Lecture 5

Energetic Particle Modes and the notion of avalanche

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Summary from Lecture 4

- Solved for the renormalized $\hat{F}_0(\omega)$ (fully NL) assuming monochromatic approximation and neglecting high-harmonic generation (stronger phase mixing).

- Further reduction of the problem was possible on the basis of scale separation:

  - description of a strong fishbone burst, with significant distortion of the "equilibrium" $\hat{F}_0$

  - break down of the description when equilibrium is completely destroyed: all particles are ejected from within $q = 1$ surface

- In this Lecture, more rigorous derivations that can be used to reconsider more qualitative (historic) descriptions presented in Lecture 4.

- New twist in this Lecture: deal with high mode number, i.e., look for most unstable TAE/EPM driven by precession resonance ($\omega = \bar{\omega}_d$) for simplicity.
  - mode structure enters importantly: one n but many m's coupled together
  - transition from local saturation to global saturation dynamics
- Understanding different NL regimes is based on the generalized form of $\hat{F}_0(\omega)$ extended to many modes.
- Note that ballooning mode structures (2 radial scale structures) enter importantly.

- Assume deeply trapped particles for simplicity of treatment: $\theta \simeq -\theta_b \sin(\omega_b t)$; $r - r_0 \simeq (v_{d0}\theta_b/\omega_b)\cos\omega_b t$, $|\omega| \simeq |\bar{\omega}_d| \ll \omega_b$. Same as for the fishbone problem, but with additional twists (see Lecture 4).

$$\delta\hat{K}_k(\omega) = -\frac{e}{m}\int_{-\infty}^{\infty}\frac{\bar{\omega}_{dk}}{x}J_0(\lambda_{Lk})J_0^2(\lambda_{dk})\frac{Q_{k,x}\hat{F}_0(\omega-x)}{\omega-\bar{\omega}_{dk}}\delta\hat{\phi}_k(x)dx$$

- Same notation as Lecture 4; $k \to m, n, \omega$, $-k \to -m, -n, -\omega^*$

$$\lambda_{Lk} = k_\perp v_\perp/\omega_c \;\; ; \;\; \lambda_{dk} = q(\mathcal{E}/\epsilon)^{1/2}(\theta_b/\omega_c)k_r \;\; ; \;\; \epsilon = r/R_0 \;\; ; \;\; \mathcal{E} = v^2/2$$

- With the inclusion of FLR/FOW and assuming monochromatic wave

$$\begin{aligned}\hat{F}_0(\omega) \;\; = \;\; & \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \sum_k \frac{c}{\omega B_0}\frac{e}{M}\frac{m}{r}\frac{\partial}{\partial r}\left\{\left[J_0^2(\lambda_{Lk})J_0^2(\lambda_{dk})\frac{Q^*_{k,\omega(\tau)}}{\omega^*(\tau)}\right.\right. \\ & \times \left.\left.\frac{\hat{F}_0\left(\omega - 2i\gamma(\tau)\right)}{\omega-\omega(\tau)+\bar{\omega}_{dk}} + J_0^2(\lambda_{L-k})J_0^2(\lambda_{d-k})\frac{Q_{k,\omega(\tau)}}{\omega(\tau)}\frac{\hat{F}_0\left(\omega-2i\gamma(\tau)\right)}{\omega+\omega^*(\tau)-\bar{\omega}_{dk}}\right]\bar{\omega}_{dk}\left|\delta\phi_{0k}(r,\tau)\right|^2\right\}\end{aligned}$$

- Note that $\sum_k$ indicates $\sum_m$, for n is fixed (single n is assumed for simplicity, but could be generalized).

- Different spatio-temporal scales enter the equation for the renormalized $\hat{F}_0(\omega)$ expression

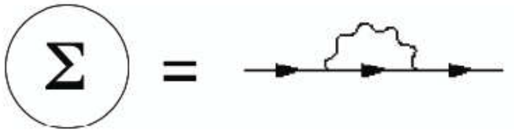
  - the finite interaction time, describing how long a resonant particle effectively exchanges energy with the wave; it can be defined as Γ^{-1}, i.e. the inverse nonlinear de-correlation time

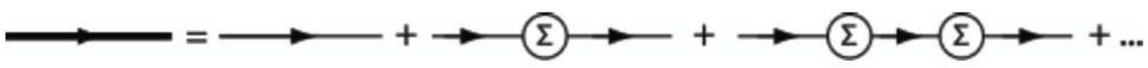
  - the finite interaction length Δr_L, corresponding to the particle radial displacement necessary for detuning the wave-particle resonance by spatial nonuniformities

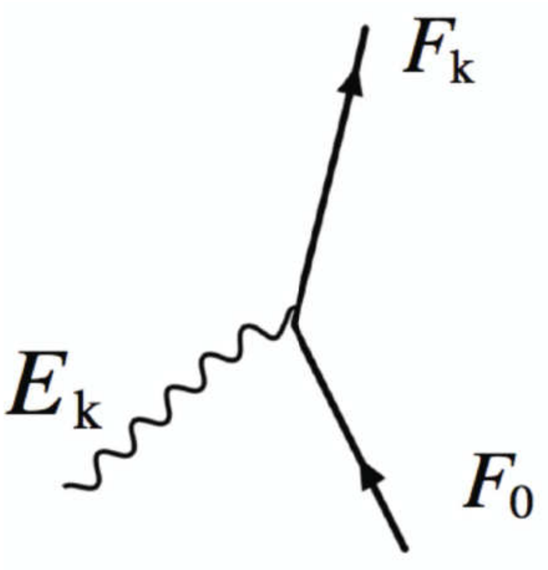
  - the characteristic scale $\sim (nq')^{-1}$ of one poloidal harmonic $\delta\phi_{0k}(r, \tau)$

  - the characteristic scale of the radial envelope

- This solution is valid for strong distortions of the equilibrium distribution function and is the analogue of the Dyson equation ($G = G_0 + G_0 \Sigma G$, with G_0/G the bare/dressed propagators), as noted, e.g., by (Al'tshul' and Karpman 65).



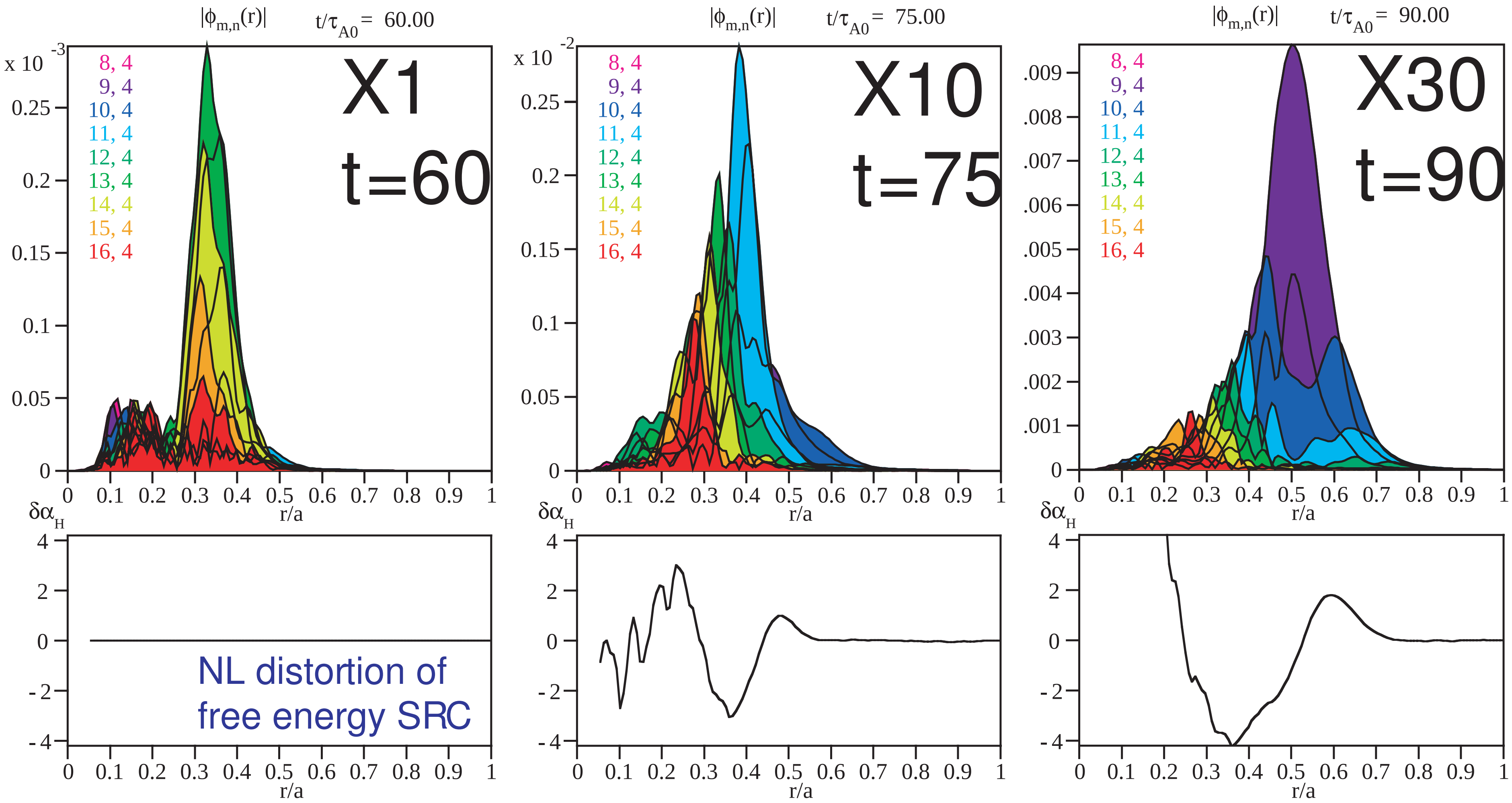




- The description does not include power spectrum generation and spatial bunching and accurate treatment of phase mixing on scales much longer than the wave-particle trapping time. Not a limitation for EPM nonlinear dynamics $\Rightarrow$ convective amplification of unstable front.

Zonca et al. IAEA, (2002)

Avalanches and NL EPM dynamics



- Importance of toroidal geometry on wave-packet propagation and shape

- For the understanding of the transition from weak to strong energetic ion transport and the corresponding change in nonlinear AE/EPM dynamics, it is crucial to note (Zonca et al 05)
  - most unstable modes are characterized by $k_\perp \rho_E \simeq 1$ (ρ_E the energetic ion typical orbit width)
  - the finite interaction length may be typically larger than the radial width of poloidal harmonics, $|\Delta r_L| \gtrsim |nq'|^{-1}$.
  - for chirping modes, with self-consistently determined time-evolving frequency (non-adiabatic regime), the Δr_L can be further extended: detuning due to energy changes is weaker than that due to spatial nonuniformities

Finite interaction length Δr_L

- Consider precession resonance, for simplicity, allowing chirping that, for most unstable EPM, nearly compensates the detuning by spatial nonuniformities. Using $n\dot{E} + \omega \dot{P}_\phi = 0$ and letting $\varepsilon = \bar{\omega}_{dk}/\omega_{*E} \approx r/R_0$

$$\Gamma \lesssim \gamma_L \approx \Delta\left(\bar{\omega}_{dk} - \omega_k(\tau)\right) \simeq \frac{(s-1)}{r}\Delta r_L\left(\bar{\omega}_{dk} - \omega_k(\tau)\right) + \bar{\omega}_{dk}\frac{\Delta E}{E} \approx \bar{\omega}_{dk}\varepsilon\frac{\Delta r_L}{r}$$

- For chirping EPM $\varepsilon \approx r/R_0$, while $\varepsilon \approx 1$ for fixed frequency modes.

$$\hat{F}_0(\omega) = \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \sum_k \frac{c}{\omega B_0}\frac{e}{M}\frac{m}{r}\frac{\partial}{\partial r}\left\{\left[J_0^2(\lambda_{Lk})J_0^2(\lambda_{dk})\frac{Q^*_{k,\omega(\tau)}}{\omega^*(\tau)}\right.\right.$$
$$\left.\left.\times\frac{\hat{F}_0\left(\omega - 2i\gamma(\tau)\right)}{\omega - \omega(\tau) + \bar{\omega}_{dk}} + J_0^2(\lambda_{L-k})J_0^2(\lambda_{d-k})\frac{Q_{k,\omega(\tau)}}{\omega(\tau)}\frac{\hat{F}_0\left(\omega - 2i\gamma(\tau)\right)}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}}\right]\bar{\omega}_{dk}\left|\delta\phi_{0k}(r,\tau)\right|^2\right\}$$

- Focus on optimal balance in $\hat{F}_0(\omega)$ expression
  - NL term must be formally $\mathcal{O}(1)$ for nonlinear dynamics to enter
  - Define Γ as the nonlinear de-correlation rate to be determined
- Estimate [...] on p. 9 and note $\Delta r_L/r \approx \epsilon^{-1}(\gamma_L/\omega_{dk}) \gtrsim \epsilon^{-1}(\Gamma/\omega_{dk})$ (see p. 9)

$$[\ldots] \approx \frac{Q_{k,\omega(\tau)}}{\omega(\tau)} \hat{F}_0 \left(\omega - 2i\gamma(\tau)\right) \frac{2\left(\omega - i\gamma(\tau)\right)}{\left(\omega - i\gamma(\tau)\right)^2 - \left(\omega - \bar{\omega}_{dk}\right)^2} \approx \frac{k_\theta}{\omega_c} \frac{\partial \hat{F}_0}{\partial r} \frac{\Gamma/\omega(\tau)}{\Gamma^2 - \left(\omega - \bar{\omega}_{dk}\right)^2}$$

- Estimate $\partial_r \approx \Delta r_L^{-1}$. The NL term in the $\hat{F}_0(\omega)$ expression becomes

$$[NL] \approx 1 \approx (ck_\theta/B_0)^2\Gamma^{-2}|\Delta r_L|^{-2}|\delta\phi_k|^2 \approx (ck_\theta/B_0)^2\Gamma^{-4}|\delta\phi_k|^2\omega_{dk}^2(\varepsilon^2/r^2)$$

$$\Rightarrow \Gamma^2 \approx |ck_\theta/B_0||\delta\phi_k|(\varepsilon/r)|\omega_{dk}| \approx \varepsilon\omega_{B0}^2$$

where ω_{B0} is the wave-particle trapping frequency with no chirping and/or adiabatic chirping.

Local saturation (weak drive)

- The typical separatrix width of particles trapped within the wave is small compared with the AE mode structure at short scales (Berk and Breizman 90), e.g., the fine TAE structure at $\Delta r/r \approx \delta/(nrq') \approx \delta/(nqs)$, $\delta \approx r/R_0$ (Cheng, Chen and Chance 85).

$$|\Delta r_L| \lesssim |\Delta r| \Rightarrow \frac{\gamma_L}{\omega_{dk}} \varepsilon^{-1} \lesssim \frac{\delta}{nqs} \Rightarrow \frac{\gamma_L}{\omega} \lesssim \frac{\varepsilon\delta}{nqs}$$

- Local mode structure becomes important as AE mode structure of the single poloidal harmonics: theories at marginal stability (Berk and Breizman 90) need refinement. Note $\varepsilon \sim (\omega/\omega_{*E}) \approx r/R_0$ for EPM, while $\varepsilon \approx 1$ for AE.

$$|\Delta r_L| \lesssim |\Delta r| \Rightarrow \frac{\gamma_L}{\omega_{dk}} \varepsilon^{-1} \lesssim \frac{1}{nqs} \Rightarrow \frac{\gamma_L}{\omega} \lesssim \frac{\varepsilon}{nqs}$$

RP: Recover (Berk and Breizman 90) analyses using these equations in the first case. Extend their analysis to the second case. What happens for strong drive?

Global saturation (strong drive)

- Assume that local saturation is not possible, i.e. (from previous page) $\varepsilon^{-1}(\gamma_L/\omega) \approx (\gamma_L/\omega)(\omega_{*E}/\omega) \gtrsim (nqs)^{-1}$

- Fast ion avalanches (see later) and EPM convective amplification is triggered when local saturation is not possible (Zonca et al 05). Still $|\Delta r_L|$ is less than the EPM radial envelope width $\approx (L_{pE}/k_\theta)^{1/2}$, with L_{pE} the characteristic energetic particle pressure scale length (Zonca and Chen 96/00)

$$\frac{r}{nqs} \lesssim |\Delta r_L| \lesssim \left| \frac{L_{pE}}{k_\theta} \right|^{1/2} \Rightarrow \frac{1}{nqs} \lesssim \frac{\gamma_L}{\omega}\frac{\omega_{*E}}{\omega} \lesssim \frac{L_{pE}/r}{|k_\theta L_{pE}|^{1/2}}$$

- For even stronger drive, strong avalanching is expected, with macroscopic distortions of the EP distribution function (Zonca et al 05)

$$\frac{\gamma_L}{\omega}\frac{\omega_{*E}}{\omega} \gtrsim \frac{L_{pE}/r}{|k_\theta L_{pE}|^{1/2}}$$

Transition from local to global saturation

- These criteria coincide with those discussed in (Zonca et al 05) for moderate mode numbers.

E: Show that, for $k_\theta L_{pE} \approx (R_0/L_{pE})^2$ and moderate mode numbers, as considered in (Zonca et al 05), one obtains same criteria for transition from weak to strong energetic particle transport.

- Transition from local to global saturation takes place for $(\gamma_L/\omega)(\omega_{*E}/\omega) \gtrsim (nqs)^{-1}$.
- Transition is accompanied by a corresponding transition from weak to strong energetic ion transport in burning plasmas.
- Transition is favored for EPM (vs. fixed frequency AE, such as TAE) and for precession resonance.

- Transit resonance and multiple toroidal mode numbers n are beyond the scope of this year's course.

- When global saturation sets in, particles stay in resonance with the mode across one characteristic poloidal mode width. Thus, they experience the *radial average* effect of the *radial envelope*.

- From now on, $(\gamma_L/\omega)(\omega_{*E}/\omega) \gtrsim (nqs)^{-1}$ so that only radial envelope equation matters (Zonca et al 05); i.e. neglect (for simplicity) the fine radial structures of TAE/EPM.

- Treat the spatial average with ballooning formalism. Note that TAE/EPM mode structure, for computing the resonant deeply trapped particles response, can be indicated as

$$\frac{e}{T_E}\delta\phi_m = \frac{A(r,\tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(nq-m)\theta} \left[\delta\phi_c(\theta,\theta_k)\cos\frac{\theta_0}{2} + \delta\phi_s(\theta,\theta_k)\sin\frac{\theta_0}{2} \right] d\theta$$

$$\simeq \frac{A(r,\tau)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(nq-m)\theta} \delta\phi_c(\theta,\theta_k) d\theta$$

- Indicating with r_m the position of the rational surface $nq = m$ and averaging on a scale $2\ell/nq' \ll (L_p/k_\theta)^{1/2}$ centered about r_m (Zonca et al 05)

$$\sum_m \frac{nq'}{2\ell} \int_{r_m-\ell/nq'}^{r_m-\ell/nq'} \hat{F}_0 J_0^2(\lambda_{Lk}) J_0^2(\lambda_{dk}) \left|\delta\phi_{0k}(r,\tau)\right|^2 dr =$$

$$= \sum_m \frac{1}{2\ell} \int_{-\ell}^{\ell} \hat{F}_0 |A|^2 \frac{T_E^2}{e^2} \frac{dx}{2\pi} \int\!\!\int_{-\infty}^{\infty} d\theta d\theta' J_0^2(\lambda_{Lk}) J_0^2(\lambda_{dk}) e^{ix(\theta'-\theta)} \delta\phi_c(\theta,\theta_k) \delta\phi_c^*(\theta',\theta_k)$$

$$= \hat{F}_0 |A|^2 \frac{T_E^2}{e^2} \int_{-\infty}^{\infty} d\theta J_0^2(\lambda_{Lk}) J_0^2(\lambda_{dk}) \left|\delta\phi_c(\theta,\theta_k)\right|^2 = \frac{T_E^2}{e^2} \hat{F}_0 \bar{J}^2 |A|^2$$

- **Summarizing:** Assume that equilibrium scale L_{pE} is larger than the scale of variation of both A and $\hat{F}_0$ and looking at the average ($|k_z| \ll k_\perp$ envelope modulation driven nonlinear dynamics) (see also Lecture 6). Moreover $Q\hat{F}_0 \simeq (k_\theta/\omega_c)\partial_r \hat{F}_0$ ($|\omega_{*E}/\omega| \approx R_0/L_{pE} \gg 1$).

$$
\begin{aligned}
\hat{F}_0(\omega) \; &= \; \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \frac{k_\theta^2}{\omega}\frac{c^2}{B_0^2}\frac{T_E^2}{e^2}\frac{\partial}{\partial r} \\
&\times \; \left\{ \left[\frac{(\bar{\omega}_{dk}/\omega^*(\tau))}{\omega - \omega(\tau) + \bar{\omega}_{dk}} + \frac{(\bar{\omega}_{dk}/\omega(\tau))}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}} \right] \frac{\partial}{\partial r}\hat{F}_0\left(\omega - 2i\gamma(\tau)\right) \bar{J}^2 \left|A(r,\tau)\right|^2 \right\}
\end{aligned}
$$

- **Definition:** To be used in the following (note the slight difference of this definition with that of Lecture 4 and the presence of FLR/FOW terms)

$$
\mathcal{I}\circ \equiv \frac{2\pi^2 e^2}{Mc^2}\frac{q}{|s|}R_0 B_0 \sum_{v_\parallel/|v_\parallel|=\pm 1} \int_0^\infty \mathcal{E}d\mathcal{E} \int d\lambda \frac{\bar{\omega}_{dk}}{k_\theta^2}\tau_b \bar{J}^2
$$

- Nonlinear envelope equation for EPM comes from the generalized fishbone like dispersion relation (Zonca and Chen 06, Chen and Zonca 07; see also Lecture 4 of Spring 2009 intensive course)

$$i\Lambda_T A(r,t) = \left(\delta W_{f,MHD} + \delta W_{f,EP} + \delta W_{KT}\right) A(r,t)$$

$$\delta W_{KT} = \mathcal{I} \circ \int_{-\infty}^{\infty} \frac{k_\theta}{\omega_c} \frac{\bar{\omega}_{dk} e^{-i\omega t}}{\bar{\omega}_{dk} - \omega - \omega(\tau)} \frac{\partial}{\partial r} \hat{F}_0(\omega) d\omega$$

$$\delta W_{f,EP} = -\mathcal{I} \circ \frac{k_\theta}{\omega_c} \frac{\bar{J}_L^2}{\bar{J}^2} \frac{\partial}{\partial r} F_0 \;\; ; \;\; \bar{J}_L^2 = \int_{-\infty}^{\infty} J_0^2(\lambda_{Lk}) \left|\delta\phi_c(\theta,\theta_k)\right|^2 d\theta$$

- Simple application for small/moderate shear s (Chen and Zonca 95/96
  - frequency dependent continuum damping Λ_T

$$\Rightarrow \;\; \Lambda_T = \left[\frac{1}{8}\left(\frac{1/4 - \Omega^2}{\epsilon_0 \Omega^2} - 1\right)\right]^{1/2} \;\; ; \;\; \Omega = \frac{\omega}{\omega_A} \;\; ; \;\; \omega_A = \frac{v_A}{qR_0} \;\; ; \;\; \epsilon_0 = 2(r/R_0 + \Delta')$$

- Consider localized modes (small $|\theta_k|$, $\theta_k = -(i/nq')\partial_r$)

$$\delta W_{f,MHD} \simeq \frac{|s|\pi}{8}\left(1 + 2\kappa(s) - \frac{\alpha}{\alpha_c}\right) - \frac{|s|\pi}{8}\kappa(s)\theta_k^2$$

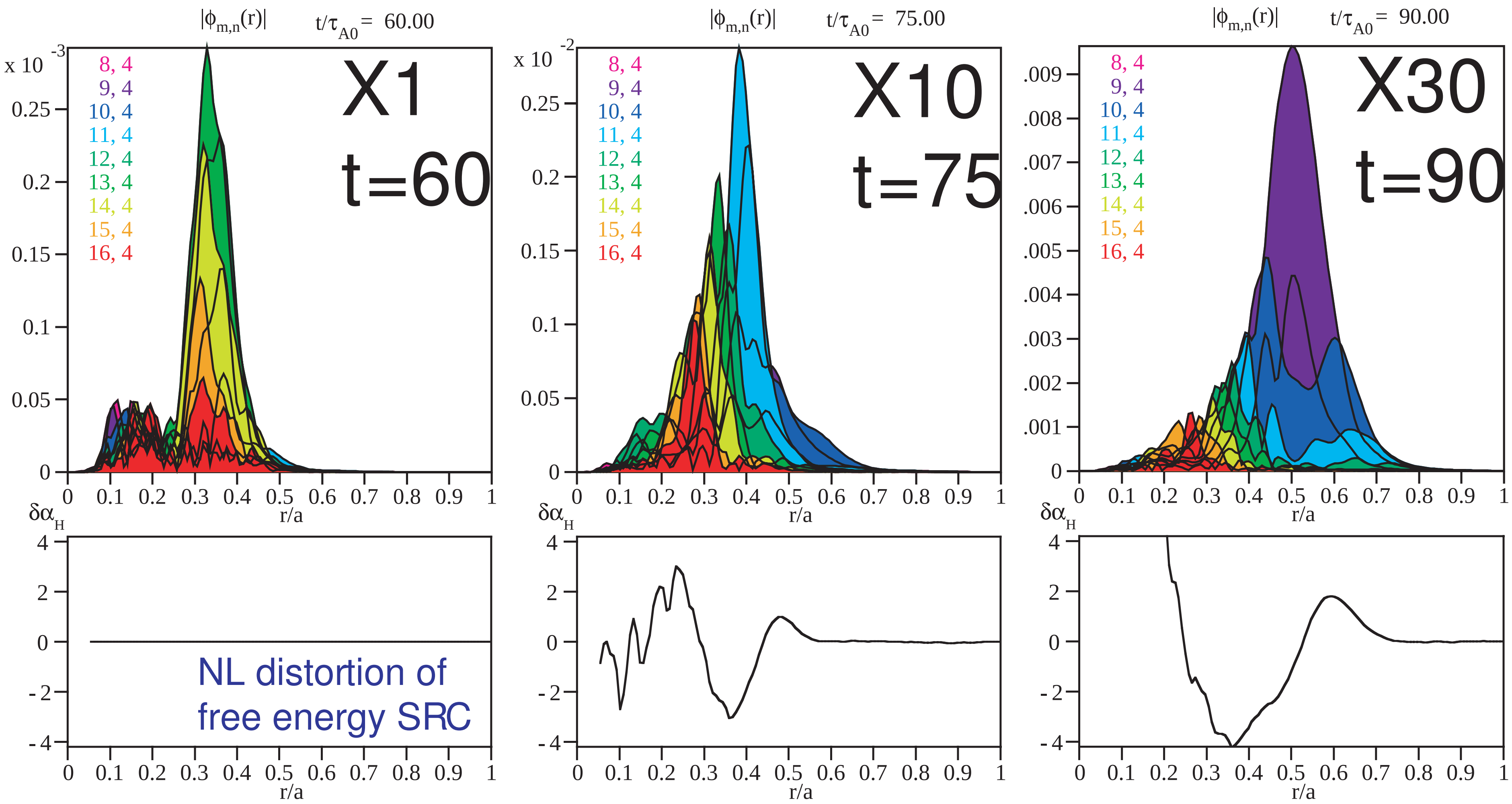
$$\alpha_c = \frac{s^2}{1+|s|} \quad ; \quad \kappa(s) \simeq \frac{1}{2}\left(1 + \frac{1}{|s|}\right) e^{-1/|s|}$$

□ Compare nonlinear equations with hybrid MHD-GKE simulations

- no sources and collisions
- assume initial unstable $F_0(0)$ and let the system evolve freely
- assume moderate mode number so that FLR/FOW effects are negligible
- initial distribution function is a slowing down

Zonca et al. IAEA, (2002)

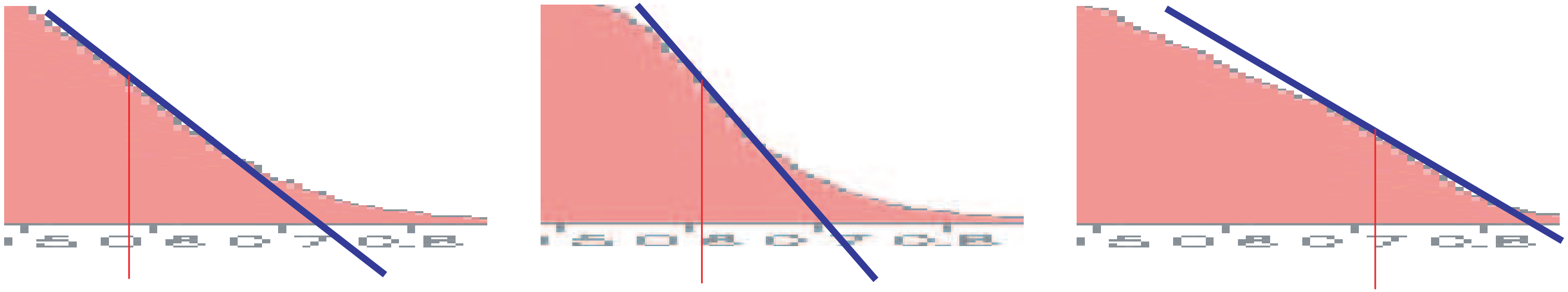
Avalanches and NL EPM dynamics

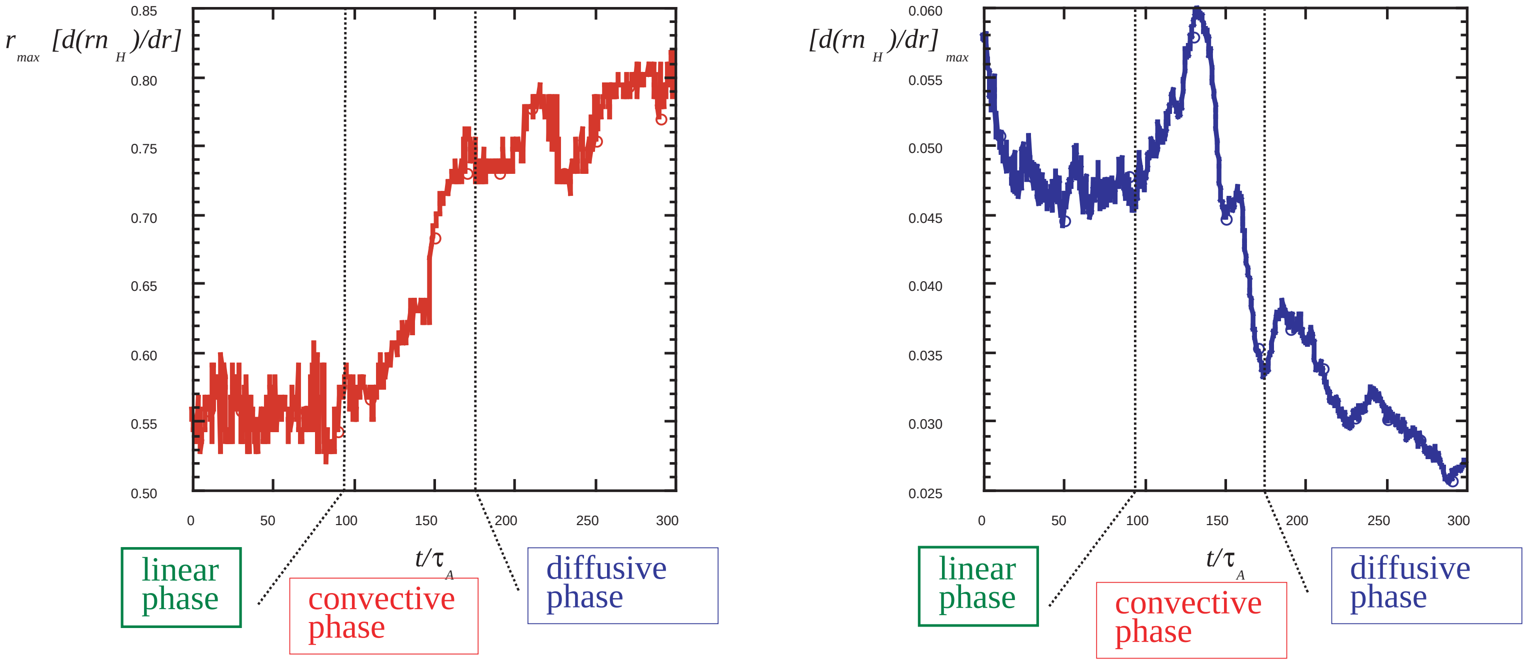


- Importance of toroidal geometry on wave-packet propagation and shape

Vlad et al. IAEA-TCM, (2003)

Propagation of the unstable front





- Gradient steepening and relaxation: spreading ... similar to turbulence

- Still need to compute the nonlinear EP response from $\hat{F}_0$.
  $\delta W_{f,EP}$ partly cancels δW_{KT}

$$
\begin{aligned}
\delta W_{f,EP} + \delta W_{KT} &= \mathcal{I} \circ \int_{-\infty}^{\infty} \frac{k_\theta}{\omega_c} \frac{\omega + \omega(\tau)}{\bar{\omega}_{dk} - \omega - \omega(\tau)} e^{-i\omega t} \frac{\partial}{\partial r} \hat{F}_0(\omega) d\omega \\
&= \text{linear}(0) + \text{nonlinear}
\end{aligned}
$$

- By formal substitution of $\hat{F}_0$ expression (see p. 16; $\mathrm{St}\hat{F}_0(\omega) = \hat{S}(\omega) = 0$)

$$
\begin{aligned}
\text{linear}(0) &= \mathcal{I} \circ \frac{k_\theta}{\omega_c} \frac{\omega(\tau)}{\bar{\omega}_{dk} - \omega(\tau)} \frac{\partial}{\partial r} F_0(0) \\
&= \frac{3\pi\epsilon^{1/2}}{4\sqrt{2}|s|} \alpha_E \frac{\omega(\tau)}{\bar{\omega}_{dF}} \left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega(\tau)} - 1 \right) + i\pi \right] \quad ; \quad \alpha_E = -R_0 q^2 (8\pi P_E')/B_0^2 \\
F_0(0) &= \frac{3P_E}{4\pi E_F} \frac{1}{(2\mathcal{E})^{3/2} + (2E_c/M)^{3/2}} \quad ; \quad E_c \equiv \text{critical energy} \ll E_F
\end{aligned}
$$

E: Derive the linear results on p. 21.

- Consider a localized Gaussian beam to show the self-trapping of the EPM by the beam spatial profile (Zonca and Chen 00)

$$\partial_r P_E = \left(1 - \frac{(r-r_0)^2}{L_{pE}^2}\right)\partial_{r_0} P_E$$

- Radial EPM envelope equations, $x = nq'(r-r_0)$

$$\begin{aligned} i\Lambda_T A \;&=\; \frac{|s|\pi}{8}\left(1 + 2\kappa(s) - \frac{\alpha}{\alpha_c}\right)A + \frac{|s|\pi}{8}\kappa(s)\frac{\partial^2 A}{\partial x^2} \\ &+ \frac{3\pi\epsilon^{1/2}}{4\sqrt{2}|s|}\alpha_{E0}\left(1 - \frac{x^2/s^2}{k_\theta^2 L_{pE}^2}\right)\frac{\omega(\tau)}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega(\tau)} - 1\right) + i\pi\right]A \end{aligned}$$

- Localized solutions exist

$$A \sim \exp\left(-\frac{(r-r_0)^2}{2\Delta^2}\right)$$

$$\Delta^{-4} = \frac{3\sqrt{2\epsilon}}{\kappa(s)}\frac{k_\theta^2}{L_{pE}^2}\alpha_{E0}\frac{\omega(\tau)}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega(\tau)}-1\right)+i\pi\right]$$

- Meso-scale mode structure with typical mode width

$$\Delta \approx (L_{pE}/k_\theta)^{1/2}\kappa(s)^{1/4}\epsilon^{-1/8}\alpha_{E0}^{-1/4}$$

E: Explain why the radial envelope width is called *meso-scale.* Derive the results above step by step.

E: Show that the localization of the free energy source (mode drive) is needed for the self-trapping of the EPM mode.

- Next: compute NL terms assuming time scale separation as for the fishbone case (see Lecture 4)

E: Follow these detailed derivations and demonstrate the model NL fishbone equations given in (Zonca et al 07).

- Perform the NL calculation assuming that the width of $\hat{F}_0(\omega)$ is less than γ, with $\omega(\tau) = \omega_0(\tau) + i\gamma(\tau)$

$$\delta W_{f,EP} + \delta W_{KT} = \mathrm{linear}(0) + \mathcal{I} \circ \int_{-\infty}^{\infty} e^{-i\omega t} d\omega \frac{\omega + \omega(\tau)}{\bar{\omega}_{dk} - \omega(\tau) - \omega} \frac{k_\theta^2}{\omega} \frac{c^2}{B_0^2} \frac{T_E^2}{e^2} \frac{\partial^2}{\partial r^2}$$

$$\times \quad \left\{ \left[\frac{(\bar{\omega}_{dk}/\omega^*(\tau))}{\omega - \omega(\tau) + \bar{\omega}_{dk}} + \frac{(\bar{\omega}_{dk}/\omega(\tau))}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}} \right] \frac{k_\theta}{\omega_c} \frac{\partial}{\partial r} \hat{F}_0 \left(\omega - 2i\gamma(\tau)\right) \left| A(r,\tau) \right|^2 \right\}$$

- Integration in ω can be performed assuming time scale separation

$$\begin{aligned}
\delta W_{f,EP} + \delta W_{KT} &= \text{linear}(0) + \mathcal{I} \circ \frac{\bar{\omega}_{dk} - \omega_0 + 3i\gamma}{(\bar{\omega}_{dk} - \omega_0)^2 + 9\gamma^2} \frac{k_\theta^2 c^2}{B_0^2} \frac{T_E^2}{e^2} \frac{\partial^2}{\partial r^2} \\
&\times \left[\frac{\bar{\omega}_{dk}}{(\bar{\omega}_{dk} - \omega_0)^2 + \gamma^2} \frac{k_\theta}{\omega_c} \frac{\partial}{\partial r} F_0(t) \left| A(r,\tau) \right|^2 \right] \\
&= \text{linear}(0) + \frac{k_\theta^2 \rho_E^2 v_E^2}{\gamma(\tau)} \mathcal{I} \circ \frac{\bar{\omega}_{dk} - \omega_0}{(\bar{\omega}_{dk} - \omega_0)^2 + 9\gamma^2} \frac{\partial^2}{\partial r^2} \\
&\quad \times \left[\frac{\gamma \bar{\omega}_{dk} (k_\theta / \omega_c)}{(\bar{\omega}_{dk} - \omega_0)^2 + \gamma^2} \frac{\partial}{\partial r} F_0(t) \left| A(r,\tau) \right|^2 \right] \\
&\quad + i \frac{k_\theta^2 \rho_E^2 v_E^2}{4\gamma^2(\tau)} \frac{\partial^2}{\partial r^2} \left[\mathcal{I} \circ \frac{\gamma \bar{\omega}_{dk} (k_\theta / \omega_c)}{(\bar{\omega}_{dk} - \omega_0)^2 + \gamma^2} \frac{\partial}{\partial r} F_0(t) \left| A(r,\tau) \right|^2 \right]
\end{aligned}$$

- Note that the real NL response is $\approx \gamma/\omega_0$ smaller than the imaginary response.

E: Can you justify why *resonant* particle response is stronger than *non-resonant*?

- These equations can be used to analyze strong nonlinear distortions. The avalanche dynamics can be estimated already looking at the early nonlinear evolution phase.

- This is achieved writing a *recursive solution* (perturbative). Note that both real and imaginary parts are important.

E: Show that the real response provides the real mode frequency ω_0 and can modify mode radial localization. However, mode localization is mostly determined by the linear drive radial profile.

- For the ground state (see p.23), the linear dispersion relation is

$$\frac{|s|\pi}{8}\left(1+2\kappa(s)-\frac{\alpha}{\alpha_c}\right)-\mathbb{R}\mathrm{e}\frac{\pi\kappa(s)}{8|s|k_\theta^2\Delta^2}+\frac{3\pi\epsilon^{1/2}}{4\sqrt{2}|s|}\alpha_{E0}\frac{\omega_0}{\bar{\omega}_{dF}}\left[\ln\left(\frac{\bar{\omega}_{dF}}{\omega_0}-1\right)\right]=0$$

$$\frac{\gamma}{\bar{\omega}_{dF}-\omega_0}=\left(\pi\frac{\omega_0}{\bar{\omega}_{dF}}-\hat{\Lambda}_T\right)-\mathbb{I}\mathrm{m}\frac{\kappa(s)/\alpha_{E0}}{3\sqrt{2\epsilon}k_\theta^2\Delta^2}\quad;\quad\hat{\Lambda}_T=\frac{4\sqrt{2}|s|}{3\pi\epsilon^{1/2}\alpha_{E0}}\Lambda_T$$

E: Show that the ground state is the most unstable mode.

- In the early nonlinear stage, the amplitude equation can be modified from the ones above and is given by ($y=r-r_0$)

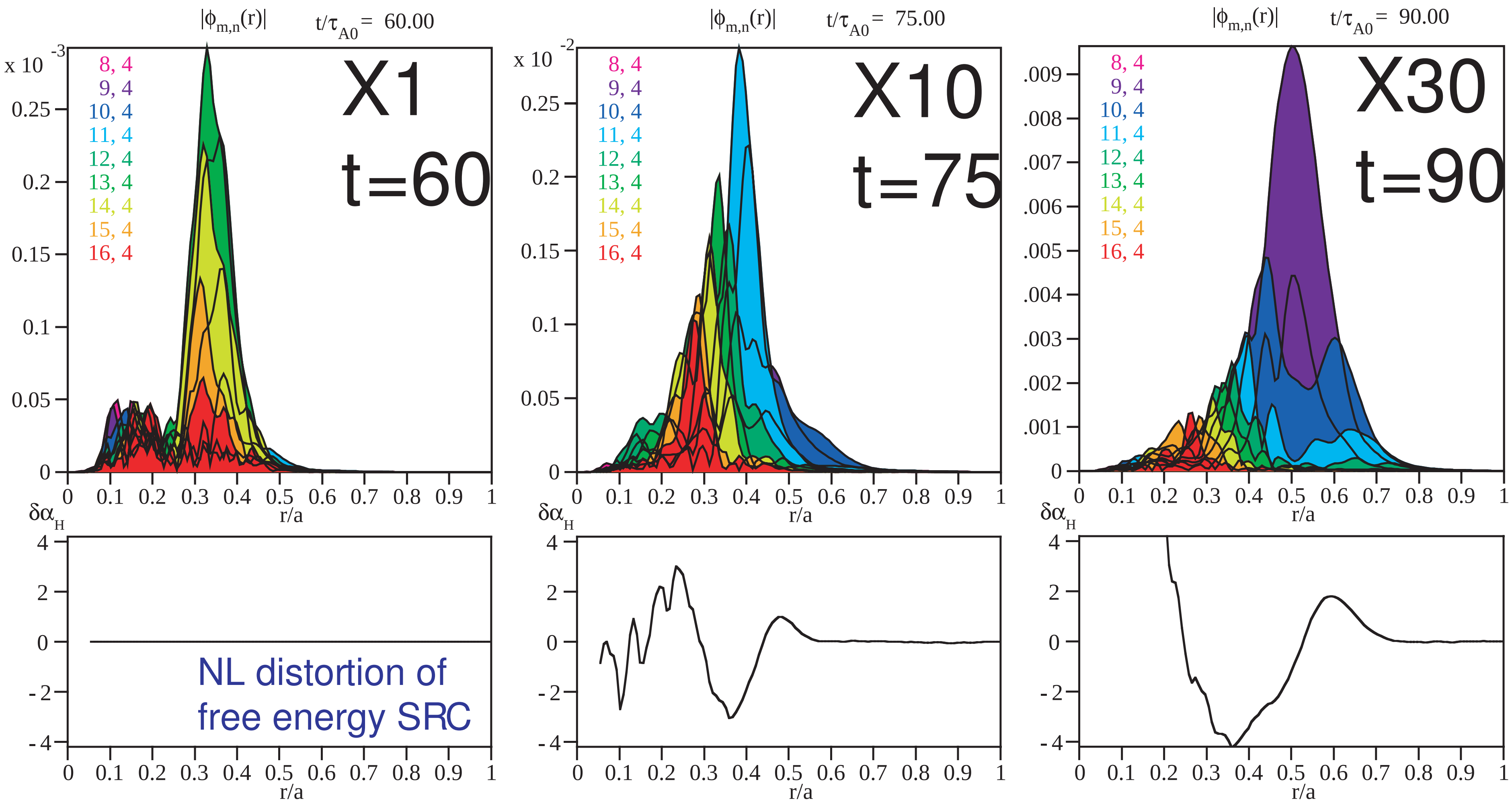
$$\begin{aligned}\frac{\partial_t|A|^2}{2(\bar{\omega}_{dk}-\omega_0)} &= \left(\pi\frac{\omega_0}{\bar{\omega}_{dF}}-\hat{\Lambda}_T\right)|A|^2-\mathbb{I}\mathrm{m}\frac{\kappa(s)/\alpha_{E0}}{3\sqrt{2\epsilon}k_\theta^2\Delta^2}|A|^2+\frac{k_\theta^2\rho_E^2v_E^2}{4\gamma^2(\tau)}|A|^2\frac{\partial^2}{\partial y^2}\left(\pi\frac{\omega_0}{\bar{\omega}_{dF}}|A|^2\right)\\ &\simeq \left(\pi\frac{\omega_0}{\bar{\omega}_{dF}}-\hat{\Lambda}_T\right)|A|^2-\mathbb{I}\mathrm{m}\frac{\kappa(s)/\alpha_{E0}}{3\sqrt{2\epsilon}k_\theta^2\Delta^2}|A|^2+\frac{k_\theta^2\rho_E^2v_E^2}{2\gamma^2(\tau)\Delta_R^2}\left(\frac{2y^2}{\Delta_R^2}-1\right)\pi\frac{\omega_0}{\bar{\omega}_{dF}}|A|^4\end{aligned}$$

- For brevity, here, the notation has been used $\mathbb{R}\mathrm{e}\Delta^{-2} = \Delta_R^{-2}$.
- Study of the NL PDE that one can derive for the EPM induced avalanche is subject of ongoing research and is beyond the scope of the present lecture.
- Easy estimate can be given in the local limit:
  - drive strength is decreased at the position where the mode is localized and is increased nearby
  - the radial displacement for the EPM to be locked into the strongest drive is (Zonca and Chen 05)

$$\frac{y}{L_{pE}} \simeq \frac{k_\theta \rho_E v_E}{\gamma \Delta_R} |A|$$

Zonca et al. IAEA, (2002)

Avalanches and NL EPM dynamics



- Importance of toroidal geometry on wave-packet propagation and shape

References and reading material

L. M. Altshul and V.I. Karpman, Zh. Eksp. Teor. Fiz. **49**. 515 (1965) [Sov. Phys. JETP **22**, 361 (1966)].

F. Zonca, S. Briguglio, L. Chen, G. Fogaccia, G. Vlad. Paper TH/4-4. Presented at the 19.th IAEA Fusion Energy Conference, Lyon, France, Oct. 14-19, (2002).

F. Zonca, S. Briguglio, L. Chen, G. Fogaccia and G. Vlad, Nucl. Fusion **45**, 477 (2005).

H.L. Berk and B.N. Breizman, Phys. Fluids B **2**, 2226 (1990).

H.L. Berk and B.N. Breizman, Phys. Fluids B **2**, 2235 (1990).

H.L. Berk and B.N. Breizman, Phys. Fluids B **2**, 2246 (1990).

C.Z. Cheng, L. Chen and M.S. Chance, Ann. Phys. **161** 21 (1985).

F. Zonca and L. Chen, Phys. Plasmas **3**, 323 (1996).

F. Zonca and L. Chen, Phys. Plasmas **7**, 4600 (2000).

F. Zonca and L. Chen, Plasma Phys. Control. Fusion **48** 537 (2006).

L. Chen and F. Zonca, Nucl. Fusion **47**, S727 (2007).

L. Chen and F. Zonca, Phys. Scr. **T60**, 81 (1995).

G. Vlad, S. Briguglio, G. Fogaccia and F. Zonca, *Alpha particle transport induced by Alfvénic instabilities in proposed burning plasma scenarios.* Presented at the 8th IAEA Technical Meeting on Energetic Particles in Magnetic Confinement Systems, 6-8 October 2003, San Diego, California.

F. Zonca et al, Nucl. Fusion **47**, 1588 (2007).

F. Zonca F. et al, arXiv:0707.2852v1 [physics.plasma-ph] (2007).