Lecture 5

Mappings and linear stability

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General notion of mapping

- Visualization of multiply periodic motions is important for increased understanding.
- Topological considerations that can be made, naturally yield to a set of difference equations, that can be seen as *mapping* of the dynamical trajectory onto a subset of the system phase space.
- Not only visualization (and understanding) are often simpler via mapping equations. Physics properties, such as numerical calculation of stochastic behaviors, are often simpler when considered for mappings.
- Conversely, regular trajectories are very conveniently described in terms of differential equations.
- This Lecture is concerned with the conversion of differential equations into mappings and vice-versa.

Problems that yield to mapping

- Mapping arise in various ways. Notably as:

  - as physical formulation of a dynamical problem, as in the case of Fermi acceleration or the twist mapping/standard mapping
  - considering successive intersections of a dynamical trajectory with a surface of section (see Lecture 2)
  - as approximation of the motion, valid for a limited time scale or range of the phase space; e.g., the periodic perturbed pendulum (see p. 26)

- This Lecture focuses on systems with two degrees of freedom, but contains useful extensions to more general cases.

Integrable systems: motion on a torus

- Consider an integrable system constituted by an oscillator with two degrees of freedom in action angle-coordinates

$$H(J_1, J_2) = E$$

- The energy E is a constant of the motion, as are the two actions.
  - the motion on the constant energy space is further restricted onto a constant action (either one) surface
  - the motion on the two dimensional surface is parameterized by the frequencies associated with each degree of freedom

$$\theta_1 = \omega_1 t + \theta_{10} \quad ; \quad \theta_2 = \omega_2 t + \theta_{20}$$

- A useful way to visualize the motion of a doubly periodic oscillator is a two dimensional torus

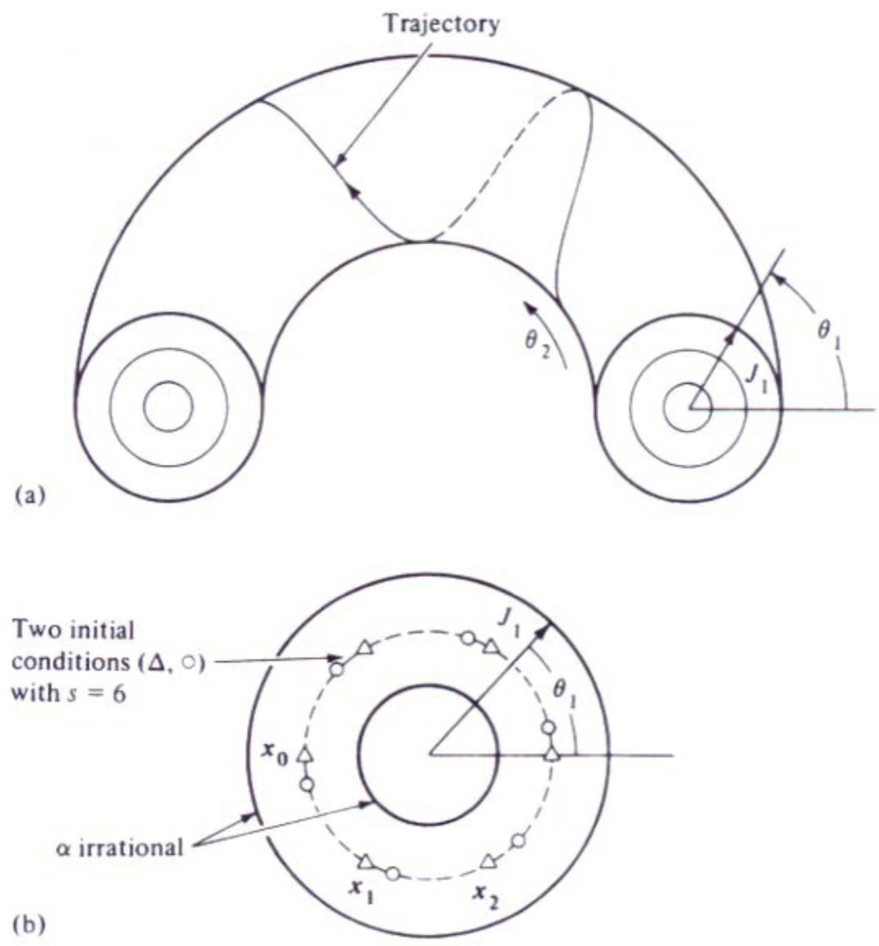


Figure from *Lieberman and Lichtenberg*, 2010.

(a) Motion on a two dimensional torus. For

$$\alpha = \omega_1/\omega_2 = r/s \quad ; \quad r, s \in \mathbb{Z}$$

the frequencies are commensurate and the motion degenerates into a periodic trajectory on the torus. Otherwise the trajectory maps onto the whole surface.

(b) Motion on a surface of section $\theta_2 =$ const and $s = 6$

- The concept of motion on a torus can be generalized for a integrable oscillator in N-degrees of freedom, with the topological construction of a N-dimensional torus.
- The consequence of this, is that any integrable motion in the respective action-angle coordinates

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{J},\boldsymbol{\theta}) \quad ; \quad \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{J},\boldsymbol{\theta})$$

can be decomposed as

$$\boldsymbol{p}(t) = \sum_{m} \boldsymbol{p}_{\boldsymbol{m}}(\boldsymbol{J}) \exp[i(\boldsymbol{m}\cdot\boldsymbol{\omega}t+\boldsymbol{m}\cdot\boldsymbol{\beta})] \quad ; \quad \boldsymbol{q}(t) = \sum_{m} \boldsymbol{q}_{\boldsymbol{m}}(\boldsymbol{J}) \exp[i(\boldsymbol{m}\cdot\boldsymbol{\omega}t+\boldsymbol{m}\cdot\boldsymbol{\beta})]$$

- Periodic solutions are obtained for $\boldsymbol{m}\cdot\boldsymbol{\omega} = 0$, i.e., $\boldsymbol{\omega} = \boldsymbol{n}\omega_0$, and the closure of the trajectory occurs after $T = 2\pi/\omega_0$.

The twist mapping

- Lecture 2 discussed the representation of trajectories in a system with two degrees of freedom by the surface of section plots. Say this is the $J_1 - \theta_1$ plane at $\theta_2 =$ const. with no loss of generality; so drop the subscript 1 as well for simplicity of notation.

- After a time $\Delta t = 2\pi/\omega_2$ the "initial" J_n, θ_n coordinates will advance to

$$J_{n+1} = J_n \;\; ; \;\; \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) \;\; ; \;\; \alpha = \omega_1/\omega_2 \equiv \text{rotation number}$$

- The mapping is called the *twist mapping* and the use of $\alpha(J_{n+1})$ is for convenience.

- The twist mapping , as two-dimensional mapping, must be area preserving (see Lecture 1 p.28)

$$\frac{\partial(J_{n+1}, \theta_{n+1})}{\partial(J_n, \theta_n)} \equiv [\theta_{n+1}, J_{n+1}] = 1$$

E: Show that a twist mapping does not need to be written in action-angle form. E.g., that – given ψ as fixed parameter – it can be written as

$$x_{n+1} = x_n \cos\psi - y_n \sin\psi \quad ; \quad y_{n+1} = x_n \sin\psi + y_n \cos\psi$$

- These results are readily generalized to a system with N degrees of freedom. Choosing $\theta_N =$ const as surface of section, the twist mapping for the remaining $N-1$ pairs of action angle coordinates is given by

$$\boldsymbol{J}_{n+1} = \boldsymbol{J}_n \quad ; \quad \boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}_{n+1})$$

Near-integrable systems

- Area preserving mapping are of particular interest and are connected with canonical mappings.

E: Demonstrate that canonical mappings, i.e. mappings that are generated by Hamilton's equations, must be area preserving.

- Consider the near-integrable Hamiltonian obtained as perturbation of a two-dimensional integrable system

$$H(\boldsymbol{J},\boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J},\boldsymbol{\theta})$$

- For this system, one expects that the twist mapping is modified into the *perturbed twist mapping*

$$J_{n+1} = J_n + \epsilon f(J_{n+1},\theta_n) \quad ; \quad \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1}) + \epsilon g(J_{n+1},\theta_n)$$

E: Show that the functions f and g are periodic in θ. Show that, since they are functions of J_{n+1} rather than J_n, the area-preserving form of the mapping is particularly simple.

E: Is there any loss of generality having chosen f and g as functions of J_{n+1} rather than J_n? Why?

- Note that the perturbed twist mapping can be obtained from the generating function

$$F_2 = J_{n+1}\theta_n + 2\pi\mathcal{A}(J_{n+1}) + \epsilon\mathcal{G}(J_{n+1},\theta_n) \quad ; \quad J_n = \frac{\partial F_2}{\partial \theta_n} \quad ; \quad \theta_{n+1} = \frac{\partial F_2}{\partial J_{n+1}}$$

- Thanks to these simple rules, the perturbed twist mapping is obtained as

$$\alpha = \frac{d\mathcal{A}}{dJ_{n+1}} \quad ; \quad f = -\frac{\partial \mathcal{G}}{\partial \theta_n} \quad ; \quad g = \frac{\partial \mathcal{G}}{\partial J_{n+1}}$$

- From these relations, one easily sees that the condition for area preservation is

$$\frac{\partial g}{\partial \theta_n} + \frac{\partial f}{\partial J_{n+1}} = 0$$

E: Show that this condition is actually giving area preservation from the definition of the perturbed twist mapping.

- For many interesting mappings, f is independent of J and $g \equiv 0$. The the perturbed twist mapping takes the form of a *radial twist mapping*

$$J_{n+1} = J_n + \epsilon f(\theta_n) \;\; ; \;\; \theta_{n+1} = \theta_n + 2\pi\alpha(J_{n+1})$$

- Particular examples of the radial twist mapping are the simplified Fermi mapping and the separatrix mapping (see p.31,32).

- The linearization of the radial twist mapping near a period 1 fixed point $J_{n+1} = J_n = J_0$ (α is an integer), with the definition of $J_n = J_0 + \Delta J_n$, the new action $I_n = 2\pi\alpha' \Delta J_n$, $K = 2\pi\alpha' \epsilon f_{\max}$ and $f^\star = f/f_{\max}$, yields the *generalized standard mapping*

$$I_{n+1} = I_n + K f^\star(\theta_n) \;\; ; \;\; \theta_{n+1} = \theta_n + I_{n+1} \mod 2\pi$$

- The generalized standard mapping is locally equivalent (in J) to any radial twist mapping. For $f^\star = \sin\theta_n$ it becomes the standard mapping used by (*Chirikov*, 1979) and (*Greene*, 1979) to study the transition from regular to chaotic motion

$$I_{n+1} = I_n + K \sin\theta_n \;\; ; \;\; \theta_{n+1} = \theta_n + I_{n+1}$$

- The generalization of the perturbed twist mapping to more than two degree of freedom is obtained with the generating function

$$F_2 = \boldsymbol{J}_{n+1} \cdot \boldsymbol{\theta}_n + 2\pi \mathcal{A}(\boldsymbol{J}_{n+1}) + \epsilon \mathcal{G}(\boldsymbol{J}_{n+1}, \boldsymbol{\theta}_n)$$

$$\boldsymbol{J}_{n+1} = \boldsymbol{J}_n + \epsilon \boldsymbol{f}(\boldsymbol{J}_{n+1}, \boldsymbol{\theta}_n) \;\; ; \;\; \boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n + 2\pi \boldsymbol{\alpha}(\boldsymbol{J}_{n+1}) + \epsilon \boldsymbol{g}(\boldsymbol{J}_{n+1}, \boldsymbol{\theta}_n)$$

- The $2N - 2$ dimensional radial twist mapping is obtained when $\mathcal{G}$ is not function of $\boldsymbol{J}_{n+1}$. In general, introducing $\boldsymbol{x} = (\boldsymbol{J}, \boldsymbol{\theta})$, the mapping representation can be written as

$$\boldsymbol{x}_{n+1} = T \boldsymbol{x}_n$$

Hamiltonian forms and mappings

- In two dimensions, the functions f and g in the perturbed twist mapping are determined from integration of the Hamilton's equations

$$\frac{dJ_1}{dt} = -\epsilon \frac{\partial H_1}{\partial \theta_1}$$

- The jump in action over one period in the θ_2 motion is $\Delta J_1 = \epsilon f(J_{n+1}, \theta_n)$, using the unperturbed motion for the integration

$$\Delta J_1 = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \theta_1}(J_{n+1}, J_2, \theta_n + \omega_1 t, \theta_{20} + \omega_2 t) dt$$

- The jump is phase can be obtained from the area preserving condition

$$g(J,\theta) = -\int^{\theta} \frac{\partial f}{\partial J} d\theta'$$

- For computation of ΔJ_1, the second degree of freedom can be expressed in generic (not necessarily action-angle) canonical coordinates.

$$\Delta J_1 = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \theta_1}(J_{n+1}, \theta_n + \omega_1 t, p_2(t), q_2(t))dt$$

- This method can be generalized to N degrees of freedom

$$\Delta \boldsymbol{J} = -\epsilon \int_0^{2\pi/\omega_2} \frac{\partial H_1}{\partial \boldsymbol{\theta}}(\boldsymbol{J}_{n+1}, J_N, \boldsymbol{\theta}_n + \boldsymbol{\omega} t, \theta_{N0} + \omega_N t)dt$$

- One can then derive the jumps of $N-1$ actions $\epsilon \boldsymbol{f} = \Delta \boldsymbol{J}$. One then finds $\boldsymbol{g} = \partial \mathcal{G}/\partial J_{n+1}$, knowing $\mathcal{G}$ from direct integration of $\boldsymbol{f} = -\partial \mathcal{G}/\partial \boldsymbol{\theta}_n$.

- One can construct the inverse procedure, e.g., construct the equations in Hamiltonian form, given the radial twist mapping.
- This can be done considering that the iteration number plays the role of time and using the function

$$\delta_1(n) = \sum_{m=-\infty}^{\infty} \delta(n-m) = 1 + 2\sum_{q=1}^{\infty} \cos 2\pi n q$$

- The radial twist mapping can then be written as

$$(dJ/dn) = \epsilon f(\theta)\delta_1(n) \;\; ; \;\; (d\theta/dn) = 2\pi\alpha(J)$$

which are the in Hamiltonian form with

$$H(J,\theta,n) = 2\pi \int^J \alpha(J')dJ' - \epsilon\delta_1(n)\int^\theta f(\theta')d\theta'$$

Linearized motion and stability

- In general, finding period k fixed points of a mapping requires that, for $\boldsymbol{x}_0 = (\boldsymbol{p}_0, \boldsymbol{q}_0)$, $T^k \boldsymbol{x}_0 = \boldsymbol{x}_0$

- The fixed points can be obtained directly from Hamilton's equations, requiring that periodic orbits exist in the surface of section $\boldsymbol{x} = (\boldsymbol{p}, \boldsymbol{q})$ (see Lecture 1)

$$\frac{\partial H}{\partial x_i} = 0$$

- In general, a small perturbation to an integrable system creates a series of alternating elliptic (stable) and hyperbolic (unstable) fixed points (see Lecture 2)

- In general, for large perturbation, one cannot use topological arguments and all fixed points may be unstable. Stability can be studied linearizing the motion around a fixed point, i.e.

$$\Delta \boldsymbol{x}_{n+k} = \boldsymbol{A} \cdot \boldsymbol{\Delta x} \quad ; \quad \boldsymbol{x} = \boldsymbol{x}_0 + \Delta \boldsymbol{x}$$

- The stability of the motion can be investigated looking at eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$, which is independent on $\Delta \boldsymbol{x}$

$$\boldsymbol{A} \cdot \boldsymbol{x} = \lambda \boldsymbol{x}$$

- Assume that the rank of the $M \times M$ matrix $\boldsymbol{A}$ is M, and – for brevity – that all eigenvalues are distinct (if not, refer to your linear algebra textbooks).

- Introduce the N dimensional null and identity matrices $\mathbf{0}$ and $\mathbf{1}$, and the $2N$ dimensional antisymmetric matrix (see Lecture 2)

$$\boldsymbol{\Gamma} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

- If the transformation induced by $\boldsymbol{A}$ is canonical,

$$\bar{\boldsymbol{x}} = \boldsymbol{A} \cdot \boldsymbol{x} \quad ; \quad A_{ij} = \frac{\partial \bar{x}_i}{\partial x_j}$$

then Poisson brackets are conserved and (see Lecture 2) the matrix $\boldsymbol{A}$ is *symplectic*

$$[\bar{x}_i, \bar{x}_j] = \sum_{k,l} A_{ik} \Gamma_{kl} A_{jl} = \Gamma_{ij} \quad ; \quad \boldsymbol{A} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{A}^T = \boldsymbol{\Gamma} \quad ; \quad \boldsymbol{A}^T \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{A} = \boldsymbol{\Gamma}$$

- When the eigenvalue problem $\boldsymbol{A} \cdot \boldsymbol{x} = \lambda \boldsymbol{x}$ is solved, with $M = 2N$ distinct eigenvalues, one can construct a matrix $\boldsymbol{X}$ with columns given by the M independent eigenvectors. If $\boldsymbol{\Lambda}$ is the diagonal matrix with $\Lambda_{ii} = \lambda_i$, the eigenvalue problem is rewritten as

$$\boldsymbol{A} \cdot \boldsymbol{X} = \boldsymbol{X} \cdot \Lambda \quad \Rightarrow \quad \Lambda = \boldsymbol{X}^{-1} \cdot \boldsymbol{A} \cdot \boldsymbol{X}$$

which shows that $\boldsymbol{X}$ diagonalizes $\boldsymbol{A}$. This also means that the transformation of coordinates $\boldsymbol{u}_k = \boldsymbol{X}^{-1} \cdot \boldsymbol{x}_k$ and its inverse $\boldsymbol{x}_k = \boldsymbol{X} \cdot \boldsymbol{u}_k$, allows the choice of $\boldsymbol{u}_k$ as the orthonormal set of M unit vectors $\boldsymbol{e}_k$. In fact,

$$\boldsymbol{\Lambda}_k \cdot \boldsymbol{u}_k = \lambda_k \boldsymbol{u}_k$$

- The properties of $\boldsymbol{A}$ imply that if λ is an eigenvalue, then $1/\lambda$ is also an eigenvalue. In fact, eigenvalues of a matrix and its transpose are the same. So, using $\boldsymbol{A} \cdot \boldsymbol{\Gamma} = \boldsymbol{\Gamma} \cdot (\boldsymbol{A}^T)^{-1}$

$$\boldsymbol{A}^T \cdot \boldsymbol{y} = \lambda \boldsymbol{y} \;\; ; \;\; (\boldsymbol{A}^T)^{-1} \cdot \boldsymbol{y} = (1/\lambda) \boldsymbol{y} \;\; ; \;\; \boldsymbol{A} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{y} = (1/\lambda) \boldsymbol{\Gamma} \cdot \boldsymbol{y}$$

  The corresponding eigenvector is $\boldsymbol{x} = \boldsymbol{\Gamma} \cdot \boldsymbol{y}$.

- In this way, one can write the $M = 2N$ λ_i eigenvalues as

$$\lambda_{i+N} = \lambda_i^{-1} \;\; ; \;\; i = 1, ..., N$$

E: Show that, for stable motion, one must have $\lim_{n \to \infty} \lambda^n$ bounded. Hint: Explain what successive applications of $\boldsymbol{A}$ would imply.

- Since $\boldsymbol{A}$ is real, complex root should show up in complex conjugate pairs. For $|\lambda| \neq 1$ and $\mathbb{I}\mathrm{m}\lambda \neq 0$ roots appear in 4-tuples

$$\lambda, \lambda^*, 1/\lambda, 1/\lambda^*$$

For $\mathbb{I}\mathrm{m}\lambda = 0$, the pairs $\lambda, 1/\lambda$ are located on the real axis. In either case, the motion is unstable.

- For $|\lambda| = 1$, the pairs $\lambda, \lambda^* = 1/\lambda$ are lying on the unit circle and the motion is stable. With all distinct roots, one requires all roots to lie on the unit circle. For multiple roots, the problem is more complicated and one generally has marginal stability.

- As consequence of the symmetries of eigenvalues, one has

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^{2N} + a_1 \lambda^{2N-1} + \ldots + a_{2N-1}\lambda + 1 = 0$$

$$a_1 = a_{2N-1} \quad ; \quad a_2 = a_{2N-2} \quad ; \quad \ldots$$

Separatrix motion and separatrix mapping

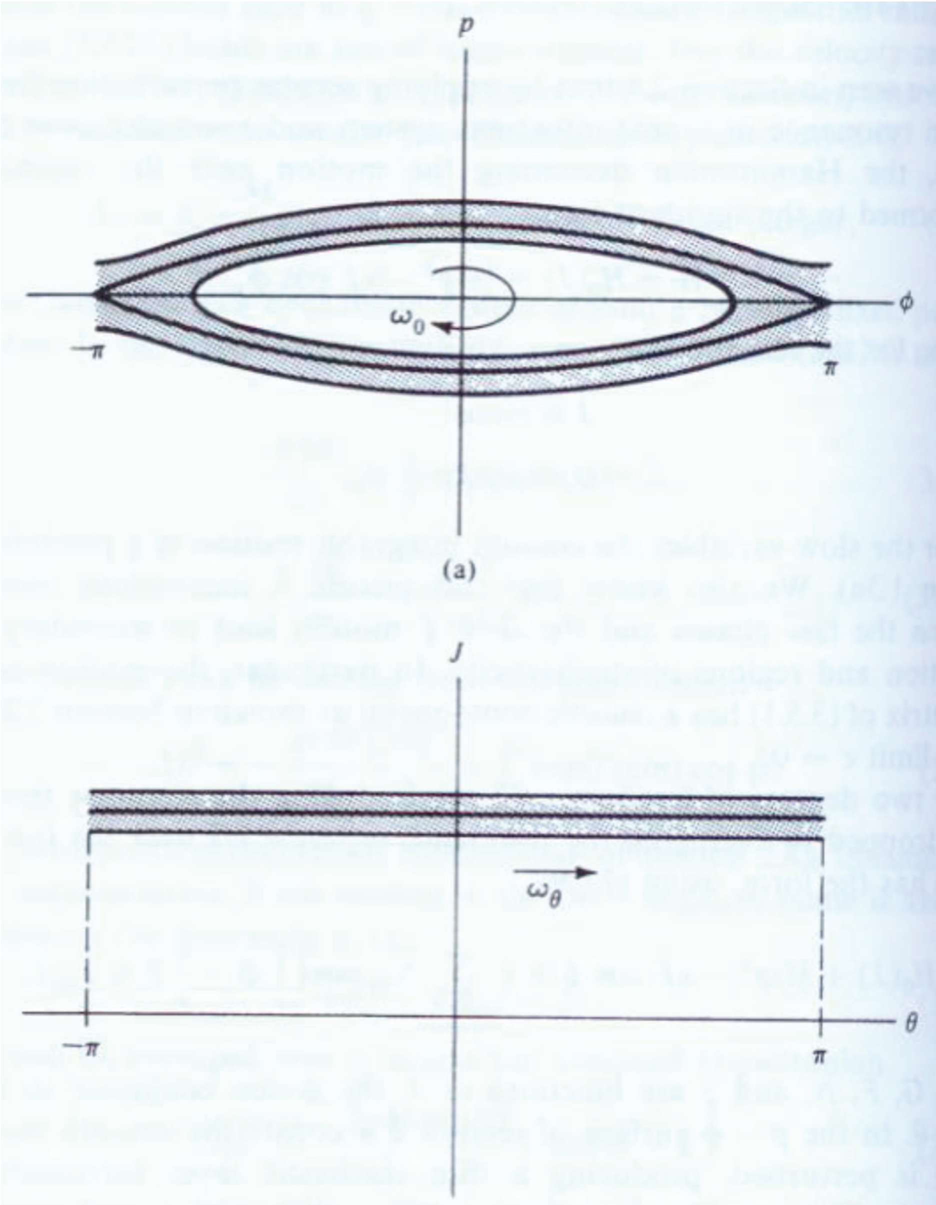
- Consider the single resonance behavior for a near integrable system investigated in Lecture 4

$$H = H_0(\boldsymbol{J}) + \frac{1}{2}Gp^2 - \epsilon F \cos\phi$$

- The analysis with secular perturbation theory yields smooth integrable motion of a pendulum in the slow variables p, ϕ and the unperturbed motion in the fast degrees of freedom

$$\boldsymbol{J} = \text{const} \quad ; \quad \boldsymbol{\theta} = \boldsymbol{\omega}_{\boldsymbol{\theta}}(\boldsymbol{J})t + \boldsymbol{\theta}_0$$

- Resonances between the fast phases and slow motion yield secondary island formation and region of stochasticity, in particular near the separatrix, even for $\epsilon \to 0$



Motion near the separatrix of a pendulum (*Lieberman and Lichtenberg*, 2010).

(a) The $p - \phi$ surface of section at $\theta = \text{const.}$

(b)The $J - \theta$ surface of section at $\phi = \text{const.}$

- For two degrees of freedom one can investigate the dynamics near the separatrix reintroducing the resonant term that were dropped when averaging the resonance $\omega_2/\omega_1 = r/s$, $r, s \in \mathbb{Z}$

$$H = H_0(J) + \frac{1}{2}Gp^2 - \epsilon F \cos\phi + \epsilon \sum_{\substack{p>1 \\ q\neq 0}} \Lambda_{pq} \cos\left(\frac{p}{r}\phi - \frac{q}{r}\theta + \chi_{pq}\right)$$

- In the $J - \theta$ surface of section at $\phi =$ const. the motion is a thin stochastic layer near $J =$ const. For convenience, one chooses $\phi = \pm\pi$ for the surface of section.
- This Hamiltonian has a perturbation in both θ and ϕ. So, a self-consistent treatment would require tracking the change $\omega_\theta(J)$

E: Discuss the connection of this self-consistent problem with that of fishbones and energetic particle modes analyzed in Lecture 4 and Lecture 5 of the Spring

2010 intensive course on *Resonant and non-resonant interactions in beam-plasma systems.*

- Dropping the self-consistency requirement, a simpler problem (retaining most important features) is obtained assuming $\omega_\theta = \Omega$, i.e. an assigned frequency as for the driven pendulum case.

$$\bar{H} = H - H_0(J) = \frac{1}{2} G p^2 - \epsilon F \cos\phi + \epsilon \sum_{\substack{p \geq 1 \\ q \neq 0}} \Lambda_{pq} \cos\left(\frac{p}{r}\phi - \frac{q}{r}\Omega t + \chi_{pq}\right)$$

E: Discuss the differences between the self-consistent and the driven-pendulum Hamiltonian.

- Unlike the self-consistent problem, where G, F, Λ and Ω are functions of J, here they are assumed constant. The part in $H_0(J)$ can be dropped, since it gives a trivial contribution.

- Note that $\dot{J} = -\partial H/\partial\theta = -\Omega^{-1}\partial\bar{H}/\partial t = -\Omega^{-1}\dot{\bar{H}}$. So the jump in action (see p.15) can be computed as $\Delta J = -\Delta\bar{H}/\Omega$.

- We compute the jump in action assuming that the dominant contribution comes from $p = q = 1$ and $\Lambda_{11} = \Lambda$. Furthermore, the frequency of small pendulum librations is $\omega_0 = (\epsilon FG)^{1/2}$, which can be considered small compared to Ω. The jump in action is obtained from p.15 as

$$\Delta J = -\epsilon\frac{\Lambda}{r}\int_{-\infty}^{\infty} dt \sin\left(\frac{\phi(\omega_0 t)}{r} - \frac{\Omega t + \theta_n}{r}\right)$$

- The unperturbed motion near the separatrix, can be described as (see Lecture 2, p.11)

$$\phi(s) = 4\tan^{-1}\left[\exp(s)\right] - \pi$$

- One can write the jump in action as function of the frequency ratio $Q_0 = \Omega/(r\omega_0)$ in terms of the *Melnikov-Arnold* integral $\mathcal{A}_m(Q_0)$

$$\Delta J = \epsilon \frac{\Lambda}{\omega_0 r} \mathcal{A}_{2r}(Q_0) \sin(\theta_n/r) \;\; ; \;\; \mathcal{A}_m(Q_0) = \int_{-\infty}^{\infty} ds \cos\left(\frac{m\phi(s)}{2} - Q_0 s \right)$$

E: Derive this result step by step

- The Melnikov Arnold integral is improper, since no limiting expression for it exists. To see this, consider

$$\mathcal{A}'_m(Q_0, s_1) = 2\int_0^{s_1} ds \cos\left(\frac{m\phi(s)}{2} - Q_0 s \right)$$

- For $s_1 \to \infty$, this integral is the sum of an oscillatory part and a jump, which comes from the region $s \lesssim Q_0^{-1}$. In this time, the instantaneous separatrix frequency has its maximum $2\omega_0$ (see Lecture 2, p.10)

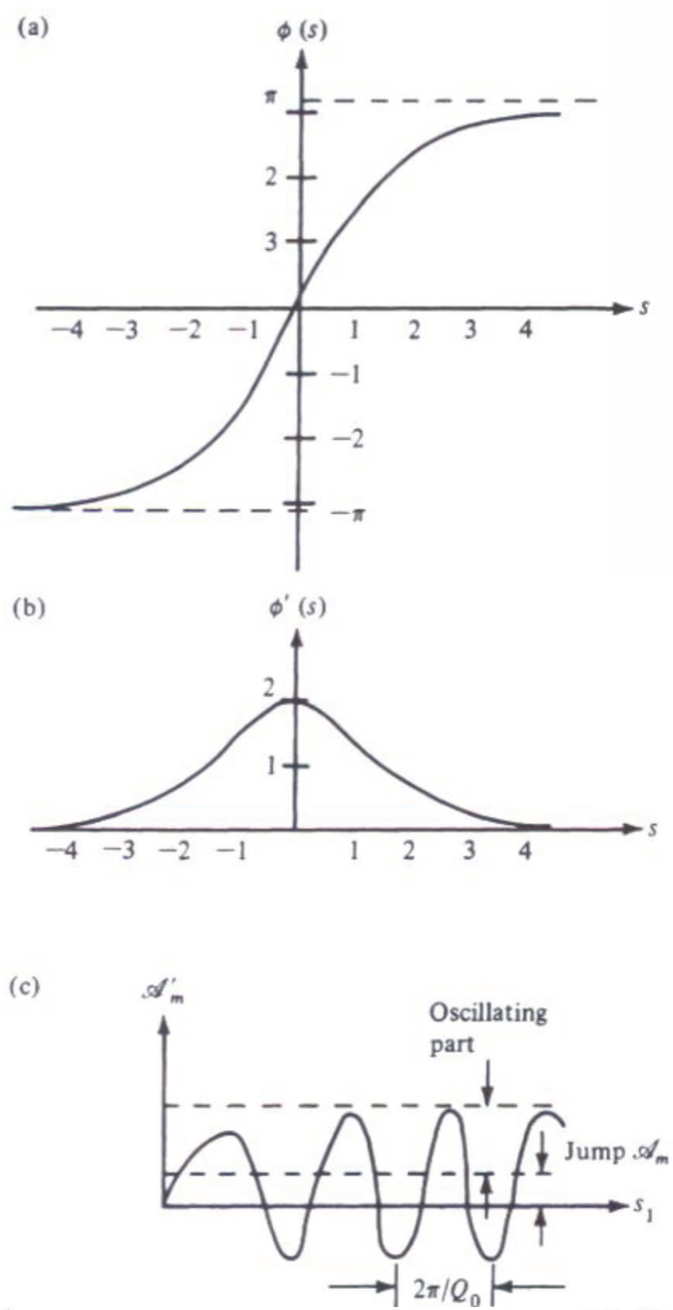


Illustration of the Melnikov-Arnold integral (*Lieberman and Lichtenberg*, 2010).

(a) The phase $\phi(s)$ for the motion near the separatrix of a pendulum.

(a) The frequency $\phi'(s)$ for the motion near the separatrix of a pendulum.

(c) Oscillating part and jump in $\mathcal{A}'_m(Q_0, s_1)$. The oscillating part averages to zero on the scale of separatrix motion.

- The jump in $\mathcal{A}_m(Q_0)$ has been evaluated by Melnikov (*Chirikov*, 1979). For m integer the jump can be evaluated exactly. For $Q_0 < 0$ it is usually very small

$$\mathcal{A}_m(Q_0) = (-1)^m \mathcal{A}_m(-Q_0) \exp(\pi Q_0)$$

- For $Q_0 > 0$ one has the expressions and recurrence relations for $m > 2$

$$\mathcal{A}_1 = \frac{2\pi \exp(\pi Q_0/2)}{\sinh(\pi Q_0)} \;\; ; \;\; \mathcal{A}_2 = 2Q_0\mathcal{A}_1 \;\; ; \;\; \mathcal{A}_m = \frac{2Q_0\mathcal{A}_{m-1}}{(m-1)} - \mathcal{A}_{m-2}$$

- For $Q_0 \gg m$, asymptotic expansion is

$$\mathcal{A}_m = \frac{4\pi(2Q_0)^{m-1}\exp(-\pi Q_0/2)}{(m-1)!}$$

- With the expression of the jump in action and the expression on p.14, one has the mapping function f for the perturbed twist map

$$f = f_0 \sin \theta_n \quad ; \quad f_0 = 8\pi \Lambda \Omega^{-1} Q_0^2 \exp(-\pi Q_0/2)$$

- The separatrix mapping is obtained determining the change in θ, $\theta_{n+1} - \theta_n = \Omega T$, with T the time for the phase ϕ to move from $-\pi$ to π (see Lecture 2, p.9)

$$T = \omega_0^{-1} \ln \frac{32}{|w|} \quad ; \quad w(J) = -\frac{\epsilon F + \Omega J}{\epsilon F}$$

$w(J)$ representing the relative deviation of the energy $-\Omega J$ from the separatrix energy ϵF.

- After changing variables from J to w, the separatrix mapping (called by Chirikov the *whisker mapping* after Arnold)

$$w_{n+1} = w_n - w_0 \sin\theta_n \quad ; \quad \theta_{n+1} = \theta_n + Q_0 r \ln\left|\frac{32}{w_{n+1}}\right|$$

$$w_0 = \Omega \frac{f}{F} = 8\pi \frac{\Lambda}{F} Q_0^2 \exp(-\pi Q_0/2)$$

E: Show that period 1 fixed point of the separatrix map are $\theta_1 = 0, \pi$ and

$$Q_0 r \ln\left|\frac{32}{w_1}\right| = 2\pi m \quad ; \quad w_1 = \pm 32 \exp\left(\frac{-2\pi m}{Q_0 r}\right) \quad ; \quad m \in \mathbb{Z}$$

- Stability requires the calculation of the eigenvalues of the matrix $\boldsymbol{A}$. Setting $\det \boldsymbol{A} = 1$, for a 2×2 matrix, the eigenvalues are given by

$$\lambda^2 - \lambda \mathrm{Tr} \boldsymbol{A} + 1 = 0$$

E: Show this last result and its connection with the fact that $\boldsymbol{A}$ is area preserving.

- The eigenvalues are

$$\lambda_{1,2} = \frac{\mathrm{Tr}\boldsymbol{A}}{2} \pm i \left[1 - \left(\frac{\mathrm{Tr}\boldsymbol{A}}{2} \right)^2 \right]^{1/2}$$

- For stability (see p.22) one must have complex roots $\lambda_{1,2} e^{\pm i\sigma}$ with $\mathrm{Tr}\boldsymbol{A} = 2\cos\sigma$ and $|\mathrm{Tr}\boldsymbol{A}| < 2$.

- In the case of the separatrix map

$$\mathrm{Tr}\boldsymbol{A} = 2 + \frac{w_0}{w_1} Q_0 r \cos\theta_1$$

- Thus, for $\theta_1 = 0$ all fixed points are unstable, while for $\theta_1 = \pi$, stability occurs for

$$w_1 > \frac{w_0 Q_0 r}{4}$$

E: Discuss this result in the light of the fact that s chaotic region always occurs near the separatrix as $w \to 0$.

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