Lecture 6

Nonlinear equilibria and long time scale behaviors

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Applications and Examples

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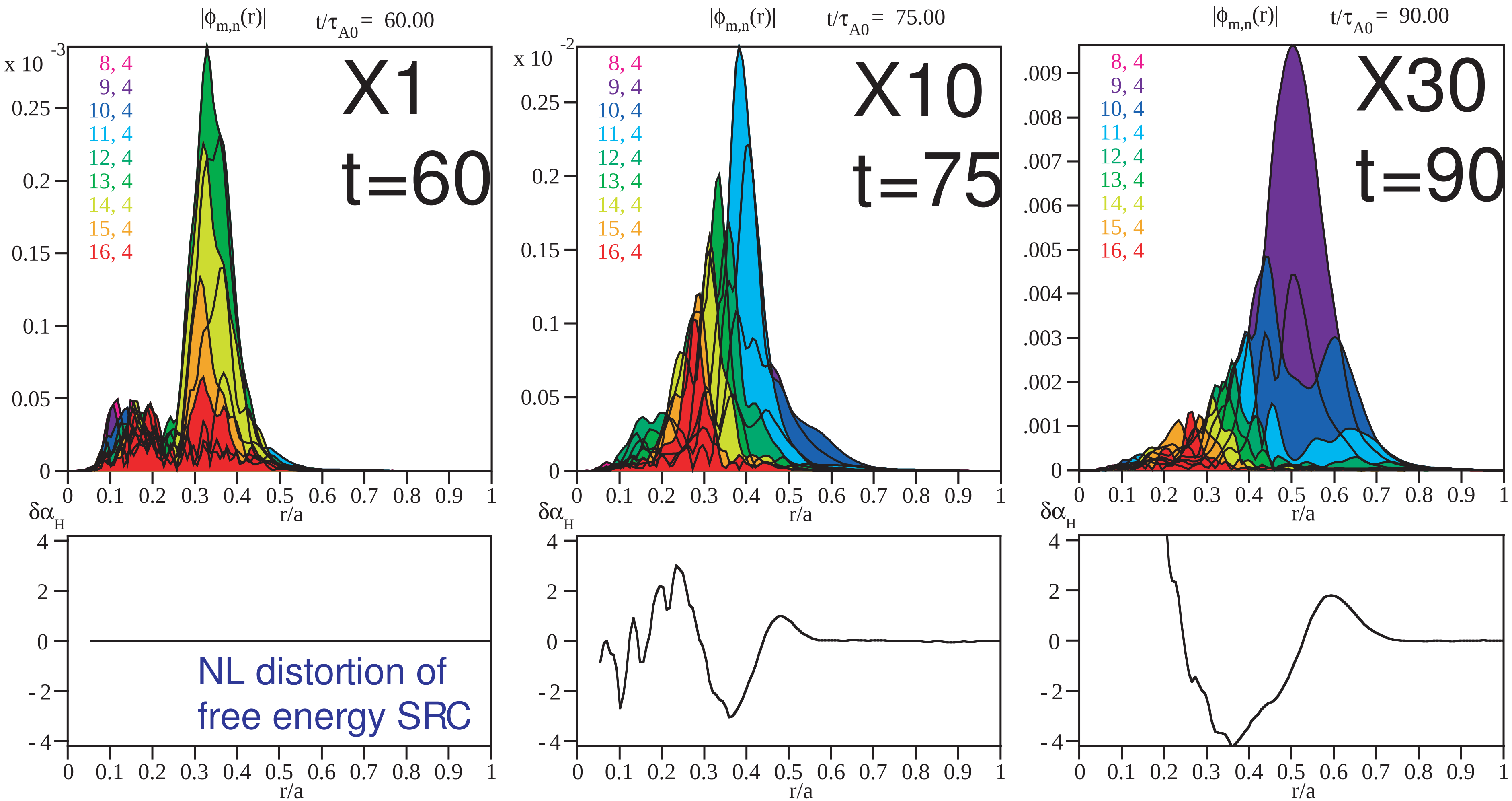
- Electron Fishbone bursty behavior and energetic electron transport
- Turbulence spreading and nonlinear ITG - Zonal Flow interactions (Chen et al. $00 \div 09$)
- In general, this method can be adopted for investigating long time-scale behavior of nonlinear dynamics that are mediated by zonal structures (envelope modulations in space, also reflecting on velocity space structures): cross-scale couplings ...

Zonca et al. IAEA, (2002)

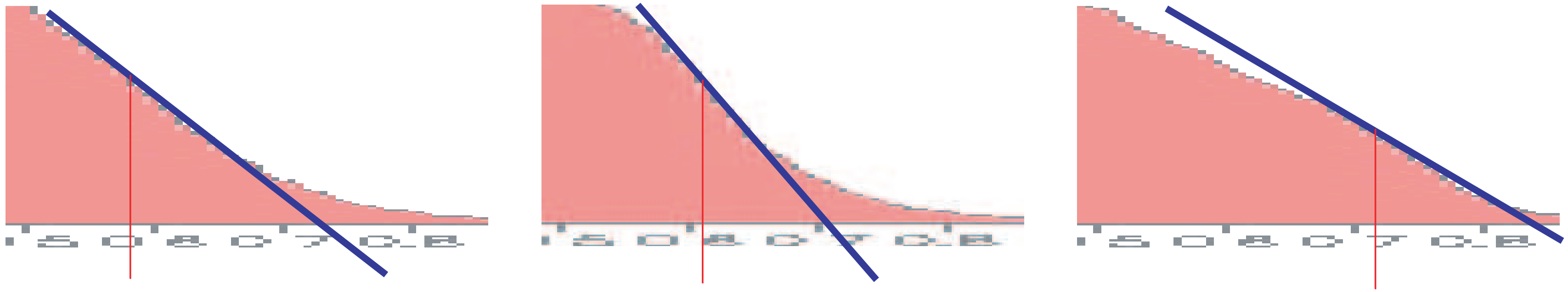
Avalanches and NL EPM dynamics

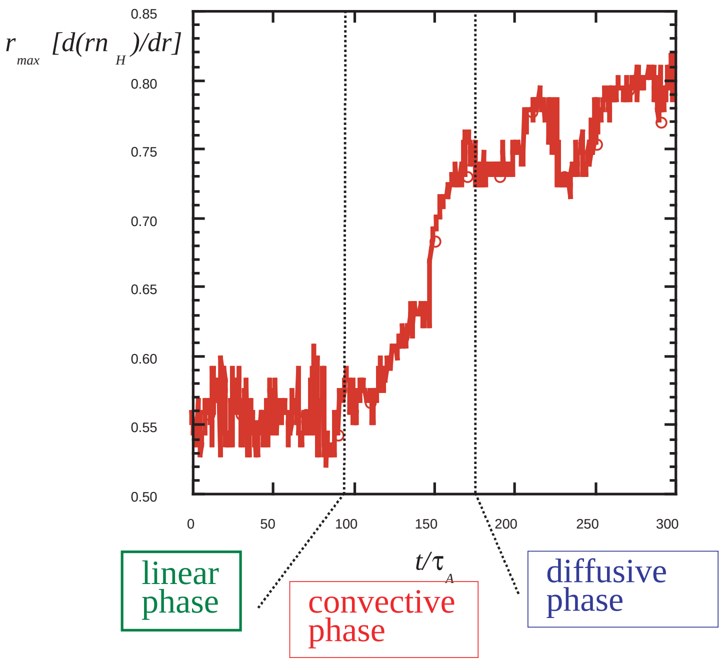


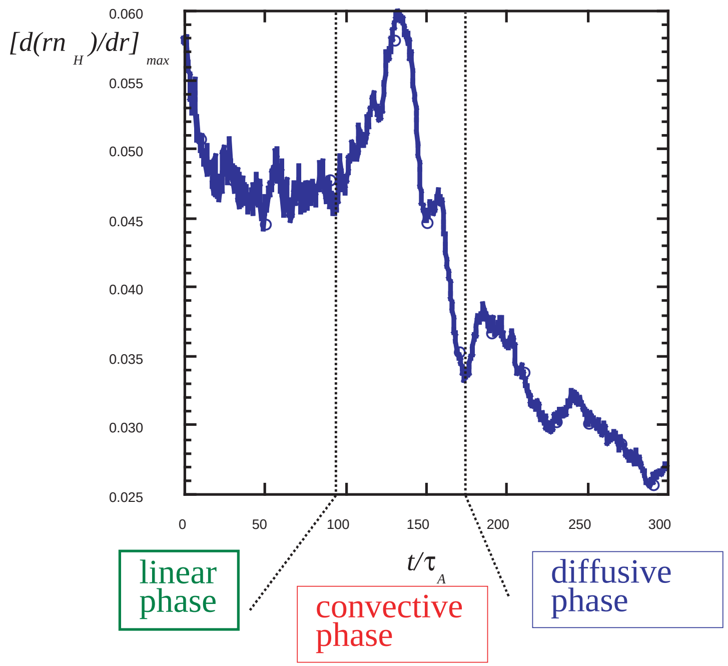
□ Importance of toroidal geometry on wave-packet propagation and shape

Vlad et al. IAEA-TCM, (2003)

Propagation of the unstable front

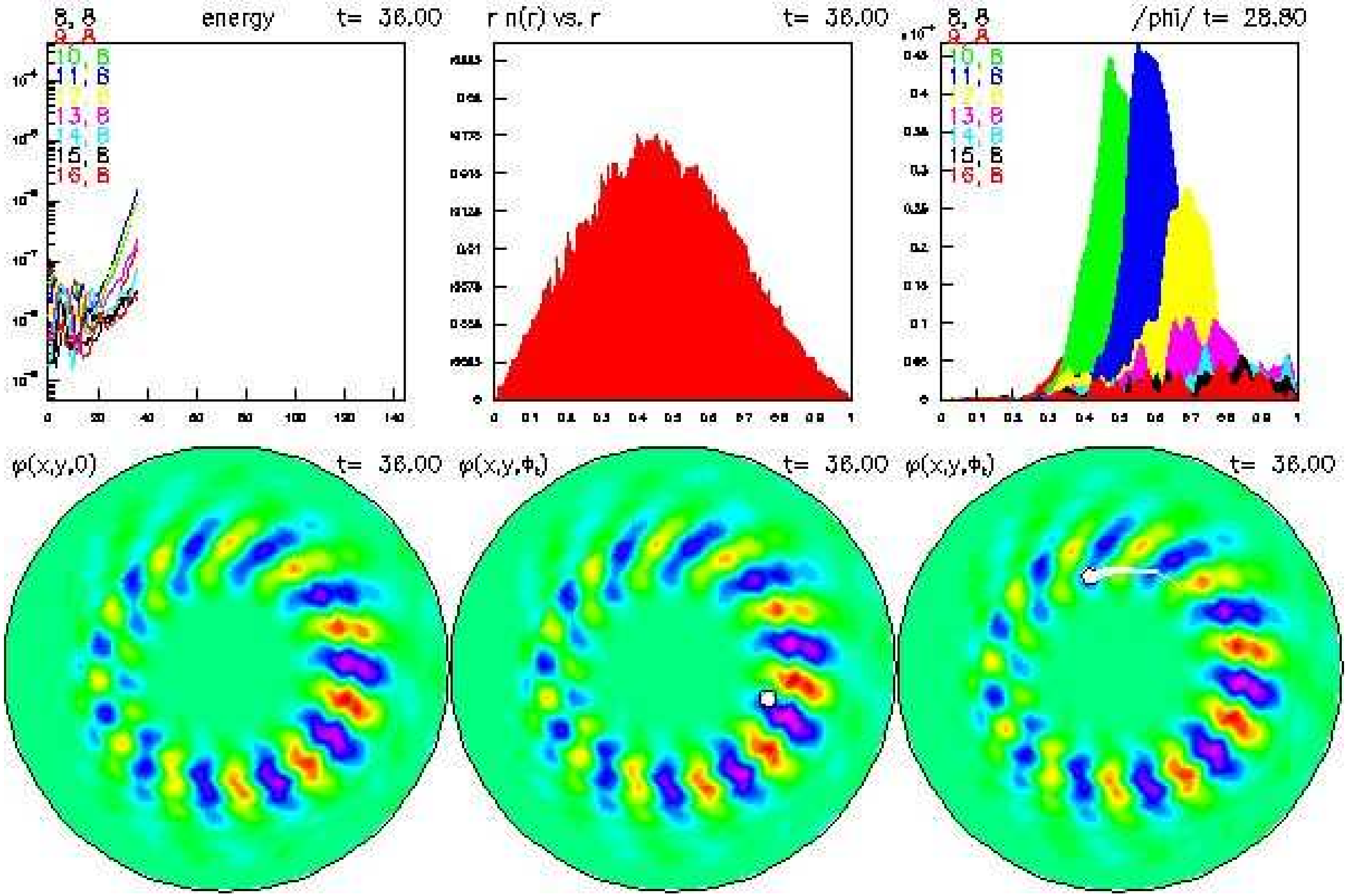




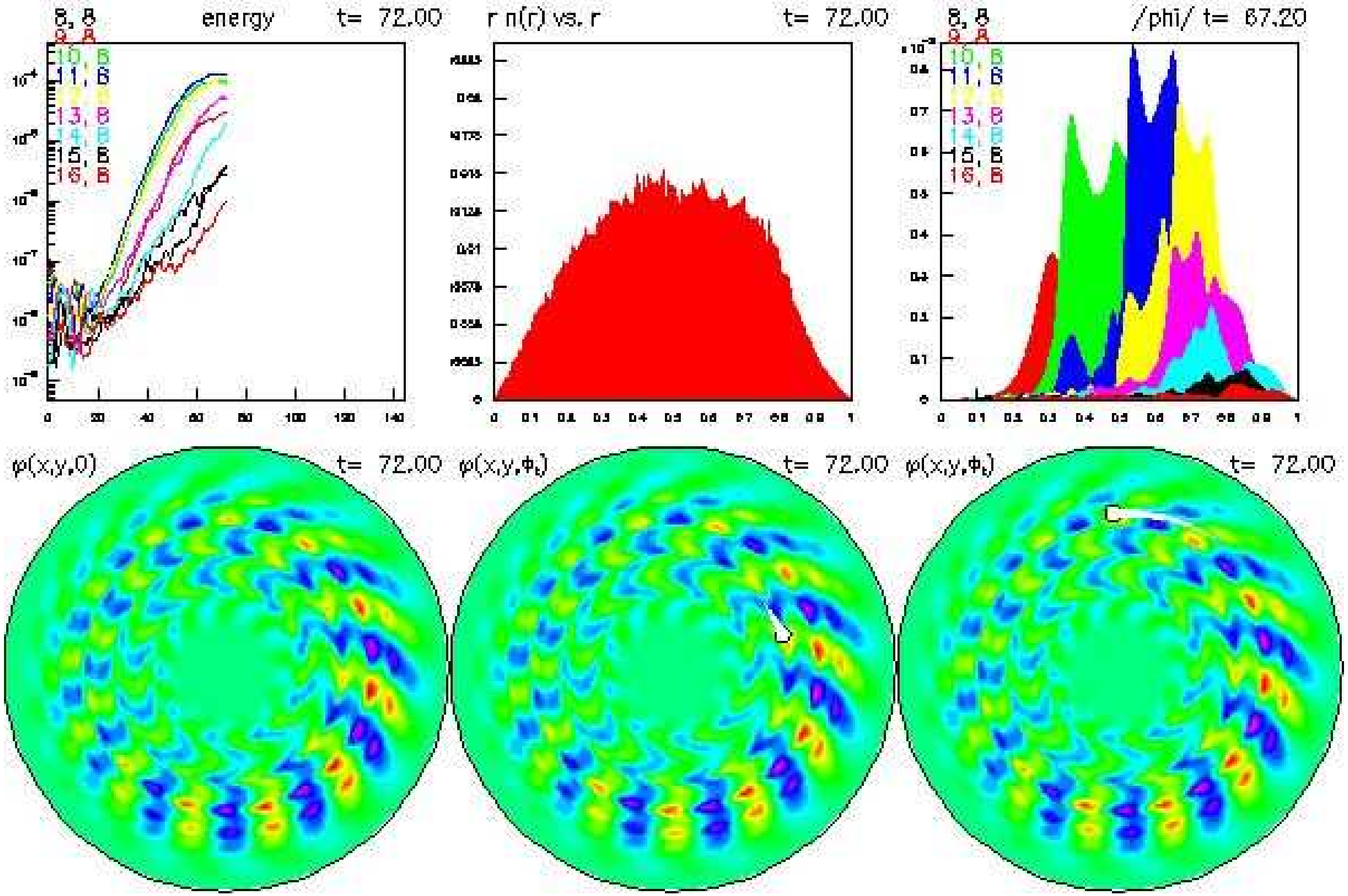


- Gradient steepening and relaxation: spreading ... similar to turbulence

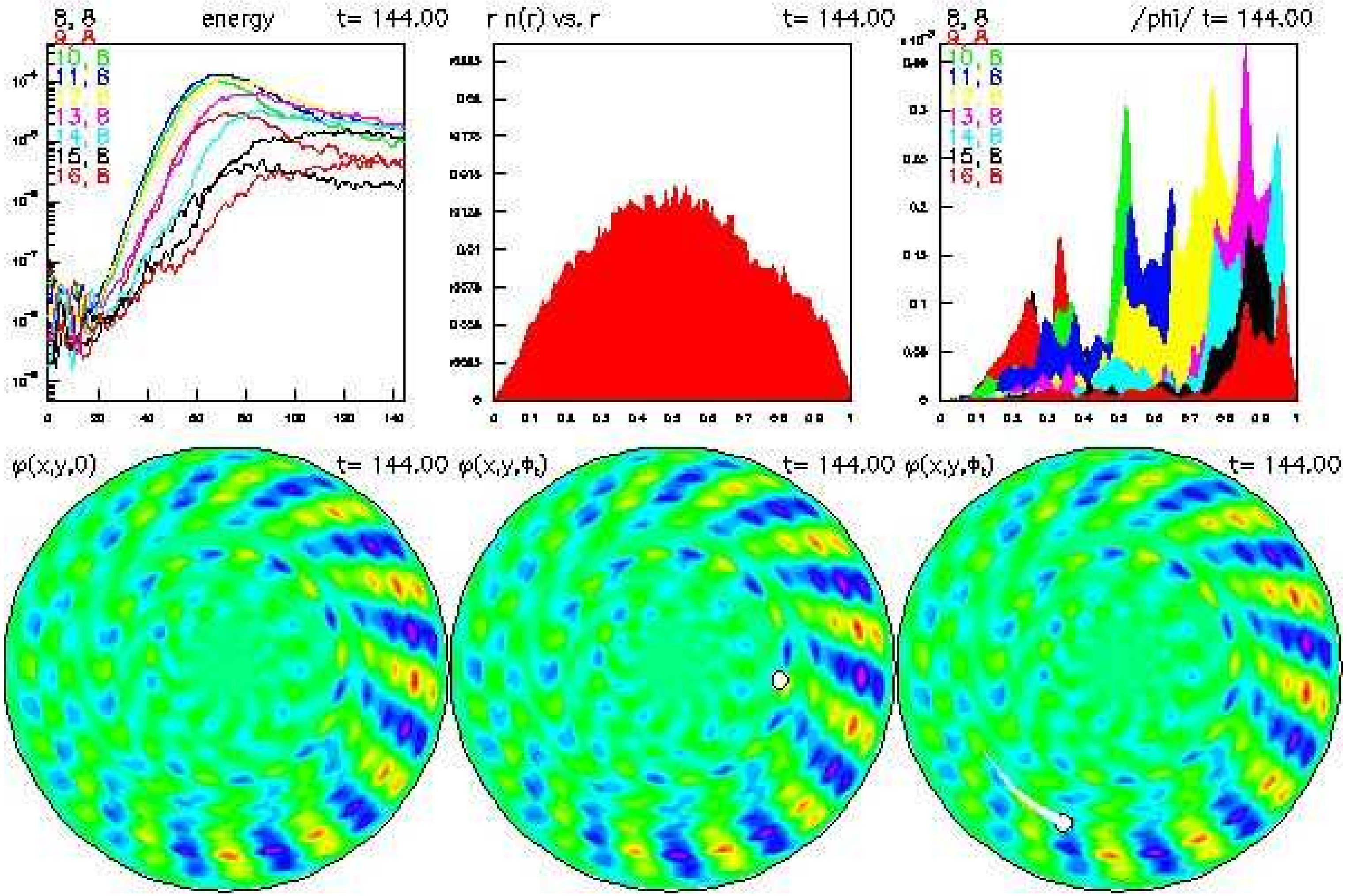
EPM modulational instability I (Zonca et al 00)



EPM modulational instability II (Zonca et al 00)

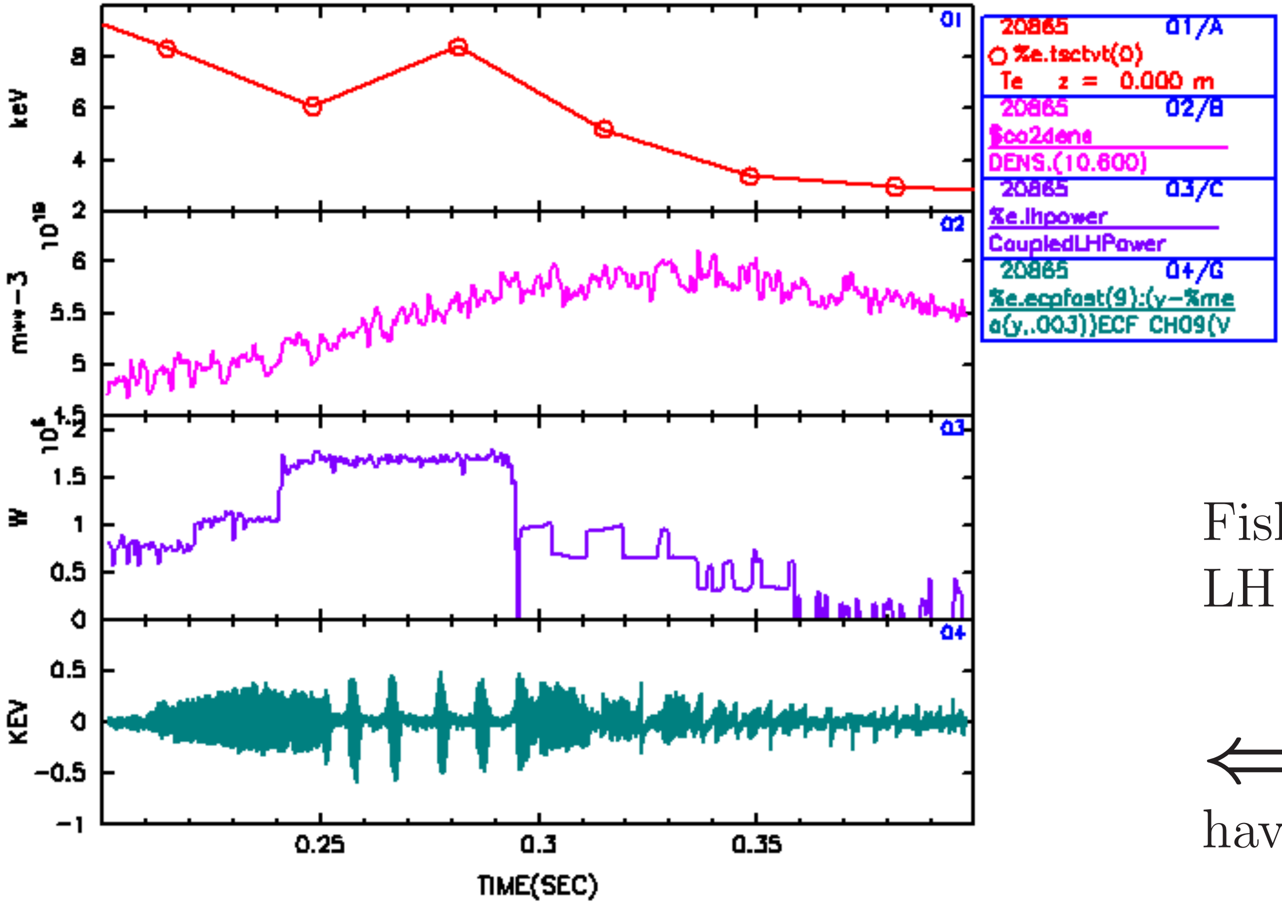


EPM modulational instability III (Zonca et al 00)



Electron Fishbone bursty behavior

P. Smeulders, et al., ECA 26B, D-5.016, 2002



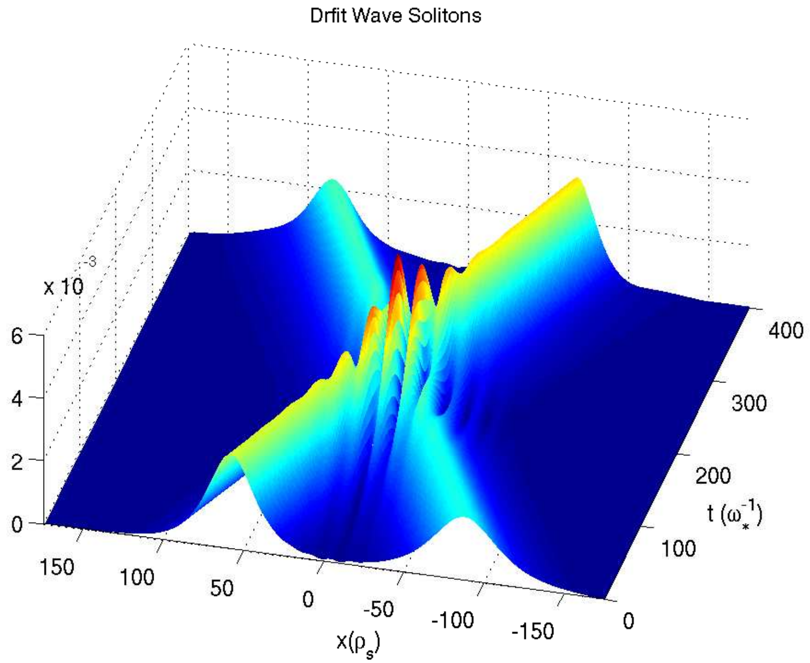
FTU observations
(IAEA 06)
Of interest for
EAST and HL-2A

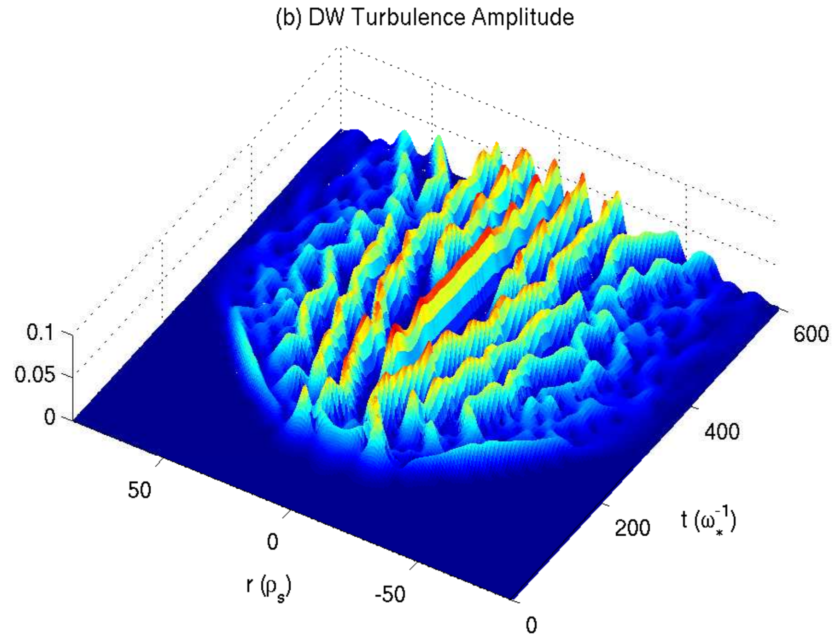
Fishbones are visible with only when LH power is on

$\Leftarrow$ Clear transition in nonlinear behavior as LH power is increased

Radial Spreading of Drift Wave-Zonal Flow Turbulence via Soliton Formation

Z. Guo and L. Chen 2008. Z. Guo et al Phys. Rev. Lett. 2009





RP: Using Lecture 5 framework, can you derive the equation for the EPM/TAE modulational instability (Zonca et al 00)? Hint: assume that the radial envelope is perturbed by $\sim \exp(ik_z r - i\omega_z t)$ and note that modulation of mode structure is modulating $\hat{F}_0$ as well.

E: Can you comment on differences/similarities of this process with that of hole-clump formation in the 1D beam-plasma system?

The form of nonlinear interactions in a torus

- Mode structures have *3 degrees of freedom*
  - toroidal mode number
  - parallel mode structure (radial width of poloidal harmonics)
  - mode radial envelope
- Nonlinear interactions take 3 forms
  - mode couplings between two n's
  - distortion of parallel mode structure
  - modulation of the radial envelope

- The focus of this intensive course was on *radial modulations*, namely nonlinear dynamics induced by distortions of the particle distribution function (spatially averaged on the periodic angular variations).
- This lecture: focus on zonal structures; i.e. modes that can alter the spatially averaged system behaviors on the periodic angular variations.
- Linear stable structures: these modes are driven only nonlinearly for symmetry reasons. Or they are driven by velocity space non-uniformity.
- Concept of zonal structures can be extended to the phase space ⇒ envelope modulations in space may also be reflecting velocity space structures with possible mutual feedbacks.

Special role of zonal structures:

- due to intrinsic coherent nature they can yield "spontaneous" modulational instability $\Rightarrow \gamma_{NL} \propto |A|$
- for the same reason, they can yield nonlinear upshift of thresholds (transport); e.g. for ITG, which is the focus of this lecture
- long time scale behaviors are very likely regulated by the dynamics of zonal structures (mutual feedback with different players/degrees of freedom otherwise weakly interacting or uncoupled) $\Rightarrow$ complex system behaviors

The form of zonal structures

- Neighboring Nonlinear Equilibria (Chen 07): assume to add a small perturbation $\delta\phi_z(r)$ to an equilibrium (Maxwellian) distribution function $F_0(\mu,\epsilon)$, $(\mu = v_\perp^2/2B,\ \epsilon = v^2/2)$. Given the new Hamiltonian $H = \epsilon + (e/m)\delta\phi_z$, the new equilibrium is:

$$F(\mu,\epsilon,r) = F_0(\mu,\epsilon) + (e/m)\delta\phi_z(\partial F_0/\partial\epsilon) + \exp\left(-i\mathbf{k}_z\cdot\mathbf{v}\times\mathbf{b}/\omega_c\right)\delta\bar{H}_z$$

- Spatio-temporal scale separation of zonal structures: efficient interaction occurs at $|\omega_z| < \gamma$ and $|k_z| \lesssim \Delta^{-1} \ll |k_\perp|$, with Δ the radial correlation length (envelope). Efficient interactions with other orderings are possible.

- Kinetic stability analyses of *neighboring nonlinear equilibria* with respect to low frequency drift waves with $|k_\perp| \gg |k_z|$, ($2\pi/|k_\perp|$ and $2\pi/|k_z|$ drift wave and zonal flow wavelengths).

- Solve for the *zonal response* $\delta\bar{H}_z$ moving to the banana frame (Tsai and Chen 93).

- Magnetic drifts
  E: Show that $\boldsymbol{v}_D = (v_\parallel/\omega_c)\boldsymbol{\nabla}\times(\boldsymbol{b}v_\parallel)$. Hint: $v_\parallel^2 = 2(\epsilon-\mu B)$.
  E: Show that, at low-β, $\boldsymbol{v}_D = (v_\parallel B_0/\omega_{c0})\boldsymbol{\nabla}(v_\parallel/B)\times\boldsymbol{b}$.

- Express magnetic drift frequency as

$$\omega_d = \frac{v_\parallel B_0}{\omega_{c0} r}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right)\left(-i\frac{\partial}{\partial r}\right) = \frac{v_\parallel B_0}{\omega_{c0} r}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right)k_z$$

- This result is general (at low-β; but it can be easily extended) for the *zonal response*: $\boldsymbol{k} = k_z\boldsymbol{\nabla}r$

- Gyrokinetic (linear) equation for $\delta\bar{H}_z$ is

$$\left(\partial_t + v_\parallel\partial_\ell + i\omega_d\right)\delta\bar{H}_z = -\frac{e}{m}\frac{\partial F_0}{\partial\epsilon}J_0(\gamma_z)\frac{\partial}{\partial t}\delta\phi_z + NL$$

$$\left[\partial_t + \frac{v_\parallel}{qR_0}\frac{\partial}{\partial\theta} + \frac{v_\parallel B_0}{\omega_{c0} r}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{B}\right)\left(\frac{\partial}{\partial r}\right)\right]\delta\bar{H}_z = -\frac{e}{m}\frac{\partial F_0}{\partial\epsilon}J_0(\gamma_z)\frac{\partial}{\partial t}\delta\phi_z + NL$$

- As usual (Rosenbluth and Hinton 98)

$$\delta \bar{H}_z = \exp\left(-iQ_z\right) H_z \quad ,$$

$$Q_z = \frac{q}{(r/R_0)} \left(\frac{v_\parallel}{\omega_c} - \frac{\overline{v_\parallel}}{\omega_c} \right) k_z \quad ,$$

with $\mathbf{b} \cdot \nabla H_z = 0$, $\overline{[...]} = \oint [...] (d\ell / v_\parallel) / \oint (d\ell / v_\parallel)$ and having also considered a high aspect-ratio tokamak equilibrium, $R_0/r \gg 1$

- Q_z indicates the generator of coordinate transformation from particle guiding-center to magnetic drift/banana-center
- H_z describes the generic and "arbitrary" nonlinear equilibrium with flow $\delta\Phi_z$. From nonlinear GKE (Frieman and Chen 82)

$$\frac{\partial}{\partial t} H_z = \frac{e}{T} F_0 \overline{\left[e^{iQ_z} J_0(\gamma_z) \frac{\partial}{\partial t} \delta\phi_z \right]} + \overline{\left[e^{iQ_z} (NL) \right]}$$

- In the e.s. limit (see also Spring 2009 Intensive Course Lecture 6)

$$
\begin{aligned}
\frac{\partial}{\partial t} H_z &= \frac{e}{T} F_0 \overline{\left[e^{iQ_z} J_0(\gamma_z) \frac{\partial}{\partial t} \delta\phi_z \right]} - \sum_{\mathbf{k}_z = \mathbf{k}' + \mathbf{k}''} \overline{\left[e^{iQ_z} \delta \bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta \bar{H}_{k''} \right]} \;, \\
&= \frac{e}{T} F_0 \overline{\left[e^{iQ_z} J_0(\gamma_z) \frac{\partial}{\partial t} \delta\phi_z \right]} + \sum_{\mathbf{k}_z = \mathbf{k}' + \mathbf{k}''} i \frac{c}{B_0} k'_\theta \frac{\partial}{\partial r} \overline{\left[e^{iQ_z} J_0(\gamma') \delta\phi_{k'} \delta \bar{H}_{k''} \right]} \;,
\end{aligned}
$$

- $k'_\phi = -k''_\phi$, $k'_\theta = -k''_\theta$, $\gamma_k = k_\perp v_\perp / \omega_c$ and $\delta \bar{\mathbf{u}}_{Ek}$ the nonlinear $\mathbf{E} \times \mathbf{B}$

$$
\delta \bar{\mathbf{u}}_{Ek} = \mathrm{i} \frac{c}{B} \left(\mathbf{b} \times \mathbf{k}_\perp \right) J_0(\gamma) \delta\phi_k
$$

- Very general equation describing the DW-ZF generation and the regulation of the DW intensity by the ZFs as well as the formation of the nonlinear equilibria due to the zonal structures on the long time scales. (L. Chen, 2C03, Sherwood 2004; and US-TTF 2004).

- No assumption of any specific wavelength ordering for neither DWs nor zonal structures: nonlinear term reduces to the well-known form of Reynolds stress (A. Hasegawa et al. 79) only in the long wavelength limit.

- Total bounce/transit averaged zonal structure response by direct construction (with $\Gamma_{dz} = \overline{\exp\left(iQ_z\right)}$)

$$\overline{\delta F_z} = -\frac{e}{T} F_0 \delta\phi_z + J_0(\gamma_z)\Gamma^*_{dz} H_z$$

- Applications and examples

  - ZF are important in regulating turbulence: "magic" state in which all free energy is converted into laminar flow? (zonal flow)

  - Flow patterns are broken by turbulence $\Leftrightarrow$ nonlinear upshift

Application to ITG

- ITG nonlinear dynamics: consider the fluid approximation for ions, $|\partial_t| \approx |\omega| \gg |v_\parallel \partial_\ell|, |\omega_d|, |\delta\bar{\mathbf{u}}_{Ek'} \cdot \nabla| \approx \gamma_\ell, \gamma_{n\ell}$, where γ_ℓ and $\gamma_{n\ell}$ are, respectively, the inverse linear and nonlinear time scales.
- For the kth normal mode of ITG we can assume, at the lowest order,

$$\delta\bar{H}_{i,k} \simeq \frac{e}{T_i} F_{i,0} \left(1 - \frac{\omega_{*i}}{\omega}\right)_k J_0(\gamma)\delta\phi_k \quad ,$$

  where $\omega_{*i} = \omega_{*ni} + \omega_{*Ti}\,(m_i\epsilon/T_i - 3/2)$, $\omega_{*ni} = T_i \mathbf{k} \times \mathbf{B}/B \cdot \nabla n_i/(n_i m_i \omega_{ci})$ and $\omega_{*Ti} = \mathbf{k} \times \mathbf{B}/B \cdot \nabla T_i/(m_i\omega_{ci})$.

- By direct substitution one derives (Chen 07)

$$\overline{\delta F_{i,z}} = -\frac{e}{T_i} F_{i,0} \left\{ \left(1 - J_0^2(\gamma_z)|\Gamma_{dz}|^2\right) \delta\phi_z + J_0(\gamma_z)|\Gamma_{dz}|^2 \left[\alpha_0 - \alpha_1 \left(\frac{m_i v^2}{2T_i} - \frac{3}{2}\right)\right] A_z \right\}$$

- Here we use the following definition, useful for symbolic manipulations

$$\left[\alpha_0 - \alpha_1\left(\frac{m_i v^2}{2T_i} - \frac{3}{2}\right)\right] A_z = \sum_{\mathbf{k}_z=\mathbf{k}'+\mathbf{k}''}\left(1 - \frac{\omega_{*i}}{\omega}\right)_{k''} \overline{\int_0^\infty dt J_0(\gamma'')\bar{\mathbf{u}}_{Ek'}\cdot\nabla\delta\phi_{k''}}$$

- These equations directly connect the nonlinear equilibria due to the zonal structures with the original plasma equilibrium via the time history of the DW turbulence. Note that the cumulative effect of ITG turbulence enters (time integration).

- In the simplest toroidal ITG case, with $\omega \simeq \omega_d(3 + 1/\tau)$ and $\tau = T_i/T_e$,

$$\begin{aligned}\alpha_0 &\simeq 1 - \omega_{*ni}/\omega \simeq 1 - (R_0/L_n)/(3 + 1/\tau)\\ \alpha_1 &\simeq \omega_{*Ti}/\omega \simeq (R_0/L_T)/(3 + 1/\tau)\end{aligned}$$

- The value of A_z, α_0 and α_1 are determined from "consistency" requirement of the NL GKE solution and the nonlinear equilibrium with flow.

- Introduce the flux surface average operator

$$\langle\langle ... \rangle\rangle = \left(\oint (d\ell/B) \right)^{-1} \left(\oint (d\ell/B) \langle ... \rangle \right) = 2\pi \left(\oint (d\ell/B) \right)^{-1} \sum_{\mathrm{sgn}(v_\parallel)=\pm} \int d\mu d\epsilon \tau_b \overline{(...)} \ ,$$

with $\langle ... \rangle$ indicating velocity space integration

$$\langle ... \rangle = 2\pi \sum_{v_\parallel/|v_\parallel|=\pm 1} \int (B/|v_\parallel|) d\mu d\epsilon$$

- Since electrons do not respond to ZF:

$$\delta n_{i,z} = \langle\langle \delta F_{i,z} \rangle\rangle = 0 \qquad \Longrightarrow \qquad \chi_i \delta\phi_z = -\alpha_0 A_z \ .$$

- In the long wavelength limit (Rosenbluth and Hinton 98)

$$\chi_i = \left\langle\left\langle (F_0/n_i) \left(1 - J_0^2(\gamma_z) |\Gamma_{dz}|^2 \right) \right\rangle\right\rangle \simeq 1.6 q^2 (R_0/r)^{1/2} k_z^2 \rho_i^2 \ .$$

- By definition, zonal temperature fluctuation is given by

$$\frac{\delta T_{i,z}}{T_i} = \left\langle\left\langle \frac{2m_i\mathcal{E}}{3T_i}\frac{\delta F_{i,z}}{n_i}\right\rangle\right\rangle = -\frac{5}{3}\chi_i\frac{e}{T_i}\delta\phi_z + (\alpha_1-\alpha_0)\frac{e}{T_i}A_z = -\left(\frac{\alpha_1}{\alpha_0}+\frac{2}{3}\right)\chi_i\frac{e}{T_i}\delta\phi_z \ .$$

- Thus, nonlinear zonal structures in the ITG case give

$$\delta T_{i,z}/(e\delta\phi_z) \simeq -(\alpha_1/\alpha_0 + 2/3)\chi_i < 0 \ ,$$

$$\frac{\alpha_1}{\alpha_0} \simeq \frac{\omega_{*Ti}}{1-\omega_{*ni}/\omega} \simeq \frac{R_0/L_T}{3+T_e/T_i - R_0/L_n} \quad \text{for ITG}$$

corresponding to temperature and potential fluctuations in anti-phase, consistently with numerical simulation results (S.E. Parker et al. 99).

- Time asymptotic behavior of the ITG-ZF system (long wavelength), describing the formation of zonal structures in the nonlinear equilibrium in the presence of zonal flows, can be written as

$$H_{i,z} = \frac{e}{T_i}\delta\phi_z F_{i,0} + \left[\frac{m_i\mathcal{E}}{T_i}\frac{(\alpha_1/\alpha_0 + \Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2}\right]\frac{\delta T_{i,z}}{T_i}F_{i,0} \quad .$$

- $\Delta(\lambda)$, with $\lambda = \mu/\mathcal{E}$, defined such

$$\Gamma_{dz} = 1 - \Delta(\lambda) m_i \mathcal{E}/T_i$$
$$\langle\langle (F_{i,0}/n_i)\Delta(\lambda) m_i\mathcal{E}/T_i\rangle\rangle = \chi_i/2 \,.$$

- This distribution function may be unstable to trapped ion ITG $\Rightarrow$ zonal structures are broken $\Rightarrow$ nonlinear threshold upshift.

- Nonlinear equilibrium paradigm for computing the nonlinear up-shift for ITG turbulent transport.

Excitation of tertiary modes

- For nonlinear equilibrium paradigm, assume:

$$\overline{\delta F_{i,z}} = \left[\frac{m_i \mathcal{E}}{T_i} \frac{(\alpha_1/\alpha_0 + 2\Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2} \right] \frac{\delta T_{i,z}}{T_i} F_{i,0} \quad .$$

E: Show this, using $H_{i,z}$ from p. 23

- Meanwhile, $\delta \bar{H}_{e,z} = -(e/T_e)\delta\phi_z$, which exactly cancels the electron adiabatic response. Thus, electron do not contribute to forming zonal structures in the ITG case; $\delta F_{e,z} = 0$.

- The quasi-neutrality condition for one ion species with $n_i = n_e = n_0$ reads

$$\left(1 + \frac{T_i}{T_e}\right) \delta\phi_k = \frac{T_i}{n_0 e^2} \left\langle e J_0(\gamma_k) \delta \overline{H}_k \right\rangle_i$$

Here, electrons are assume adiabatic for the $n \neq 0$ modes, $\delta \bar{H}_{e,k} = 0$.

- $\delta\overline{H}_{i,k}$ is obtained from the nonlinear gyrokinetic equation (Frieman and Chen 82).
- Assume low frequency DW, $|\omega\tau_b| \ll 1$, with $|k_\perp| \gg |k_z|$.
  - The frequency ordering is consistent with previous investigations of zonal structure stability in the presence of K–H modes (Rogers et al 00; Hinton et al 00).
  - The condition on the wave-vector is given by the requirement that modes are localized within the potential wells associated with the zonal structures.

- Drop ion subscript i (for simplicity) unless needed and define

$$\delta \bar{H}_k = e^{-iQ_k} \delta H_{bk} \quad ,$$

where Q_k indicates the generator of coordinate transformation from particle guiding-center to magnetic drift/banana-center.

- Solve for drift wave response in the nonlinear equilibrium modified by zonal structures:

$$(\bar{\omega}_d - \omega)_k \, \delta H_{bk} = -\frac{e}{T} F_0 \, (\omega - \omega_*)_k \, \overline{\left[e^{iQ_k} J_0(\gamma_k) \delta\phi_k \right]} + i \overline{\left[e^{iQ_k} \delta \bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta \bar{H}_{k''} \right]}$$

$$i \overline{\left[e^{iQ_k} \delta \bar{\mathbf{u}}_{Ek'} \cdot \nabla \delta \bar{H}_{k''} \right]} = i \delta \bar{\mathbf{u}}_{Ez} \cdot \boldsymbol{\nabla} \delta H_{bk} + \overline{\left[e^{iQ_k} J_0(\gamma_k) \delta\phi_k \right]} \frac{c}{B} k_\theta \frac{\partial}{\partial r} \delta \bar{H}_z$$

$$(\bar{\omega}_d - \omega + \omega_z) \, \delta H_{bk} = \left(\omega_* - \omega + \omega_{*T}^{n\ell} + \omega_z \right) \frac{e}{T} F_0 \overline{\left[e^{iQ_k} J_0(\gamma_k) \delta\phi_k \right]}$$

- Bounce averaged ion response (nonlinear):

$$\delta H_{bk} = \left[1 + \frac{\omega_* + \omega_{*T}^{n\ell} - \bar{\omega}_d}{\bar{\omega}_d + \omega_z - \omega}\right] \overline{[J_0(\gamma_k) e^{iQ_k} \delta\phi_k]} \frac{e}{T_i} F_0 \quad ,$$

$$\omega_{*T}^{n\ell} = \left[\frac{m_i \mathcal{E}}{T_i} \frac{(\alpha_1/\alpha_0 + \Delta(\lambda)/\chi_i)}{(\alpha_1/\alpha_0 + 2/3)} - \frac{3}{2}\right] J_0(\gamma_z) \frac{ck_\theta}{eB_0} \frac{\partial}{\partial r} \delta T_{i,z} \quad ,$$

$$\omega_z = \frac{c}{B_0} k_\theta J_0(\gamma_z) \frac{\partial}{\partial r} \delta\phi_z \quad ,$$

- Definitions:
  - $\omega_{*T}^{n\ell}$ is the nonlinear diamagnetic frequency associated with the zonal structures
  - ω_z is the $\mathbf{E} \times \mathbf{B}$ frequency shift due to ZF

- By direct substitution, and using $|\bar{\omega}_d| \ll |\omega_*|$, the quasi-neutrality condition becomes:

$$\left(1+\frac{T_i}{T_e}\right)\delta\phi_k = 4\pi\int d\mu d\epsilon \frac{B}{|v_\parallel|}\frac{F_0}{n_i}\left\{J_0(\gamma_k)e^{-iQ_k}\left[1+\frac{\omega_*+\omega_{*T}^{n\ell}}{\bar{\omega}_d+\omega_z-\omega}\right]\overline{[J_0(\gamma_k)e^{iQ_k}\delta\phi_k]}\right\} \quad .$$

- The solution of the integral equation exist since it is possible to construct a variational principle

$$V = \frac{N(\delta\phi_k^*, \delta\phi_k)}{D(\delta\phi_k^*, \delta\phi_k)}$$

- $N(\delta\phi_k^*, \delta\phi_k)$ and $D(\delta\phi_k^*, \delta\phi_k)$ are bilinear in $\delta\phi_k^*$, $\delta\phi_k$ and V has an extremum for $N(\delta\phi_k^*, \delta\phi_k) = D(\delta\phi_k^*, \delta\phi_k)$ if the quasi-neutrality condition is satisfied.

$$N(\delta\phi_k^*, \delta\phi_k) = 4\pi \int d\mu d\epsilon \frac{F_0}{n_i} \left(\int \frac{d\ell}{|v_\parallel|} / \oint \frac{d\ell}{B} \right) \left[\left(1 + \frac{T_i}{T_e} \right) \overline{|\delta\phi_k|^2} - \left| \overline{J_0(\gamma_k) e^{iQ_k} \delta\phi_k} \right|^2 \right] ,$$

$$D(\delta\phi_k^*, \delta\phi_k) = 4\pi \int d\mu d\epsilon \frac{F_0}{n_i} \left(\int \frac{d\ell}{|v_\parallel|} / \oint \frac{d\ell}{B} \right) \frac{\omega_* + \omega_{*T}^{n\ell}}{\bar{\omega}_d + \omega_z - \omega} \left| \overline{J_0(\gamma_k) e^{iQ_k} \delta\phi_k} \right|^2 .$$

- The contribution of circulating particles to the non-adiabatic response δH_{bk} is vanishingly small. E: Show this!
- The main contribution comes from trapped and barely circulating particles.
- Mode structures:
  - Finite orbit widths of trapped as well as barely circulating particles give the radial dispersion of the waves
  - The radial profile of ω_z and $\partial_r \delta T_{i,z}$ provide the radial potential well

- The condition for bound states to exist is that the mode be localized near a minimum of the potential well at radial positions where (Rogers *et al.* 00)

$$\partial_r \omega_z = \partial_r^2 \delta\phi_z = \partial_r^2 \delta T_{i,z} = 0 \quad .$$

- The condition is readily given in the long wavelength limit.

- The radial scale is obtained by balancing radial dispersiveness and potential well variations near its minimum and is given by

$$|k_r|^2 \rho_{bi}^2 \approx |k_z|^2 \Delta r^2 \approx |k_z/k_r|^2 \quad ; \qquad \rho_{bi} = q\rho_i (R_0/r)^{1/2} \quad .$$

- Consistency of the long wavelength limit and mode localization:

$$|k_r|\rho_{bi} \approx |k_z \rho_{bi}|^{1/2} \ll 1 \quad ; \text{ long wavelength ;}$$
$$|k_z/k_r| \ll 1 \quad ; \text{ mode localization .}$$

Marginal stability condition: local estimate

- A simple estimate of the low-frequency trapped ion temperature gradient driven modes (TITG) threshold can be obtained assuming deeply trapped particles, for which

$$\bar{\omega}_d = \bar{\omega}_{dT}(\epsilon/v_T^2) = -(k_\theta \epsilon)/(R_0 \omega_{c0})$$

- From the variational principle $N(\delta\phi_k^*, \delta\phi_k) = D(\delta\phi_k^*, \delta\phi_k)$, neglecting FLR/FOW effects and considering also that, typically, $\alpha_1/\alpha_0 \gg 1$

$$\overline{[J_0(\gamma_k) e^{iQ_k} \delta\phi_k]} \rightarrow \overline{\delta\phi_k}$$

$$\frac{\omega_* + \omega_{*T}^{n\ell}}{\bar{\omega}_d + \omega_z - \omega} \simeq \frac{(m_i \epsilon/T_i - 3/2)}{\bar{\omega}_{dTi}(m\epsilon/T_i) + \omega_z - \omega} \frac{ck_\theta}{eB_0} \frac{\partial}{\partial r} (T_i + \delta T_{i,z})$$

- At marginal stability, the TITG mode frequency satisfies the condition:

$$\omega = \omega_z + \frac{3}{2}\bar{\omega}_{dTi} \quad ; \quad \frac{\omega_* + \omega_{*T}^{n\ell}}{\bar{\omega}_d + \omega_z - \omega} \simeq -\frac{R_0}{T_i}\frac{\partial}{\partial r}\left(T_i + \delta T_{i,z}\right)$$

- Introduce the notation $\kappa^2 = 2(r/R_0)\lambda/[1-(1-r/R_0)\lambda]$:
  - $\kappa^2 < 1$ $[0 \leq \lambda = (\mu B/\epsilon) < (1-r/R_0)]$ indicates circulating particles;
  - trapped particles have $\kappa^2 > 1$ $[(1-r/R_0) < \lambda = (\mu B/\epsilon) \leq (1+r/R_0)]$.

- Mode structure decomposition:

$$\delta\phi_k = e^{in\phi}\sum_m e^{-im\theta}\delta\phi_m(r) \quad ; \quad \overline{\delta\phi_k} = e^{in(\phi-q\theta)}\sum_l C_l(\kappa)\delta\phi_l(r)$$

$$C_l(\kappa) = \frac{1}{2\pi W(\kappa)}\oint d\theta \frac{\cos(nq-l)\theta}{\left(1-\kappa^2\sin^2(\theta/2)\right)^{1/2}} \; , \qquad \begin{cases} W(\kappa) = \dfrac{2}{\pi}\mathbb{K}(\kappa) & (\kappa^2 < 1) \\ W(\kappa) = \dfrac{2}{\pi\kappa}\mathbb{K}(1/\kappa) & (\kappa^2 > 1) \end{cases}$$

- From the same variational principle at threshold, *i.e.* $\omega \simeq \omega_0 = \omega_z + (3/2)\overline{\omega}_{dTi}$, the critical gradient obeys the condition

$$\left(1+\frac{T_i}{T_e}\right)\delta\phi_m(r) < \left(\frac{2r}{R_0}\right)^{1/2}\int_0^\infty \frac{d\kappa}{\kappa^2}W(\kappa)C_m(\kappa)\left[1-\frac{R_0}{T_i}\frac{\partial}{\partial r}\left(T_i+\delta T_{i,z}\right)\right]\sum_l C_l(\kappa)\delta\phi_l(r) \ .$$

- Generally $\overline{|\delta\phi_k|^2} > |\overline{\delta\phi_k}|^2 \Rightarrow$ the critical gradient is minimized for modes $\delta\phi_k = \delta\phi_m(r)\exp(-im\theta)$ with $|m-nq| \ll 1$.
- q is fixed by the radial localization condition discussed above $\Rightarrow$ the most favorable m is that which minimizes $|m-nq|$ for a given n. This yields:
- Simplified expression for "Dimits" shift in the local limit, connected with the time history of ITG turbulence via neighboring equilibrium expression

$$\left(-\frac{R_0\partial_r\left[\left(T_i+\delta T_{i,z}\right)\right]}{T_i}\right)_{\rm crit} = \frac{(1+T_i/T_e)}{(2r/R_0)^{1/2}\Lambda_m}; \qquad \Lambda_m = \int_0^\infty \frac{d\kappa}{\kappa^2}W(\kappa)C_m^2(\kappa) = O(1) \ .$$

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