Lecture 4

Secular perturbation theory

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Removal of resonances: two degrees of freedom

- Near resonances of the unperturbed (integrable) Hamiltonian, resonant denominators appear when computing the first order adiabatic invariant calculated by classical perturbation theory.
- This problem is solve by introducing a canonical transformation to a frame that rotates with the resonant frequency.
- What is left is then the slow oscillation of new variables about their value at resonance.
- Assume a Hamiltonian in action-angle coordinates ($\boldsymbol{n} = (l, m)$)

$$H = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{J}, \boldsymbol{\theta}) = H_0(\boldsymbol{J}) + \epsilon \sum_{\boldsymbol{n}} H_{\boldsymbol{n}}(\boldsymbol{J}) e^{i\boldsymbol{n}\cdot\boldsymbol{\theta}}$$

- A primary resonance exists if

$$\frac{\omega_2}{\omega_1} = \frac{r}{s} \; : \; r, s \in \mathbb{Z} \;\; ; \;\; \omega_1(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_1} \;\; ; \;\; \omega_2(\boldsymbol{J}) = \frac{\partial H_0}{\partial J_2}$$

- The notion of resonance can be extended from that of primary resonance to include also the case of secondary resonance created by harmonic frequencies of an island oscillation generated by the primary resonance.
- The resonance is eliminated with the canonical transformation

$$F_2 = (r\theta_1 - s\theta_2)\hat{J}_1 + \theta_2\hat{J}_2$$

$$J_1 = \frac{\partial F_2}{\partial \theta_1} = r\hat{J}_1 \;\; ; \;\; J_2 = \frac{\partial F_2}{\partial \theta_2} = \hat{J}_2 - s\hat{J}_1 \;\; ; \;\; \hat{\theta}_1 = \frac{\partial F_2}{\partial \hat{J}_1} = r\theta_1 - s\theta_2 \;\; ; \;\; \hat{\theta}_2 = \frac{\partial F_2}{\partial \hat{J}_2} = \theta_2$$

- The new variable $\hat{\theta}_1$ describes the slow deviation from resonance: $\dot{\hat{\theta}}_1 = r\dot{\theta}_1 - s\dot{\theta}_2$
- The choice of θ_2 is arbitrary. However, for computation of higher order resonances, is convenient to choose θ_2 as the slowest of the two original frequencies.
- The new Hamiltonian is then written as

$$\hat{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{l,m} H_{l,m}(\hat{\boldsymbol{J}}) e^{(i/r)[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]}$$

- One can apply classical adiabatic invariant theory (see Lecture 3 p.27) and average on the fast $\hat{\theta}_2$ variation in order to eliminate one degree of freedom

$$\bar{H} = \bar{H}_0(\hat{\boldsymbol{J}}) + \epsilon \bar{H}_1(\hat{\boldsymbol{J}}, \hat{\theta}_1) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \left\langle \bar{H}_1(\hat{\boldsymbol{J}}, \hat{\theta}_1) \right\rangle_{\hat{\theta}_2} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{p=-\infty}^{\infty} H_{-pr,ps}(\hat{\boldsymbol{J}}) e^{-ip\hat{\theta}_1}$$

- After averaging, the new Hamiltonian describes a system with only one degree of freedom; thus, it is integrable.
- The averaging is valid near resonance, where $|\dot{\hat{\theta}}_2| \gg |\dot{\hat{\theta}}_1|$.
- Since the new Hamiltonian does not depend on $\hat{\theta}_2$, one readily has $\hat{J}_2 =$ const.

$$\hat{J}_2 = J_2 + \frac{s}{r} J_1 = \text{const.}$$

- For the unperturbed Hamiltonian at the primary resonance, satisfied for a particular $\boldsymbol{J}$, periodic solutions are degenerate in $\boldsymbol{\theta}$ (they exist for all $\boldsymbol{\theta}$).
- The perturbation removes this degeneracy. In fact, periodic solutions exist for a stationary point or set of points in the $\hat{J}_1 - \hat{\theta}_1$ plane, for which

$$\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0 \quad ; \quad \frac{\partial \bar{H}}{\partial \hat{\theta}_1} = 0$$

- The Fourier harmonics $H_{-pr,ps}$ generally fall off rapidly as p increases. A good approximation of the Hamiltonian is obtained looking at the integrable motion with $p = 0, \pm 1$ only

$$\bar{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos\theta_1$$

- Fixed points are located at

$$\frac{\partial \bar{H}}{\partial \hat{\theta}_1} = 0 \quad \Rightarrow \quad \hat{\theta}_{10} = 0, \pi$$

- The resonance condition in $\hat{J}_{10}$ is obtained at

$$\frac{\partial \bar{H}}{\partial \hat{J}_1} = 0$$

- At lowest and first order, respectively

$$\frac{\partial \hat{H}_0}{\partial \hat{J}_{10}} = r\frac{\partial H_0}{\partial J_1} - s\frac{\partial H_0}{\partial J_2} = r\omega_1 - s\omega_2 = 0$$

$$\delta \hat{J}_{10}\frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} + \epsilon\frac{\partial H_{0,0}}{\partial \hat{J}_{10}} \pm 2\epsilon\frac{\partial H_{r,-s}}{\partial \hat{J}_{10}} = 0$$

- The Hamiltonian is said to be accidentally degenerate if resonance is satisfied for particular values of J_1, J_2. In this case, the unperturbed Hamiltonian in the new coordinates is function of both actions

$$\hat{H}_0 = \hat{H}_0(\hat{J}_1, \hat{J}_2)$$

- The Hamiltonian is said to be intrinsically degenerate if resonance is satisfied for all values of J_1, J_2. In this case, $H_0 = H_0(sJ_1 + rJ_2)$ and the unperturbed Hamiltonian in the new coordinates is independent of $\hat{J}_1$

$$\hat{H}_0 = \hat{H}_0(\hat{J}_2)$$

- Secondary resonances are almost always accidental.

Accidental degeneracy

- Given the Hamiltonian

$$\bar{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos\theta_1$$

one can estimate the excursions in the $\hat{J}_1 - \hat{\theta}_1$ plane

$$\dot{\hat{J}}_1 = \mathcal{O}(\epsilon H_{r,-s}) \quad ; \quad \dot{\hat{\theta}}_1 = \mathcal{O}(1)$$

- It is legitimate to expand about the stationary point in $\hat{J}_1$, i.e. $\Delta\hat{J}_1 = \hat{J}_1 - \hat{J}_{10}$

$$\bar{H}_0(\hat{\boldsymbol{J}}) = \hat{H}_0(\hat{\boldsymbol{J}}_0) + \frac{\partial \hat{H}_0}{\partial \hat{J}_{10}} \Delta\hat{J}_1 + \frac{1}{2}\frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} \Delta\hat{J}_1^2 + \ldots$$

where the linear term must be dropped for the definition of $\hat{J}_{10}$

- Neglecting constant terms and keeping lowest order terms in ϵ and $\Delta\hat{J}_1$, the universal description of the motion near a resonance is given by the *standard Hamiltonian*

$$\Delta\bar{H} = \frac{1}{2}G\Delta\hat{J}_1^2 - F\cos\hat{\theta}_1$$

with the *nonlinear parameter* G and the perturbation strength F given by

$$G(\hat{\boldsymbol{J}}_0) = \frac{\partial^2\hat{H}_0}{\partial\hat{J}_{10}^2} \quad ; \quad F(\hat{\boldsymbol{J}}_0) = -2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}_0)$$

- For $GF > 0$, the stable fixed point is at $\hat{\theta}_1 = 0$ and the frequency of librations near the stable point is (see Lecture 2 p.9)

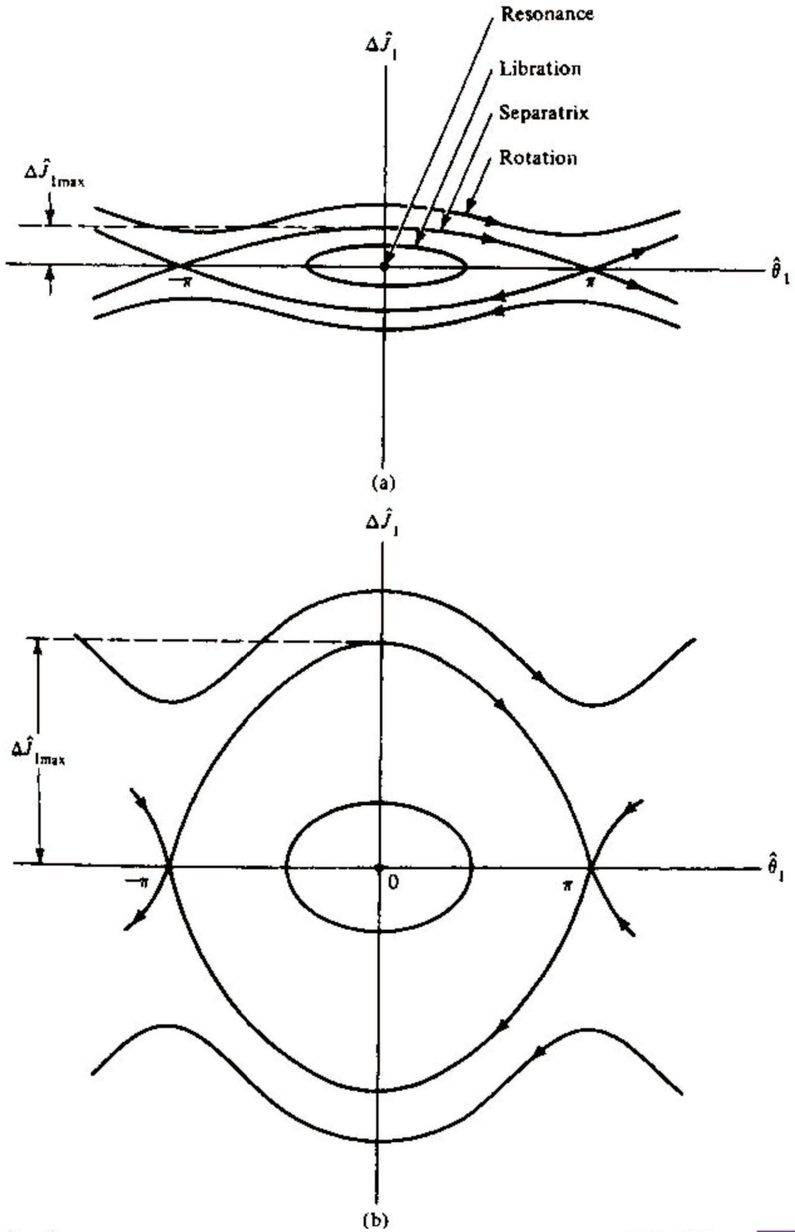
$$\hat{\omega}_1 = (FG)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$

- The maximum excursion $\Delta\hat{J}_{1\max}$ is given by half the separatrix width and is given by (see Lecture 2 p.8)

$$\Delta\hat{J}_{1\max} = 2\left(\frac{F}{G}\right)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$

- Similarly, near the stable fixed point, trajectories are elliptic with

$$\frac{\Delta\hat{J}_1}{\Delta\hat{\theta}_1} = \left(\frac{F}{G}\right)^{1/2} = \mathcal{O}[(\epsilon H_{r,-s})^{1/2}]$$



Adiabatic motion near an isolated resonance (*Lieberman and Lichtenberg*, 2010).

(a) accidental degeneracy: maximum excursion in action and libration frequency are $\mathcal{O}(\epsilon^{1/2})$

(b) intrinsic degeneracy: maximum excursion in action is large while libration frequency is low $\mathcal{O}(\epsilon)$

Intrinsic degeneracy

- Reconsider the Hamiltonian

$$\bar{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos\theta_1$$

where now, for intrinsic degeneracy, $\hat{H}_0(\hat{\boldsymbol{J}}) = \hat{H}_0(\hat{J}_2)$. Thus, the estimate of excursions in the $\hat{J}_1 - \hat{\theta}_1$ plane are

$$\dot{\hat{J}}_1 = \mathcal{O}(\epsilon H_{r,-s}) \;\; ; \;\; \dot{\hat{\theta}}_1 = \mathcal{O}(\epsilon H_{0,0}, \epsilon H_{r,-s})$$

- This shows that the excursions in $\hat{J}_1$ and $\hat{\theta}_1$ are of the same order.

- The character of the solution can be studied by expansion of the Hamiltonian near the stable fixed point $\hat{\theta}_{10} = 0$.

$$H_{0,0}(\hat{\boldsymbol{J}}) = H_{0,0}(\hat{\boldsymbol{J}}_0) + \frac{\partial H_{0,0}}{\partial \hat{J}_{10}} \Delta \hat{J}_1 + \frac{1}{2} \frac{\partial^2 H_{0,0}}{\partial \hat{J}_{10}^2} \Delta \hat{J}_1^2 + \ldots$$

$$H_{r,-s}(\hat{\boldsymbol{J}}) = H_{r,-s}(\hat{\boldsymbol{J}}_0) + \frac{\partial H_{r,-s}}{\partial \hat{J}_{10}} \Delta \hat{J}_1 + \frac{1}{2} \frac{\partial^2 H_{r,-s}}{\partial \hat{J}_{10}^2} \Delta \hat{J}_1^2 + \ldots$$

Here, first order terms drop out for the first order resonance condition at the elliptic fixed point (see p.7 with (+) sign)

- Ignoring constants, one can obtained the linearized new Hamiltonian by expanding $\cos \hat{\theta}_1 = 1 - \Delta \hat{\theta}_1^2/2 + \ldots$

$$\Delta \bar{H} = \frac{1}{2} G \Delta \hat{J}_1^2 + \frac{1}{2} F \Delta \hat{\theta}_1^2$$

$$G(\hat{\boldsymbol{J}}_0) = \epsilon \frac{\partial^2 H_{0,0}}{\partial \hat{J}_{10}^2} + 2\epsilon \frac{\partial^2 H_{r,-s}}{\partial \hat{J}_{10}^2} \quad ; \quad F(\hat{\boldsymbol{J}}_0) = -2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}_0)$$

- The frequency of librations near the stable point is (see Lecture 2 p.9)

$$\hat{\omega}_1 = (FG)^{1/2} = \mathcal{O}(\epsilon H_{r,-s})$$

- The maximum excursion $\Delta\hat{J}_{1\max}$ is of order unity, while near the stable fixed point, trajectories are elliptic with

$$\frac{\Delta\hat{J}_1}{\Delta\hat{\theta}_1} = \left(\frac{F}{G}\right)^{1/2} = \mathcal{O}(1)$$

- The transition from accidental to intrinsic degeneracy is controlled by

$$G(\hat{\boldsymbol{J}}_0) = \frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} + \epsilon \frac{\partial^2 H_{0,0}}{\partial \hat{J}_{10}^2} + 2\epsilon \frac{\partial^2 H_{r,-s}}{\partial \hat{J}_{10}^2}$$

with the first term on the RHS passing to the limit of zero.

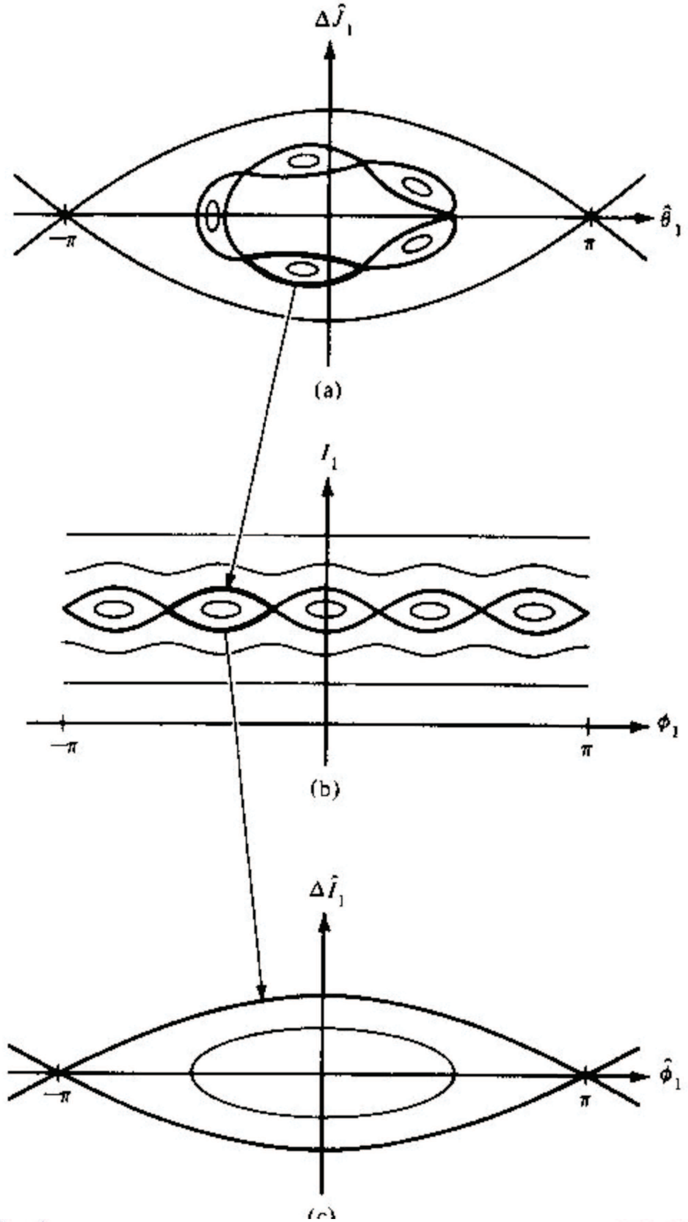
Higher order resonances

- If ϵ is not sufficiently small, secondary resonances are present in the Hamiltonian

$$\hat{H} = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \hat{H}_1(\hat{\boldsymbol{J}}, \hat{\boldsymbol{\theta}}) = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon \sum_{l,m} H_{l,m}(\hat{\boldsymbol{J}}) e^{(i/r)[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]}$$

that destroy the adiabatic invariant $\hat{J}_2$.

- These secondary resonances are between harmonics of the motion in the $\hat{J}_1 - \hat{\theta}_1$ plane with the fundamental frequency ω_2.
- In the adiabatic limit, these resonances give rise to island chains, while the resonances can be treated as previously.



Motion near a secondary resonance (*Lieberman and Lichtenberg*, 2010).

(a) secondary island chain for $5\hat{\omega}_1 = \hat{\omega}_2$

(b) transformation to action-angle coordinates of the unperturbed primary libration

(c) transformation to rotating coordinate system associated with secondary librations

- Consider the Hamiltonian

$$K = \hat{H}_0(\hat{\boldsymbol{J}}) + \epsilon H_{0,0}(\hat{\boldsymbol{J}}) + 2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}) \cos\theta_1 + \epsilon_2 \sum_{l,m}{}' H_{l,m}(\hat{\boldsymbol{J}}) e^{(i/r)[l\hat{\theta}_1 + (ls+mr)\hat{\theta}_2]}$$

where ϵ_2 is a new ordering parameter for investigating the effects of secondary resonances $ls + mr \neq 0$ represented by the second sum.

- Consider primary librations near the elliptic fixed point, i.e.

$$\bar{H} = \bar{H}(\hat{J}_{10}, \hat{J}_2) + \frac{1}{2} G \Delta \hat{J}_1^2 + \frac{1}{2} F \theta_1^2 - \frac{1}{4!} F \theta_1^4$$

with G and F functions of $\hat{J}_{10}, \hat{J}_2$ given by

$$G = \frac{\partial^2 \hat{H}_0}{\partial \hat{J}_{10}^2} + \epsilon \frac{\partial^2 H_{0,0}}{\partial \hat{J}_{10}^2} + 2\epsilon \frac{\partial^2 H_{r,-s}}{\partial \hat{J}_{10}^2} \quad ; \quad F = -2\epsilon H_{r,-s}(\hat{\boldsymbol{J}}_0)$$

- We can transform to $I_1 - \phi_1$ action-angle coordinates of the unperturbed primary libration with the generating function

$$F_1 = \frac{1}{2} R \hat{\theta}_1^2 \cot \bar{\phi}_1 \quad ; \quad R = \left(\frac{F}{G} \right)^{1/2}$$

$$\bar{I}_1 = -\frac{\partial F_1}{\partial \bar{\phi}_1} = \frac{1}{2} R \frac{\theta_1^2}{\sin^2 \bar{\phi}_1} \quad ; \quad \Delta \hat{J}_1 = \frac{\partial F_1}{\partial \hat{\theta}_1} = R \hat{\theta}_1 \cot \bar{\phi}_1$$

followed by another transform with the near-identity generating function

$$F_2 = I_1 \bar{\phi}_1 - \frac{G I_1^2}{192 \hat{\omega}_1} (8 \sin 2\bar{\phi}_1 - \sin 4\bar{\phi}_1) \quad ; \quad \hat{\omega}_1 = (FG)^{1/2}$$

$$\phi_1 = \frac{\partial F_2}{\partial I_1} = \bar{\phi}_1 - \frac{G I_1}{96 \hat{\omega}_1} (8 \sin 2\bar{\phi}_1 - \sin 4\bar{\phi}_1) \quad ; \quad \bar{I}_1 = \frac{\partial F_2}{\partial \bar{\phi}_1} = I_1 - \frac{G I_1^2}{48 \hat{\omega}_1} (4 \cos 2\bar{\phi}_1 - \cos 4\bar{\phi}_1)$$

E: Derive these last results step by step. Hint: see Lecture 2.

- After inversion, and letting $I_2 = \hat{J}_2, \phi_2 = \theta_2$

$$\bar{\phi}_1 = \phi_1 - \frac{GI_1}{96\hat{\omega}_1}(8\sin 2\phi_1 - \sin 4\phi_1) \;\; ; \;\; I_1 = \bar{I}_1 - \frac{GI_1^2}{48\hat{\omega}_1}(4\cos 2\bar{\phi}_1 - \cos 4\bar{\phi}_1)$$

$$K = K_0(I_1, I_2) + \epsilon_2 K_1(I_1, I_2, \phi_1, \phi_2) \;\; ; \;\; K_0(I_1, I_2) = \bar{H}(\hat{J}_{10}, I_2) + \hat{\omega}_1 I_1 - \frac{1}{16} G I_1^2$$

$$K_1(I_1, I_2, \phi_1, \phi_2) = \sum_{l,m}{}' H_{l,m}(\hat{J}_{10}, I_2) \exp\left[\frac{il}{r}\left(\frac{2I_1}{R}\right)^{1/2} \sin\phi_1\right] e^{i(ls/r+m)\phi_2}$$

- Note that, here, the term $\propto GI_1^2$ is formally $\mathcal{O}(\epsilon)$ since $I_1 = \mathcal{O}(\epsilon^{1/2})$

- The expression of K_1 can be made more transparent, expanding the sinusoidal phase in terms of Bessel functions as

$$K_1(I_1, I_2, \phi_1, \phi_2) = \sum_{l,m,n}{}' \Gamma_{l,m,n}(\hat{J}_{10}, \boldsymbol{I}) \exp\left[in\phi_1 + i\left(m + l\frac{s}{r}\right)\phi_2\right]$$

$$\Gamma_{l,m,n} = H_{l,m}(\hat{J}_{10}, I_2) J_n\left[\frac{l}{r}\left(\frac{2I_1}{R}\right)^{1/2}\right]$$

- This form shows that resonances between ϕ_1 and ϕ_2 are possible, which are eliminated with the standard procedure, noting that resonance implies

$$\frac{\hat{\omega}_2}{\hat{\omega}_1} = \frac{p}{q} \;:\; p, q \in \mathbb{Z} \;\;;\;\; \hat{\omega}_1 = \frac{\partial K_0}{\partial I_1} = \mathcal{O}(\epsilon^{1/2}) \;\;;\;\; \hat{\omega}_2 = \frac{\partial K_0}{\partial I_2} = \omega_2 = \mathcal{O}(1)$$

- The resonance is eliminated, as above, with the canonical transformation

$$F_2 = (p\phi_1 - q\phi_2)\hat{I}_1 + \phi_2\hat{I}_2$$

$$I_1 = \frac{\partial F_2}{\partial \phi_1} = p\hat{I}_1 \;\; ; \;\; I_2 = \frac{\partial F_2}{\partial \phi_2} = \hat{I}_2 - q\hat{I}_1 \;\; ; \;\; \hat{\phi}_1 = \frac{\partial F_2}{\partial \hat{I}_1} = p\phi_1 - q\phi_2 \;\; ; \;\; \hat{\phi}_2 = \frac{\partial F_2}{\partial \hat{I}_2} = \phi_2$$

- Averaging on the fast $\hat{\phi}_2$ variations, one obtains the condition

$$nq = -\left(m + l\frac{s}{r}\right)p$$

$$n = -jp \;\; ; \;\; l = kr \;\; ; \;\; m = jq - ks \;\; ; \;\; j, k \in \mathbb{Z}$$

- The new Hamiltonian is then written as

$$\bar{K} = \bar{K}_0(\hat{I}_1, \hat{I}_2) + \epsilon_2 \bar{K}_1(\hat{I}_1, \hat{I}_2, \hat{\phi}_1)$$

$$\bar{K}_1 = \sum_j K_{-jp,jq} \exp(-ij\hat{\phi}_1) = \sum_j \left(\sum_k \Gamma_{kr,jq-ks,-jp} \right) \exp(-ij\hat{\phi}_1)$$

- The new invariant of the motion is

$$\hat{I}_2 = I_2 + \left(\frac{p}{q} \right) I_1$$

- The method of removing second order resonances is the same as for primary resonances, but their properties are quite different.

- The largest term in the secondary resonance sources is at $|j| = |k| = 1$. For $q = 1$, i.e. resonance with the fundamental frequency of $\phi_2 = \hat{\theta}_2$ oscillation, and $p = \mathcal{O}(\epsilon^{-1/2})$, $I_1/R = \mathcal{O}(1)$, we have $J_p(\sqrt{2I_1/R}) = \mathcal{O}[(I_1/R)^{p/2}/p!] = \mathcal{O}[1/(\epsilon^{-1/2})!]$

- The estimate of the size of the oscillation in $\hat{I}_1$ is much smaller than the

$$\hat{I}_{1\max} < \mathcal{O}[1/(\epsilon^{-1/2})!]^{1/2}$$

- This means that secondary island resonances become rapidly negligible away from the separatrix: note the scaling $J_p(\sqrt{2I_1/R}) \propto (I_1/R)^{p/2}$.
- This is an implicit indication of the stability of second order islands at modest value of the perturbation strength (see Lecture 6).

Resonant wave particle interactions

- Reconsider the case analyzed in Lecture 3 of wave particle interaction of a charge moving in a constant $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ field, i.e. $\boldsymbol{A}(\boldsymbol{x}) = -B_0 y\hat{\boldsymbol{x}}$

$$H = H_0 + \epsilon H_1 \;\;;\;\; H_0 = \frac{1}{2M}\left|\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right|^2 \;\;;\;\; H_1 = e\Phi_0 \sin(k_z z + k_\perp y - \omega t)$$

- This system eventually reduces to a two degrees of freedom system in action angle coordinates, whose unperturbed motion has characteristic frequencies

$$\omega_\phi = \frac{\partial \bar{H}}{\partial P_\phi} = \Omega \;\;;\;\; \omega_\psi = \frac{\partial \bar{H}}{\partial P_\psi} = k_z^2 \frac{P_\psi}{M} - \omega = k_z v_z - \omega$$

$$\bar{H} = \frac{k_z^2 P_\psi^2}{2M} + \Omega P_\phi - \omega P_\psi + \epsilon e\Phi_0 \sum_m J_m(k_\perp \rho)\sin(\psi - m\phi)$$

- Resonance between the two degrees of freedom occurs when $\omega_\psi - m\omega_\phi = 0$, i.e.

$$P_\psi = \frac{M}{k_z^2}(\omega + m\Omega) \quad , \quad \text{for } k_z \neq 0 \;\; ; \;\; \omega + m\Omega = 0 \quad , \quad \text{for } k_z = 0$$

- From these condition it is possible to see that $k_z \neq 0$ corresponds to accidental degeneracy, while $k_z = 0$ corresponds to intrinsic degeneracy.

E: Prove the last statement.

- Accidental degeneracy. Assuming resonance for $m = l$, i.e. $\omega_\psi - l\omega_\phi = 0$, one can transform to the primary resonance rotating frame

$$F_2 = (\psi - l\phi)\hat{P}_\psi + \phi\hat{P}_\phi \;\; ; \;\; (P_\phi = \hat{P}_\phi - l\hat{P}_\psi \;\; ; \;\; P_\psi = \hat{P}_\psi)$$

E: Derive all the details of the canonical transformation with F_2.

- The new Hamiltonian is, given $\rho = (2/M\Omega)^{1/2}(\hat{P}_\phi - l\hat{P}_\psi)^{1/2}$

$$\hat{H} = \frac{k_z^2 \hat{P}_\psi^2}{2M} + \Omega(\hat{P}_\phi - l\hat{P}_\psi) - \omega\hat{P}_\psi + \epsilon e\Phi_0 \sum_m J_m(k_\perp\rho)\sin\left[\hat{\psi} - (m-l)\hat{\phi}\right]$$

and, after averaging on $\hat{\phi}$ and shifting the phase $\hat{\psi} \to \hat{\psi} + \pi/2$

$$\bar{H} = \frac{k_z^2 \hat{P}_\psi^2}{2M} + \Omega(\hat{P}_\phi - l\hat{P}_\psi) - \omega\hat{P}_\psi + \epsilon e\Phi_0 J_l(k_\perp\rho)\cos\hat{\psi}$$

- Once more, this is the pendulum Hamiltonian, with $G = k_z^2/M$ and $F = -\epsilon e\Phi_0 J_l(k_\perp\rho)$. Therefore (see pp.10-11), the frequency near the elliptic singular point and the peak amplitude of $\hat{P}_\psi$ oscillation near the separatrix are both $\mathcal{O}(\epsilon^{1/2})$ and given by

$$\hat{\omega}_\psi = \left|\frac{\epsilon e\Phi_0 k_z^2 J_l(k_\perp\rho)}{M}\right|^{1/2} \quad ; \quad \Delta\hat{P}_{\psi\max} = \frac{2\hat{\omega}_\psi}{G}$$

- The separation between adjacent resonances is (see p.27) $\delta \hat{P}_\psi = M\Omega/k_z^2$ and the ratio of the maximum separatrix width to the resonance separation is given by

$$\frac{2\Delta \hat{P}_{\psi\max}}{\delta \hat{P}_\psi} = \frac{4\hat{\omega}_\psi}{\Omega}$$

- Intrinsic degeneracy. For $k_z = 0$ one needs to expand $\Delta \hat{P}_\psi$ and $\delta\hat{\psi}$ the Hamiltonian

$$\bar{H} = \frac{k_z^2 \hat{P}_\psi^2}{2M} + \Omega(\hat{P}_\phi - l\hat{P}_\psi) - \omega \hat{P}_\psi + \epsilon e \Phi_0 J_l(k_\perp \rho) \cos\hat{\psi}$$

about the elliptic singular point, finding

$$G = \epsilon e \Phi_0 \frac{\partial^2 J_l(k_\perp \rho_0)}{\partial \hat{P}_{\psi 0}^2} \quad ; \quad F = \epsilon e \Phi_0 J_l k_\perp \rho_0)$$

- Unlike the case of accidental degeneracy, $\hat{\omega}_\psi = \mathcal{O}(\epsilon)$ and $\Delta \hat{P}_{\psi\max} = \mathcal{O}(1)$.
- When we consider small but finite k_z^2 in the Hamiltonian expansion, then

$$G = \frac{k_z^2}{M} + \epsilon e \Phi_0 \frac{\partial^2 J_l(k_\perp \rho_0)}{\partial \hat{P}_{\psi 0}^2}$$

- Transition from accidental to intrinsic degeneracy occurs at

$$\frac{k_z^2}{M} \lesssim \epsilon e \Phi_0 \left| \frac{\partial^2 J_l(k_\perp \rho_0)}{\partial \hat{P}_{\psi 0}^2} \right|$$

- Second order resonance. Following the general procedure (see p.20), one can move to I, θ action-angle variables for describing the $\Delta \hat{P}_\psi - \hat{\psi}$ oscillations

$$\bar{K} = \bar{K}_0(I, \hat{P}_\phi) + \epsilon K_1 \quad ; \quad \hat{\psi} = \hat{\psi}_0 + \left(\frac{2I}{R}\right)^{1/2} \sin\theta + \text{h.o.t.}$$

$$\bar{K}_0 = \hat{\omega}_\psi I - \frac{\epsilon}{16} G I^2 + \ldots \quad ; \quad K_1 = e\Phi_0 \sum_{m \neq l} J_m(k_\perp \rho) \sin\left[\hat{\psi}_0 + \left(\frac{2I}{R}\right)^{1/2} \sin\theta - (m-l)\hat{\phi}\right]$$

- Using Bessel function expansion (see p.22) and taking the most important secondary resonance $m = l + 1$ near a given term n of the Bessel expansion, one finds

$$K_1 = K_{1,l+1,n} = e\Phi_0 J_{l+1}(k_\perp \rho) J_n\left[\left(\frac{2I}{R}\right)^{1/2}\right] \sin(\hat{\psi}_0 + n\theta - \hat{\phi})$$

- Moving to the secondary resonance rotating frame

$$\hat{\theta} = n\theta - \hat{\phi} + \hat{\psi}_0 - \frac{\pi}{2} \quad ; \quad I = n\hat{I}$$

the new pendulum Hamiltonian is

$$\Delta\bar{K} = \frac{1}{2}G_s\Delta\hat{I}^2 - F_s\cos\hat{\theta} \quad ; \quad G_s = -\frac{G}{8} \quad ; \quad F_s = -\epsilon K_{1,l+1,n}$$

- The frequency near the elliptic singular point and the peak amplitude of $\Delta\hat{I}$ oscillation near the separatrix are given by

$$\hat{\omega}_s = (F_s G_s)^{1/2} \quad ; \quad \Delta\hat{I}_{\max} = \frac{2\hat{\omega}_s}{G_s}$$

- Note that

$$n\hat{\omega}_\psi - \Omega = 0 \;\; ; \;\; (n+1)(\hat{\omega}_\psi + G_s \delta I) - \Omega = 0$$

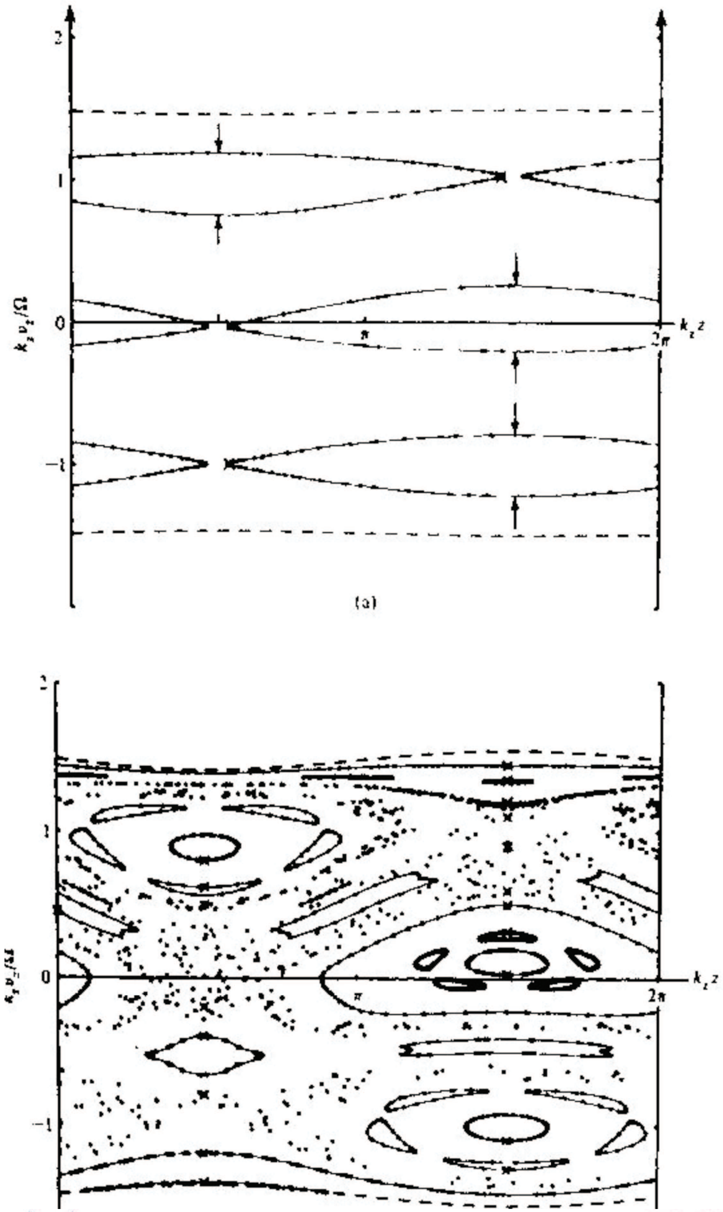
- Thus, the separation between adjacent resonances is $\delta \hat{I} = \hat{\omega}_s / G_s$ and the ratio of the maximum separatrix width to the resonance separation is given by

$$\frac{2\Delta \hat{I}_{\max}}{\delta \hat{I}} = \frac{4\hat{\omega}_s}{\hat{\omega}_\psi}$$

- This relation is identical to that obtained for the primary resonance. So, by induction, we can consider that this relation has a universal form.

- Another point to notice is that secondary (higher order) resonances are accidental.

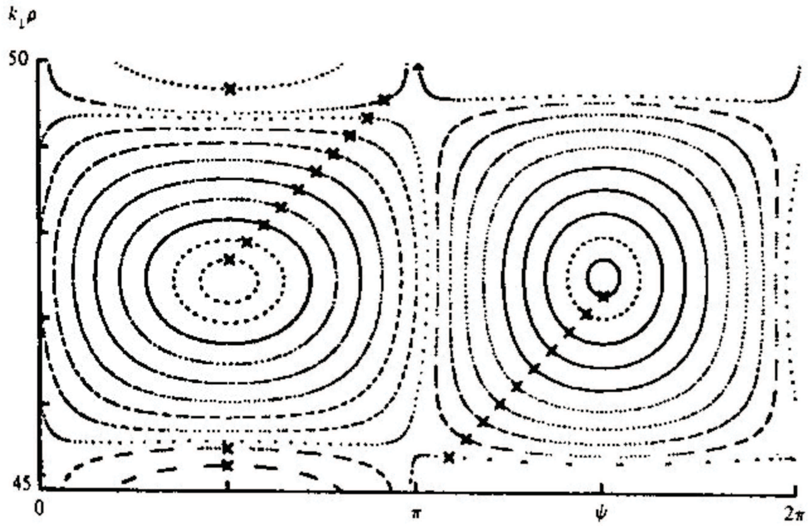
E: Recover all the missing steps for the derivation of the second-order resonance for the case that was just discussed.



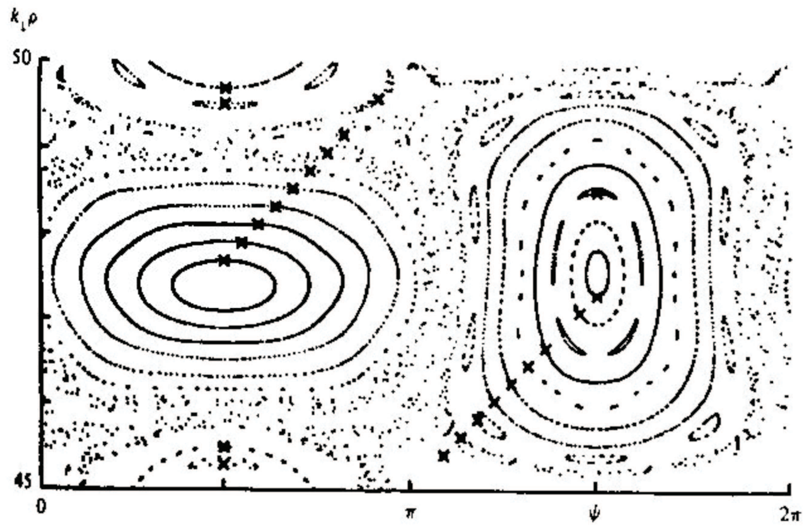
Surface of section plots for $k_z v_z/\Omega \propto P_\psi$ vs. $\psi = k_z z$ for the wave-particle resonance considered above. (*Smith and Kaufman*, 1975).

(a) Three resonances for weak perturbation $k_z^2 e\Phi_0/M\Omega^2 = 0.025$

(b) Same three resonances for strong perturbation $k_z^2 e\Phi_0/M\Omega^2 = 0.1$



(a)



(b)

Surface of section plots of $k_\perp\rho$ vs. ψ (note that this is not the actual surface of section plot P_ψ vs. ψ, see Lecture 3) (*Karney*, 1977).

(a) Resonance condition $\omega/\Omega = 30$ for small amplitude. (x) denote initial conditions in the integration of Hamilton's equations

(b) Same as (a) but with larger amplitude.

Lie transformation methods

- The main problem with perturbation theories based on canonical transformation is that generating functions mix old and new coordinates.
- Consider the phase space and let $\boldsymbol{x} = (\boldsymbol{p}, \boldsymbol{q})$.
- The *Lie generating function* is defined as $w(\bar{\boldsymbol{x}}, \epsilon)$ satisfying

$$\frac{d\bar{\boldsymbol{x}}}{d\epsilon} = [\bar{\boldsymbol{x}}, w]$$

- These equations are just Hamilton's equations of motion, given the Hamiltonian w and the time ϵ. Thus, the definition of w generates a canonical transformation for any ϵ that, for any initial point $\boldsymbol{x}$, defines

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)$$

- This defines a transformation from $\boldsymbol{x}$ to $\bar{\boldsymbol{x}}$, which satisfies the Poisson brackets

$$[\bar{q}_i, \bar{q}_j] = [\bar{p}_i, \bar{p}_j] = 0 \;\; ; \;\; [\bar{q}_i, \bar{p}_j] = \delta_{ij}$$

- Next, one defines the *evolution operator* T which maps any function g evaluated at $\bar{\boldsymbol{x}}$ to the function f evaluated in $\boldsymbol{x}$

$$f = Tg \;\; ; \;\; f(\boldsymbol{x}) = g[\bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)]$$

- If g is the identity function $g(\boldsymbol{x}) = \boldsymbol{x}$, then

$$\bar{\boldsymbol{x}} = T\boldsymbol{x}$$

- Introducing the *Lie operator* $L = [w, \,]$, then T satisfies

$$\frac{dT}{d\epsilon} = -TL \quad ; \quad T = \exp\left[-\int^{\epsilon} L(\epsilon')d\epsilon' \right]$$

- For autonomous systems

$$\bar{H}[\bar{\boldsymbol{x}}(\boldsymbol{x}, \epsilon)] = H(\boldsymbol{x}) \quad ; \quad \bar{H} = T^{-1}H$$

- For time dependent systems, this equation is generalized to (*Dewar*, 1976)

$$\bar{H} = T^{-1}H + T^{-1}\int_0^{\epsilon} T(\epsilon')\frac{\partial w}{\partial t}d\epsilon'$$

E: Follow the detailed derivation of this result in (*Dewar*, 1976).

- These equations can be solved by means of perturbation series (*Deprit*, 1969)

$$w = \sum_{n=0}^{\infty} \epsilon^n w_{n+1} \; ; \; L = \sum_{n=0}^{\infty} \epsilon^n L_{n+1} \; ; \; L_n = [w_n, \,]$$

$$T = \sum_{n=0}^{\infty} \epsilon^n T_n \; ; \; H = \sum_{n=0}^{\infty} \epsilon^n H_n \; ; \; \bar{H} = \sum_{n=0}^{\infty} \epsilon^n \bar{H}_n$$

- By direct substitution into $dT/d\epsilon = -TL$, one obtains

$$T_n = -\frac{1}{n} \sum_{m=0}^{n-1} T_m L_{n-m} \; ; \; T_0 = 1$$

E: Use the identity $TT^{-1} = 1$ to show $dT^{-1}/d\epsilon = LT^{-1}$ and then to derive the relation

$$T_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_m^{-1} \quad ; \quad T_0^{-1} = 1$$

- Given the form of T_n the rule to derive T_n^{-1} is: replace L_n by $-L_n$ and invert the order of all noncommuting L operators. Note that in general T's and L's do not commute, i.e. $L_i L_j \neq L_j L_i$.
- Up to third order one has

$$T_1 = -L_1 \quad ; \quad T_2 = -\frac{1}{2} L_2 + \frac{1}{2} L_1^2 \quad ; \quad T_3 = -\frac{1}{3} L_3 + \frac{1}{6} L_2 L_1 + \frac{1}{3} L_1 L_2 - \frac{1}{6} L_1^3$$

E: Given the expressions of T up to third order, derive the corresponding expressions of T^{-1} using the rule above. Show that this is consistent with the equation $dT^{-1}/d\epsilon = LT^{-1}$.

- Differentiating the $\bar{H}$ expression and using $dT/d\epsilon = \partial T/\partial\epsilon = -TL$

$$\frac{\partial T}{\partial \epsilon}\bar{H} + T\frac{\partial \bar{H}}{\partial \epsilon} = \frac{\partial H}{\partial \epsilon} + T\frac{\partial w}{\partial t} \quad ; \quad \frac{\partial w}{\partial t} = \frac{\partial \bar{H}}{\partial \epsilon} - L\bar{H} - T^{-1}\frac{\partial H}{\partial \epsilon}$$

- Inserting the series expansions, one obtains

$$\frac{\partial w_n}{\partial t} = n\bar{H}_n - \sum_{m=0}^{n-1} L_{n-m}\bar{H}_m - \sum_{m=1}^{n} mT^{-1}_{n-m}H_m$$

- Separating the first term in the first sum and using $L_n\bar{H}_0 = L_nH_0 = [w_n, H_0]$, the equation above is rewritten as

$$D_0 w_n = n(\bar{H}_n - H_n) - \sum_{m=0}^{n-1}(L_{n-m}\bar{H}_m + mT^{-1}_{n-m}H_m) \quad ; \quad D_0 = \frac{\partial}{\partial t} + [\,, H_0]$$

- The system of equations can be solved, proceeding order by order: one chooses $\bar{H}_n$ to eliminate secularities in w_n and then solves for w_n.
- Up to third order one has

$$D_0 w_1 = \bar{H}_1 - H_1 \quad ; \quad D_0 w_2 = 2(\bar{H}_2 - H_2) - L_1(\bar{H}_1 + H_1)$$

$$D_0 w_3 = 3(\bar{H}_3 - H_3) - L_1(\bar{H}_2 + 2H_2) - \frac{1}{2} L_2 (2\bar{H}_1 + H_1) - \frac{1}{2} L_1^2 H_1$$

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