IFTS Intensive Course on Advanced Plasma Physics-Spring 2011, Non-linear charged particle dynamics (part I)

Mid-term take home exam

18–30 May 2011, IFTS – ZJU, Hangzhou

1. Consider the harmonic (linear) oscillator, whose Hamiltonian, for constant G, F, α, is given by

$$H = G\frac{p^2}{2} + F\frac{q^2}{2} = \alpha$$

   Derive action-angle coordinates for this system and discuss the properties of the motion.

2. Write the Hamiltonian for the central force problem and show that it is integrable. Show that there exists an isolating integral of the motion in addition to the total energy and that the motion can be reduced to quadratures.

3. Consider the motion of a particle in an axisymmetric toroidal equilibrium with $\boldsymbol{B} = F(\psi)\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi$, with ϕ the toroidal angle and $2\pi\psi$ the poloidal magnetic flux contained between the magnetic axis and the flux surface ψ. Three obvious invariants of motion are the energy E, the magnetic moment μ and the canonical toroidal angular momentum P_ϕ.

   (i) Write the explicit form of P_ϕ and show that the particle motion in a cross section $\phi =$ const. must be a closed trajectory.

   (ii) Write the formal expression of the frequency characterizing the motion of the particles on the close trajectory. For trapped particles this is the bounce frequency, while for circulating particles this is the transit frequency. Show that the frequency of this periodic motion is given by (b for bounce and t for transit)

$$\omega_{b,t} = 2\pi \left(\frac{\partial J}{\partial E}\right)^{-1} \quad ; \quad J = m \oint v_\parallel d\ell$$

with m the particle mass, ℓ the arc length along the magnetic field line and J the *second invariant*.

(iii) In a large aspect ration tokamak, $B \simeq B_0(1 - r/R_0 \cos\theta)$, with θ the poloidal angle, r the minor radial coordinate and R_0 the major radius at the magnetic axis. Using the variables $\mathcal{E} = v^2/2$ and

$$\kappa^2 = \frac{2(r/R_0)\lambda}{1 - (1 - r/R_0)\lambda}$$

with $\lambda = \mu B_0/\mathcal{E}$, calculate the expression of bounce and transit frequency in terms of elliptic integrals.

Reading assignment: The construction of an adiabatic invariant, if it actually exists, effectively reduces the system from N to $N-1$ degrees of freedom. Read the work (R. J. Hastie, J. B. Taylor and F. A. Hass (1967), *Ann. Phys.* **41**, 302), where the authors show that, when studying the physics of magnetized plasmas, this fact is connected with the definition of a hierarchy of invariants. Write a brief summary of this paper and discuss the physics underlying these invariants.

4. Use the general form of the Deprit perturbation series given in Lecture 4 and give the nonlinear libration motion up to second order for the nonlinear pendulum Hamiltonian

$$H = G\frac{p^2}{2} + F(1 - \cos\phi)$$

using a series expansion of $\cos\phi$.

5. Consider the perturbed twist map discussed in Lecture 5. Show that the linearized motion around a period 1 fixed point is given by

$$\begin{pmatrix} \Delta J_{n+1} \\ \Delta\theta_{n+1} \end{pmatrix} = \boldsymbol{A} \cdot \begin{pmatrix} \Delta J_n \\ \Delta\theta_n \end{pmatrix}$$

Derive the explicit form of $\boldsymbol{A}$ and the area preserving condition of the map by imposing $\det \boldsymbol{A} = 1$.

6. Following the general method illustrated in Lecture 5 and introducing the "time-variable" n, show that the Hamiltonian for the standard mapping is

$$H = \frac{I^2}{2} + K \cos\theta \sum_{m=-\infty}^{\infty} \exp(i2\pi mn)$$

Use the criterion of slow variation $|d\theta/dn| \ll 2\pi$ to show that the Hamiltonian reduces to

$$H = \frac{I^2}{2} + K \cos\theta + 2K \cos\theta \cos 2\pi n$$

Explain why the last term can be considered a perturbation that averages to zero. Derive the frequency of the small librations about the elliptic fixed point and the maximum excursion of ΔI.