\langle\delta\phi\rangle - \frac{m\bar{\mu}}{B_\parallel^*}\boldsymbol{b}\times\boldsymbol{\nabla}B_0\cdot\boldsymbol{\nabla}\langle\delta A_\parallel\rangle \\
&\quad -\frac{\langle\delta A_\parallel\rangle}{B_\parallel^*}\boldsymbol{\nabla}\times\boldsymbol{b}\cdot m\bar{\mu}\boldsymbol{\nabla}B_0 \\
&= -e\left(\frac{1}{\mathcal{J}B}\frac{\partial}{\partial\theta}\langle\delta\phi\rangle + \frac{1}{c}\frac{\partial}{\partial t}\langle\delta A_\parallel\rangle\right) \\
&\quad -\frac{c}{B_\parallel^*}mv_\parallel\boldsymbol{b}\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\langle\delta\phi\rangle - \frac{m\bar{\mu}}{B_\parallel^*}\boldsymbol{b}\times\boldsymbol{\nabla}B_0\cdot\boldsymbol{\nabla}\langle\delta A_\parallel\rangle + \mathcal{O}\left(\frac{r}{nR_0}\right) \quad .
\end{aligned}
$$

- As before, the terms of $\mathcal{O}(r/nR_0)$ are typically negligible, but they should be accounted for in the precise derivation of

$$\dot{H}_0 = (\bar{p}_\parallel/m)\delta\dot{\bar{p}}_\parallel + m\bar{\mu}\delta\dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla} B_0$$

since they give rise to obvious cancellations.

E: Show that the expression for $\dot{H}_0$ can be written as

$$\dot{H}_0 = e\dot{\boldsymbol{X}} \cdot \langle \delta \boldsymbol{E} \rangle + e\left(\partial_t + \dot{\boldsymbol{X}} \cdot \boldsymbol{\nabla}\right) \frac{v_\parallel}{c} \langle \delta A_\parallel \rangle \quad ,$$

with $\dot{\boldsymbol{X}}$ given the corresponding expression on p.11 and

$$\langle \delta \boldsymbol{E} \rangle = -\boldsymbol{\nabla} \langle \delta\phi \rangle - \frac{\boldsymbol{b}}{c}\frac{\partial}{\partial t} \langle \delta A_\parallel \rangle \quad .$$

- With all these elements, we can compute $\dot{P}_\phi$ as

$$\begin{aligned}
\dot{P}_\phi &= \left(-\frac{e}{c} + m v_\parallel \frac{F_\psi}{B_0}\right)\delta\dot{\psi} + m\delta\dot{v}_\parallel \frac{F}{B_0} - m v_\parallel \frac{F}{B_0^2}\delta\dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}B_0 \\
&= -e\left(1 + \frac{mcv_\parallel}{eB_\parallel^*}\frac{4\pi F P_\psi}{B_0^2}\frac{F^2}{B_0^2R^2}\right)\left(\frac{\partial}{\partial\xi}\left\langle\delta\phi\right\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\xi}\left\langle\delta A_\parallel\right\rangle\right) \\
&\quad -\frac{e}{c}\frac{F}{B_0}\left(\frac{\partial}{\partial t} + \dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\right)\left\langle\delta A_\parallel\right\rangle - \frac{e}{c}\left\langle\delta A_\parallel\right\rangle\dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\left(\frac{F}{B_0}\right) \quad .
\end{aligned}$$

- Here, we have used the equilibrium condition (Grad-Shafranov equation) plus the relationship

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b} + F_\psi + \frac{4\pi F P_\psi}{B_0^2} = 0 \ ,$$

which readily follows from the expression of parallel equilibrium current density and the equilibrium condition.

- Note, also, that the last term in the expression for $\dot{P}_\phi$ is due to the terms of $\mathcal{O}(r/nR_0)$, which are typically dropped in the equations of motions of pp.22-24. These terms are important for exact conservation properties, e.g. that the quantity

$$\Pi_\phi \equiv P_\phi + (e/c)(F/B_0) \left\langle \delta A_\parallel \right\rangle$$

should be conserved for axisymmetric perturbations.

- Note that for $\beta \sim \epsilon_B \sim \rho_L/R \ll 1$ the $\propto P_\psi$ can be dropped consistently with the present assumption to derive evolution equations up to order ϵ. Thus, we can write

$$\dot{\Pi}_\phi \simeq -e\left(\frac{\partial}{\partial\xi}\langle\delta\phi\rangle - \frac{v_\parallel}{c}\frac{\partial}{\partial\xi}\langle\delta A_\parallel\rangle\right) \quad .$$

- The expression for $\dot{J}$ is also readily derived as

$$\dot{J} = \frac{\partial J}{\partial P_\phi}\dot{P}_\phi + \frac{\partial J}{\partial H_0}\dot{H}_0 = -\frac{\partial H_0/\partial P_\phi}{\partial H_0/\partial J}\dot{P}_\phi + \frac{1}{\partial H_0/\partial J}\dot{H}_0 = \frac{2\pi}{\omega_b}\left(-\bar{\omega}_d\dot{P}_\phi + \dot{H}_0\right)$$

Conservation of the extended phase space Hamiltonian

- The extended phase space Hamiltonian $K = H_0 + e\langle\delta\psi\rangle - \bar{H}$ is conserved (see e.g. Lecture 1 of Spring 10 intensive course).

- Therefore, we have

$$\dot{\bar{H}} = \dot{H}_0 + \left(\frac{\partial}{\partial t} + \dot{\boldsymbol{X}}\cdot\boldsymbol{\nabla}\right) e\left\langle\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel\right\rangle = e\frac{\partial}{\partial t}\left(\langle\delta\phi\rangle - \frac{v_\parallel}{c}\left\langle\delta A_\parallel\right\rangle\right) \quad .$$

- From a comparison of this expression and that for $\dot{\Pi}_\phi$, we readily derive

$$\frac{\partial}{\partial t}\dot{\Pi}_\phi + \frac{\partial}{\partial \xi}\dot{\bar{H}} = 0 \quad ,$$

which readily yields $\Pi_\phi - (n/\omega)\bar{H}$ =const for a single n fixed ω mode.

References and reading material

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