Lecture 4

Beam modes in tokamaks: the fishbone burst cycle

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Peculiarities of beam-plasma interactions in toroidal fusion devices

- Most dangerous waves belong to the drift brach, for which $k_\perp \rho \lesssim 1$ while parallel wavelength is set by curvature, and there are continuous spectra $(\gamma_d(\omega))$.
- This Lecture: using the conservation of extended phase space Hamiltonian, $n\dot{E} = \omega \dot{P}_\phi$ (n is the toroidal mode number) it is possible to demonstrate that (ω_d is the magnetic drift frequency, R_0 the curvature radius)
$$\frac{\Delta r}{r} \simeq \frac{n\bar{\omega}_d}{\omega} \frac{R_0}{r} \frac{\Delta E}{E} \simeq \frac{\omega_*}{\omega} \frac{\Delta E}{E}$$
- Beam modes (EPM) are (radially) self-trapped by the beam (Chen and Zonca 95, 96, 00): consequences on the nonlinear dynamics (Lecture 5).
- The mode-particle pumping (White et al 83) is a secular radial loss process which is connected with resonant wave-particle interactions and naturally yields a non-adiabatic frequency chirping $\dot{\omega} \sim \omega_B^2$.

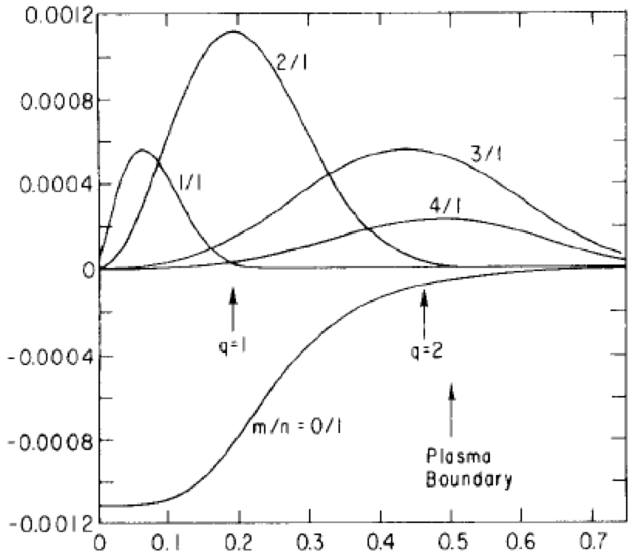
Mode-particle pumping (fishbone)

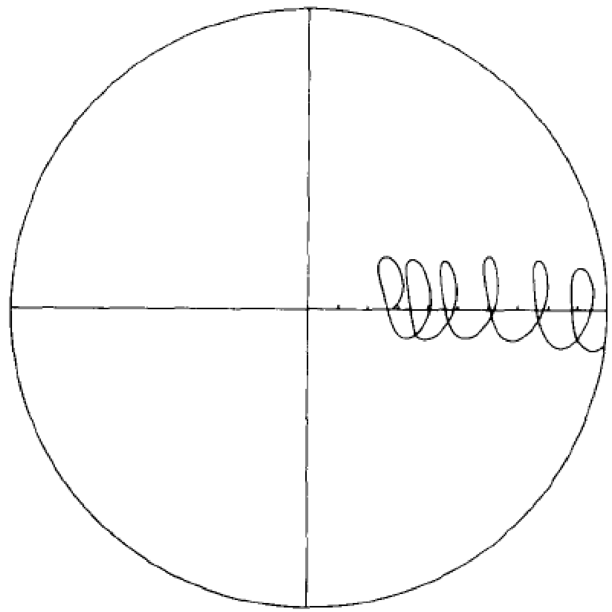
- Mode-particle pumping: (White et al. 83) trapped particles MHD $(\delta\phi, \delta A_{\|})$ with $\delta\phi = \delta\phi_0(r)\sin(n\varphi - m\theta - \omega t + \psi)$

$$r \simeq r_0 + \frac{v_{d0}}{\omega_b}\theta_b\cos(\omega_b t) + \langle\Delta r\rangle \qquad \langle\dot{\Delta r}\rangle = \frac{c}{B}\frac{m}{r_0}\left[\frac{N\omega_b + (m/q)\bar{\omega}_d}{N\omega_b + n\bar{\omega}_d}\right]\delta\phi_0 J_N(m\theta_b)\cos\psi$$

$$\theta \simeq -\theta_b\sin(\omega_b t) + \langle\Delta\theta\rangle \qquad \dot{\psi} = \frac{\langle\Delta r\rangle}{r_0}(s-1)n\bar{\omega}_d$$

$$\omega = n\bar{\omega}_d + N\omega_b$$





- For deeply trapped particles in a tokamak (White et al. 83):

$$\dot{\phi} = \bar{\omega}_d \qquad \theta = -\theta_b \sin(\omega_b t) \qquad v_\parallel = R\dot{\phi} = R\bar{\omega}_d$$

- For shear Alfvén waves $\delta E_\parallel \simeq 0$, $(\delta\phi, \delta A_\parallel)$ are given by $\delta\phi = \delta\phi_0(r)\sin\xi$, $\delta A_\parallel = (nq-m)/(qR)(c/\omega)\delta\phi$, $\xi = n\phi - m\theta - \omega t + \xi_0$

- With fluctuation, $\dot{\phi} = \bar{\omega}_d + (s-1)\bar{\omega}_d(\Delta r/r)$, $s = rq'/q$, $\theta = -\theta_b\sin(\omega_b t) + \Delta\theta$, $r = r_0 + (v_{d0}\theta_b/\omega_b)\cos(\omega_b t) + \Delta r$, $\xi = (n\bar{\omega}_d - \omega)t + m\theta_b\sin(\omega_b t) - m\Delta\theta + n\Delta\phi$

$$\dot{\Delta r} = \frac{c}{B}\cos\xi\frac{m}{r}\left[1 - \left(n - \frac{m}{q}\right)\frac{\bar{\omega}_d}{\omega}\right]\delta\phi_0$$

$$r\dot{\Delta\theta} = \frac{c}{B}\sin\xi\frac{\partial}{\partial r}\left\{\left[1 - \left(n - \frac{m}{q}\right)\frac{\bar{\omega}_d}{\omega}\right]\delta\phi_0\right\}$$

- The fishbone characteristic frequency is $|\omega| \sim \bar{\omega}_d \ll \omega_b$. Both magnetic moment and second (action) invariant $J = M \oint v_\parallel dl$, while $P_\phi = M(Rv_\parallel - \omega_c \psi/B) \simeq -M\omega_c\psi/B$ is broken (Chen and Zonca 07) due to precession resonance ($\mathbf{B} = \mathbf{F}(\psi)\nabla\phi + \nabla\phi \times \nabla\psi$).

E: Write the Hamiltonian for a particle moving in the shear Alfvén Wave described in the previous page. Show that the "extended phase-space" Hamiltonian is conserved: $n\dot{H} - \omega\dot{P}_\phi = 0$. Use this result to show that, for the precession resonance, particles are displaced radially more than they loose energy

$$\frac{\Delta r}{r} \simeq \frac{n\bar{\omega}_d}{\omega}\frac{R_0}{r}\frac{\Delta E}{E} \simeq \frac{\omega_*}{\omega}\frac{\Delta E}{E}$$

- The bounce averaged expression of the secular particle radial motion yields the resonance condition $\omega = n\bar{\omega}_d + N\omega_b$.

$$\langle \dot{\Delta r} \rangle = \frac{c}{B}\frac{m}{r_0}\left[\frac{N\omega_b + (m/q)\bar{\omega}_d}{N\omega_b + n\bar{\omega}_d}\right]\delta\phi_0 J_N(m\theta_b)\cos\psi$$

$$r\langle\dot{\Delta\theta}\rangle = \frac{c}{B}\frac{\partial}{\partial r}\left\{\left[\frac{N\omega_b + (m/q)\bar{\omega}_d}{N\omega_b + n\bar{\omega}_d}\right]\delta\phi_0 J_N(m\theta_b)\right\}\sin\psi$$

E: Derive these results. Hint: use the identity $e^{\pm iz\sin x} = \sum_{N=-\infty}^{\infty} J_N(z)e^{\pm iNx}$ and average the particle motion on many bounce times.

E: Can you recognize the dynamics of particle trapping in the wave? Show that $\dot{\psi} = n\langle\dot{\Delta\phi}\rangle = (s-1)n\bar{\omega}_d\langle\Delta r/r\rangle$ and that the particle trapping frequency in the wave (deeply trapped) is

$$\omega_B^2 = \left|\frac{c}{B}\frac{m}{r_0}(s-1)\frac{n\bar{\omega}_d}{r}\left[\frac{N\omega_b + (m/q)\bar{\omega}_d}{N\omega_b + n\bar{\omega}_d}\right]\delta\phi_0 J_N(m\theta_b)\right|$$

E: For the precession resonance, show that $\dot{\omega} = (s-1)n\bar{\omega}_d\langle\dot{\Delta r}/r_0\rangle \sim \omega_B^2$. Hint:

$$\langle\dot{\Delta r}\rangle = \frac{r_0}{(s-1)n\bar{\omega}_d}\omega_B^2\cos\psi$$

E: Show that the width of the trapped (within the wave) particle orbit is $\Delta r_B \sim r_0\omega_B/[n\omega_D(s-1)]$. Show that the radial displacement of a resonant particle in one growth time is $\langle \Delta r \rangle \sim (\omega_B/\gamma)\Delta r_B$. Use these results to speculate how far a particle can move before it gets really trapped in the wave. Can you give a simple argument explaining why a particle cannot be trapped if $\dot{\omega} \sim \omega_B^2$?

- These exercises illustrate that within a fishbone burst, the frequency of the mode is expected to chirp downward, mostly because of radial outward particle displacement while they transfer energy to the growing wave. Since $\dot{\omega} \sim \omega_B^2$, the mode can freely grow several e-folding times before trapping becomes important.

- In the remaining part of the lecture, the fishbone burst nonlinear dynamics is analyzed assuming a single mode is dominant and able to strongly distort the distribution function. The nonlinear dynamics is due to the interplay of the fast particle radial profile and the fishbone burst (fixed radial structure). The effect of radial mode structure on EPM dynamics is presented in Lecture 5.

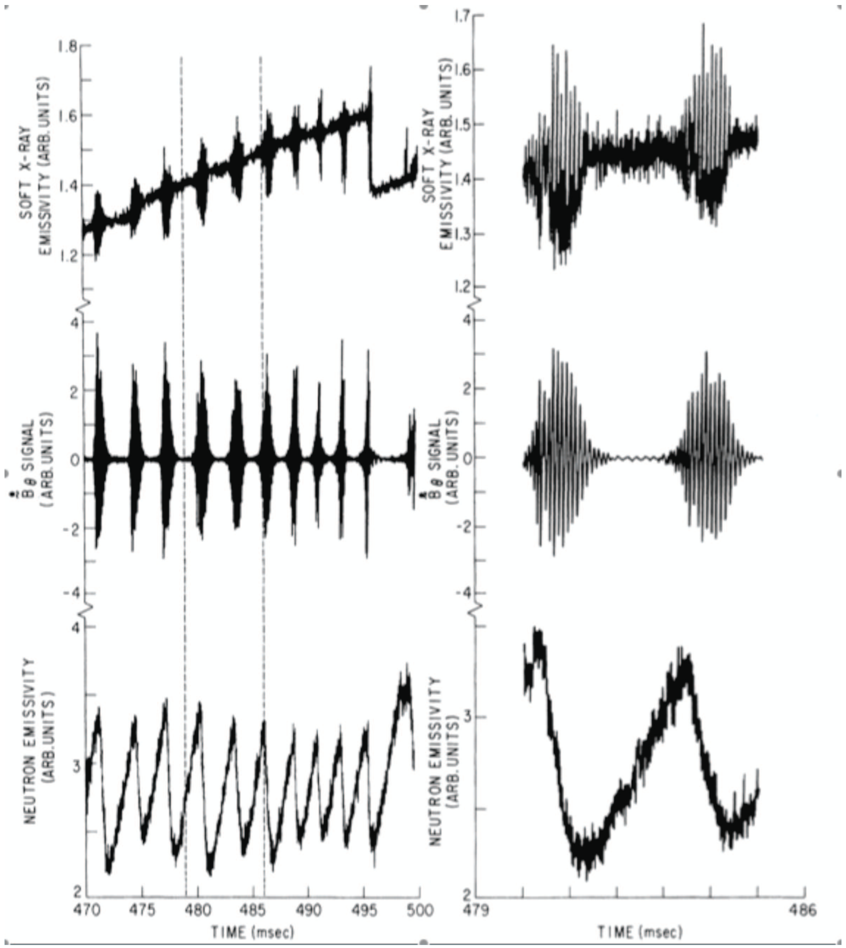
E: Follow White et al 83 and calculate the mode-particle pumping for (well-circulating particles). Hint: $\phi = q\omega_t t + \Delta\phi$, $\theta = \omega_t t - (\rho_d/r)\sin(\omega_t t) + \Delta\theta$, $r = r_0 + \rho_d \cos(\omega_t t) + \Delta r$, $\omega_t = v_\parallel/(qR)$, $\rho_d = (2E/M)^{1/2} q/\omega_c$

E: Show that the resonance condition is $\omega = [(nq - m) + N]\omega_t$ and

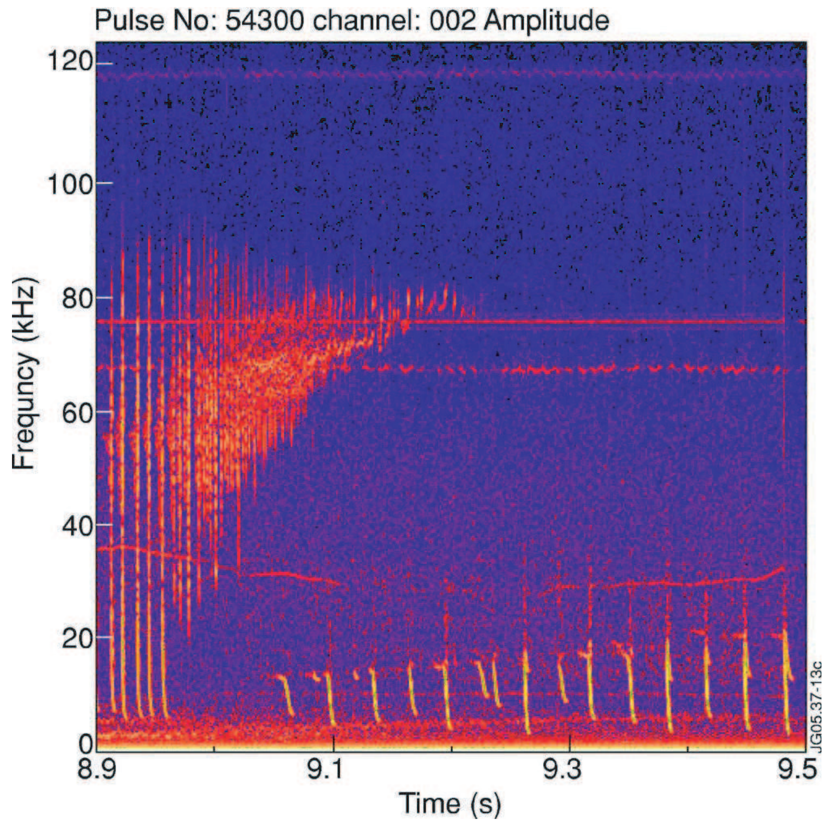
$$\langle \dot{\Delta r} \rangle = \frac{c}{B}\frac{m}{r_0}\left[\frac{N}{N + nq - m}\right]\delta\phi_0 J_N(m\rho_d/r)\cos\psi \qquad \dot{\psi} = -s\frac{\Delta r}{r}\omega_t$$

What is the trapping frequency of a particle in the well of the wave?

Fishbone burst observation



PDX: K. McGuire et al, Phys. Rev. Lett. **50**, 891 (1983).



JET: F. Nabais et al, Phys. Plasmas **12**, 102509 (2005).

Derivation of the fishbone dispersion relation

- This derivation is a summary of that given by Chen et al 84, where it is shown that, for a radial displacement $\xi = e^{-i\omega_0 t + i\zeta - i\theta}\xi_0 = -(c/\omega_0 B_0)e^{-i\omega_0 t+i\zeta-i\theta}(\delta\phi_0(r)/r)$

$$i|s|\frac{\omega_0}{\omega_A} = \delta\hat{W} = \delta\hat{W}_f + \delta\hat{W}_k \qquad \omega_A = v_A/(q(r_s)R_0) = v_A/R_0$$

- Generalization of this dispersion relation is discussed in Lecture 4 of Spring 2009 Lecture Notes

$$i|s|\Lambda\xi = \delta\hat{W}\xi = \left(\delta\hat{W}_f + \delta\hat{W}_k\right)\xi$$

- Simplest expression of $\delta\hat{W}_f$ is given by Bussac et al 75

$$\delta\hat{W}_f = 3\pi\Delta q_0\left(13/144 - \beta_{ps}^2\right)\left(r_s^2/R_0^2\right)$$

with $\beta_{ps} = -(R_0/r_s^2)^2\int_0^{r_s} r^2(d\beta/dr)dr$, $\Delta q_0 = 1 - q(r=0)$ and $\beta = 8\pi P/B_0^2$

- The expression of $\delta\hat{W}_k$ is given by Chen et al 84

$$\delta\hat{W}_k\xi = -4\frac{\pi^2}{B_0^2}m\omega_c\frac{R_0}{r_s^2}\int_0^{r_s}\frac{r^2}{q}dr\int \mathcal{E}d\mathcal{E}d\lambda\sum_{v_\parallel/|v_\parallel|=\pm1}\overline{e^{i\theta}\omega_d e^{-iq(r)\theta}}\tau_b\delta K$$

- Definitions: m is the energetic particle mass, $\omega_c = (eB/mc)$ is the cyclotron frequency, $\mathcal{E} = v^2/2$, $\lambda = \mu B_0/\mathcal{E} = (B_0/B)v_\perp^2/v^2$, $\mathbf{B}\cdot\nabla(\zeta - q(r)\theta) = 0$, ζ is the "toroidal angle" chosen such that (r,θ,ζ) is a toroidal flux coordinate system with straight field lines ($q = q(r)$), $\overline{(\ldots)} = (\oint v_\parallel^{-1}d\ell)^{-1}\oint v_\parallel^{-1}(\ldots)d\ell$ denotes bounce-averaging, ℓ is the arc length along the equilibrium $\mathbf{B}$-field, τ_b is the bounce/transit time for magnetically trapped/circulating particles, ω_d is the magnetic drift frequency and $QF_0 = (\omega\partial_{\mathcal{E}} + \hat{\omega}_*)F_0$, $\hat{\omega}_*F_0 = \omega_c^{-1}(\mathbf{k}\times\mathbf{B}/B)\cdot\nabla F_0$, with $F_0 = F_0(\mathcal{E},\lambda,v_\parallel/|v_\parallel|)$ the fast particle equilibrium distribution function.

- Representation of energetic particle distribution function neglecting finite orbit widths and separating both adiabatic ($\propto \partial_{\mathcal{E}} F_0$) as well as convective ($\propto QF_0$) responses:

$$\delta f = \frac{e}{m}\frac{\partial F_0}{\partial \mathcal{E}}\delta\phi + \delta H = \frac{e}{m}\frac{\partial F_0}{\partial \mathcal{E}}\delta\phi - \frac{e}{m}\frac{QF_0}{\omega}\delta\phi + \delta K$$

- Governing equation for (bounce averaged) δK, with $\delta\phi = \delta\phi_0(r,t)\exp \mathrm{i}(\zeta - \theta)$

$$(i\partial_t - \bar{\omega}_d)\,\delta K = -\frac{e}{m}\frac{QF_0}{\omega}\overline{e^{\mathrm{i}q(r)\theta}\omega_d e^{-\mathrm{i}\theta}}\delta\phi_0(r,t)e^{\mathrm{i}(\zeta - q(r)\theta)}$$

- Note that these equations are already nonlinear for the case of a single wave, provided that F_0 is interpreted as slowly (with respect to ω) varying equilibrium, due to sources and wave particle nonlinear interactions (single mode and strong equilibrium distortion).

- Denote with subscript k the Fourier-space projection with $n = m = 1, \omega$ and with subscript $-k$ the Fourier-space projection with $n = m = -1, -\omega^*$. Evolution equation for F_0 from nonlinear gyrokinetic equation (Frieman and Chen 82) with external source $S(\mathcal{E}, \lambda, v_\parallel/|v_\parallel|, t)$.

$$\frac{\partial F_0}{\partial t} = S - \frac{c}{B_0}\frac{i}{r}\frac{\partial}{\partial r}\left(\delta K_{-k}\delta\phi_k - \delta K_k \delta\phi_{-k}\right)$$

Note that only trapped particles are considered for simplicity, for which $\overline{v_\parallel} = 0$. See Lecture 6 of Spring 2009 Lecture Notes

E: Show that the equations given above can describe the fishbone burst cycle when single mode is dominant and able to strongly distort the distribution function. Hint: See Lecture 5.

E: What is the meaning of ω in the evolution equation for δK? Can you formulate the concept of time scale separation?

E: Show that on the r.h.s. of the evolution equation for F_0 there is a term

$$\frac{c^2}{B_0^2|\omega|^2} k_\theta \frac{\partial}{\partial r}\left(k_\theta \frac{\partial F_0}{\partial r} \partial_t |\delta\phi_k|^2 \right)$$

How does this term behave under time-reversal? What is the meaning of this property? Can you justify neglecting it given the results of this lecture?

- Summary: equations for the fishbone burst cycle

$$i|s|\Lambda\xi = \delta\hat{W}\xi = \left(\delta\hat{W}_f + \delta\hat{W}_k\right)\xi$$

$$\delta\hat{W}_k\xi = -4\frac{\pi^2}{B_0^2}m\omega_c\frac{R_0}{r_s^2}\int_0^{r_s}\frac{r^2}{q}dr\int \mathcal{E}d\mathcal{E}d\lambda\sum_{v_\parallel/|v_\parallel|=\pm 1}\overline{e^{i\theta}\omega_d e^{-iq(r)\theta}}\tau_b\delta K$$

$$(i\partial_t - \bar{\omega}_d)\,\delta K = -\frac{e}{m}\frac{QF_0}{\omega}\overline{e^{\mathrm{i}q(r)\theta}\omega_d e^{-\mathrm{i}\theta}}\delta\phi_0(r,t)e^{\mathrm{i}(\zeta-q(r)\theta)}$$

$$\frac{\partial F_0}{\partial t} = S - \frac{c}{B_0}\frac{i}{r}\frac{\partial}{\partial r}\left(\delta K_{-k}\delta\phi_k - \delta K_k\delta\phi_{-k}\right)$$

- Nonlinear interplay between mode dynamics (dispersiveness) and particle transport is crucial. Here the mode structure is fixed by ideal MHD (kink) response. Mode structure plays an important role for EPM nonlinear evolution (see Lecture 5)

Fourier-Laplace solution of nonlinear equations

- Use Fourier-Laplace transform for solving nonlinear equations. Start with δK

$$\delta K(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \delta\hat{K}(\omega) d\omega$$

$$\delta\hat{K}(\omega) = \frac{1}{2\pi} \int_0^{\infty} e^{i\omega t} \delta K(t) dt$$

E: Show that with these definitions causality is preserved ($\delta K(t) = 0$ for $t < 0$) if ω integration path is above all poles and branch points of $\delta\hat{K}(\omega)$.

- Assume $\delta K(0) = 0$ and let $\hat{\omega}_d = \overline{\exp(iq\theta)\omega_d \exp(-i\theta)}$. $\delta\hat{K}(\omega)$ is:

$$\delta\hat{K}(\omega) = -\frac{1}{2\pi} \frac{e/m}{\omega - \bar{\omega}_{dk}} \int_0^{\infty} e^{i\omega t} \frac{\hat{\omega}_d}{\Omega} Q F_0 \delta\phi dt = -\int_{-\infty}^{\infty} \frac{e}{m} \frac{\hat{\omega}_{dk}}{x} \frac{Q_{k,x}\hat{F}_0(\omega - x)\delta\phi_k(x)}{\omega - \bar{\omega}_{dk}} dx$$

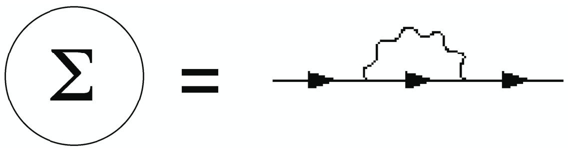
- Note three important points:
  - Ω in the first integral has the meaning of frequency, to be intended as operator acting on $\delta\phi$, i.e. it formally acts at $i\partial_t$
  - in the second (convolution) integral, the frequency and wave-vector values are indicated on which the $Q_{k,x}$ operator (acting on $\hat{F}_0$) is to be computed
  - in the second (convolution) integral, the wave-vector value is indicated on which the $\bar{\omega}_{dk}$ operator is to be computed
- The solution for $\hat{F}_0$ is obtained in the same way:

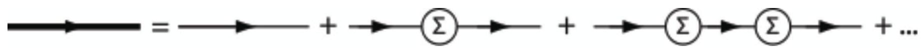
$$\hat{F}_0(\omega) = \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \frac{c}{\omega B_0}\frac{1}{r}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\left[\delta\phi_k(x)\delta K_{-k}(\omega - x) - \delta\phi_{-k}(x)\delta K_k(\omega - x)\right] dx$$

- Substituting the expression for $\delta\hat{K}(\omega)$ and with some little algebra, the equation for $\hat{F}_0(\omega)$ is

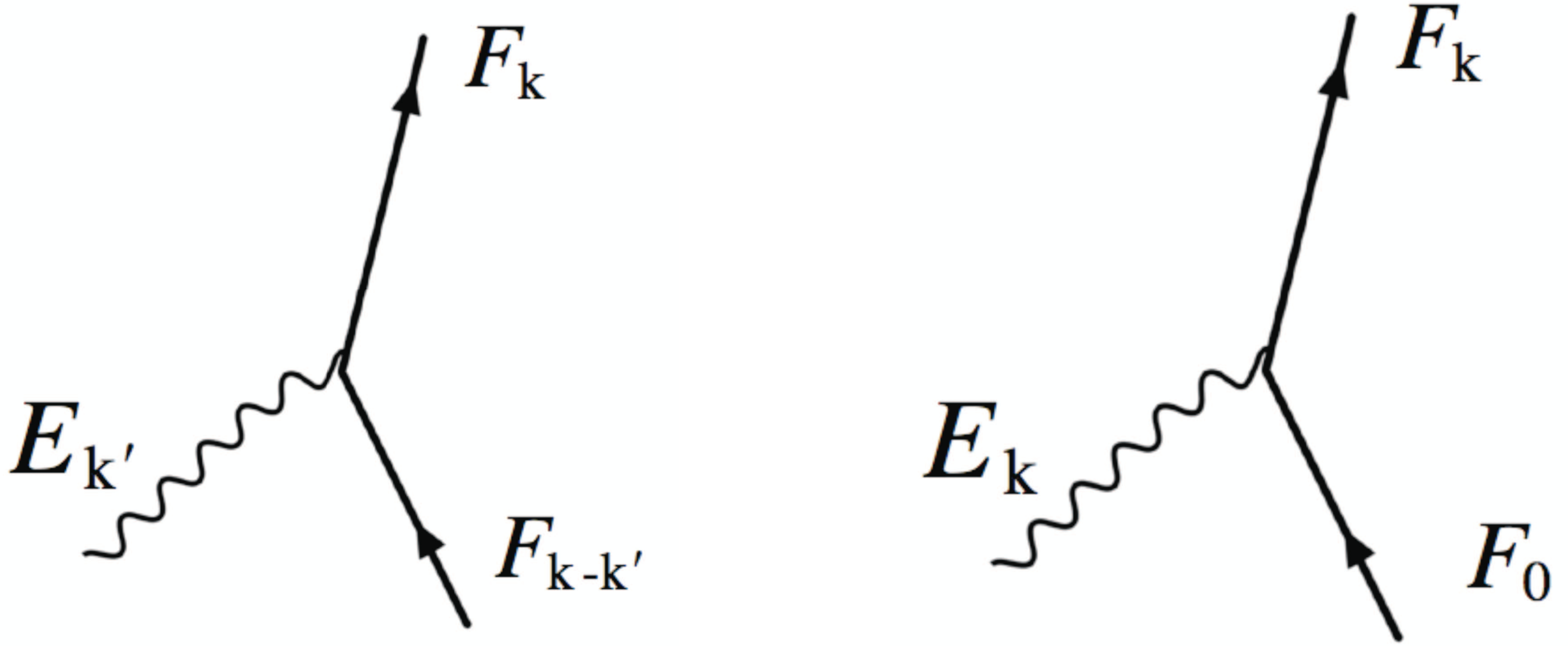
$$\hat{F}_0(\omega) = \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) - \frac{c}{\omega B_0}\frac{e}{m}\frac{1}{r}\frac{\partial}{\partial r}\iint_{-\infty}^{\infty}\left[\delta\phi_k(x)\frac{\hat{\omega}_{d-k}}{x'}\frac{Q_{-k,x'}\hat{F}_0(\omega - x - x')\delta\phi_{-k}(x')}{\omega - x - \bar{\omega}_{d-k}}\right.$$
$$\left. -\delta\phi_{-k}(x)\frac{\hat{\omega}_{dk}}{x'}\frac{Q_{k,x'}\hat{F}_0(\omega - x - x')\delta\phi_k(x')}{\omega - x - \bar{\omega}_{dk}}\right] dx dx'$$

- This solution is valid for strong distortions of the equilibrium distribution function and is the analogue of the Dyson equation ($G = G_0 + G_0\Sigma G$, with G_0/G the bare/dressed propagators), as noted, e.g., by Al'tshul' and Karpman 65.





Renormalized F_0 and Feynman diagrams



RP: Revisit RP at p.24 of Lecture 3 in the light of this discussion

Monochromatic wave case

- Summarizing
  - The general equations for the fishbone burst cycle dominated by one single wave are those at p.15.
  - The single wave hypothesis is supported by numerical simulations (Fu et al 06), showing that nonlinear mode coupling modifies saturation amplitude but not the underlying physics
  - The single wave hypothesis is supported by prior analytic works (O'Neil 1965 and Al'tshul' and Karpman 1965), showing that the single wave case adequately describes nonlinear dynamics due to wave-particle trapping but does not properly describe phase space mixing or spatial bunching. These processes should be of lesser importance for the fishbone burst cycle since nonlinear dynamics is dominated by the interplay of wave-dispersiveness and fast particle transport
  - The renormalized F_0 solution is still so complex that it requires numerical solution. Progress can be made introducing the monochromatic wave approximation (Al'tshul' and Karpman 1965)

- Definition of monochromatic wave:

$$\delta\phi_k = \delta\bar{\phi}_0(r)\exp\left(i\zeta - i\theta - i\int_0^t \omega(t')dt'\right)$$

- For the limit $|\dot{\omega}(t)| \ll |\gamma(t)\omega(t)|$ and assuming $|\gamma/\omega| \ll 1$, it is always possible to introduce temporal scale separation such that $\partial_\tau \ll \partial_t$

$$\delta\phi_k = \delta\phi_{0k}(r,\tau)\exp\left(-i\omega(\tau)t\right) \qquad \delta\phi_{0k}(r,\tau) \simeq \delta\bar{\phi}_{0k}(r)\exp\left(i\zeta - i\theta - i\int_0^\tau \omega(\tau')d\tau' + i\omega(\tau)\tau\right)$$

E: Show that the correct constraint is $|\dot{\omega}(t)| \ll |\gamma(t)\omega(t)|$ for $|\gamma/\omega| \ll 1$ and not $|\dot{\omega}(t)| \ll |\omega(t)|^2$, which more generally applies for $|\gamma/\omega| \sim 1$. Show that $|\dot{\gamma}(t)| \ll \gamma^2(t)$. Hint: Note that $|\gamma/\omega| \ll 1$ applies for fishbones.

E: Show that the monochromatic wave assumption applies for the strong fishbone chirping $\dot{\omega} \approx \omega_B^2$.

- The proper Fourier representations of the monochromatic wave are, given $\delta\phi_{0-k}(r,\tau) = \delta\phi_{0k}(r,\tau)^*$

$$\delta\hat{\phi}_k(\omega) = \frac{i\delta\phi_{0k}(r,\tau)}{2\pi[\omega - \omega(\tau)]} \qquad \delta\hat{\phi}_{-k}(\omega) = \frac{i\delta\phi_{0-k}(r,\tau)}{2\pi[\omega + \omega^*(\tau)]}$$

E: Show this result.

- Using these expressions in the renormalized F_0 equation, one can carry out readily the $dxdx'$ integration and obtain:

$$\hat{F}_0(\omega) = \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \frac{c}{\omega B_0}\frac{e}{m}\frac{1}{r}\frac{\partial}{\partial r}\left\{\left[\frac{Q^*_{k,\omega(\tau)}}{\omega^*(\tau)}\frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega - \omega(\tau) + \bar{\omega}_{dk}}\right.\right.$$
$$\left.\left. + \frac{Q_{k,\omega(\tau)}}{\omega(\tau)}\frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}}\right]\hat{\omega}_{dk}\left|\delta\phi_{0k}(r,\tau)\right|^2\right\}$$

E: Show this result. Hint: $\bar{\omega}_{d-k} = -\bar{\omega}_{dk}$ and $Q_{-k,-\omega^*(\tau)} = -Q^*_{k,\omega(\tau)}$.

- Two page summary equations for the strongly driven "monochromatic" fishbone burst, assuming (see Lecture 1) the MHD response comes from a linear background plasma

$$i|s|\Lambda\xi = \delta\hat{W}_{f,MHD}\xi + \left(\delta\hat{W}_{f,EP} + \delta\hat{W}_{k,EP}\right)\xi$$

- Introducing $\delta\phi_{0k}(r,\tau) = -(B_0/c)r\omega(\tau)\xi_{0k}(\tau)$, connecting the scalar potential fluctuation with the MHD radial displacement

$$\delta\hat{W}_{f,EP}\xi = \mathcal{I}\circ\int_{-\infty}^{\infty}\frac{(1/r)}{\omega_c}\frac{\partial}{\partial r}\hat{F}_0(\omega-\omega(\tau))e^{-i\omega t}\xi_{0k}(\tau)d\omega$$

$$\delta\hat{W}_{k,EP}\xi = \mathcal{I}\circ\int_{-\infty}^{\infty}\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}\hat{F}_0(\omega-\omega(\tau))}{\bar{\omega}_{dk}-\omega}e^{-i\omega t}\xi_{0k}(\tau)d\omega$$

$$\mathcal{I}\circ \equiv 4\frac{\pi^2}{B_0^2} m\omega_c^2 \frac{R_0}{r_s^2} \int_0^{r_s} \frac{r^3}{q} dr \int \mathcal{E} d\mathcal{E} d\lambda \sum_{v_\parallel/|v_\parallel|=\pm 1} \overline{e^{i\theta}\omega_d e^{-iq(r)\theta}} \tau_b$$

$$\hat{F}_0(\omega) = \frac{i}{\omega} \mathrm{St} \hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega} F_0(0) + \frac{\omega_c}{\omega}\frac{1}{r}\frac{\partial}{\partial r}\left\{ \left[\omega(\tau) Q^*_{k,\omega(\tau)} \frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega - \omega(\tau) + \bar{\omega}_{dk}} \right.\right.$$

$$\left.\left. + \omega^*(\tau) Q_{k,\omega(\tau)} \frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}} \right] \hat{\omega}_{dk} r^2 \left|\xi_{0k}(\tau)\right|^2 \right\}$$

- Note that the equation for the renormalized $\hat{F}_0$ is slightly generalized to include a generic collision operator in velocity space represented by $\mathrm{St}\hat{F}_0$.

  E: Derive the summary equations filling in the missing algebra and show that the inclusion of the collision operator is trivial.

  E: Discuss the physical meaning/role of $\gamma(\tau) = \Im\omega(\tau)$ in the equation for the renormalized $\hat{F}_0$.

- Numerical solution of the present system describing the nonlinear dynamics of a strongly driven "monochromatic" fishbone burst can be used in different applications:

  - Nonlinear electron - fishbone, of interest in FTU, TS, HL-1M, HL-2A: application to ion fishbone requires inclusion of finite Larmor radius and finite orbit widths, i.e. a further generalization of the above equations.
  - Nonlinear evolution of EPM bursts, where the interplay between mode dispersiveness and particle transport involves the radial mode structure as well (Lecture 5)
  - Weak nonlinear evolution of a monochromatic Alfvén Eigenmode. This implies setting $\gamma(\tau) \to 0$ in the $\hat{F}_0$ equation, obtaining a description of mode saturation due to wave-particle trapping. As noted by O'Neil 65 and Al'tshul' and Karpman 65, this approximation (single wave) adequately describes nonlinear dynamics due to wave-particle trapping but does not properly describe phase space mixing or spatial bunching, though it improves the solution based on Berk et al 96,97.

- With extension to treat transit resonance, the same approach could be used to analyze the self-consistent nonlinear problem of geodesic acoustic mode excitation by energetic particles, accounting for resonant excitation, transport, sources and sinks.

- Further simplification is possible based on time scale separation (Zonca et al 07). This approximation involves that fluctuation induced transport decorrelates wave and particles (chirping is a key element) but does not destroy wave-particle resonance. Formally one obtains a quasilinear equation.

RP: Solve anyone of the problems mentioned in the list above.

The quasilinear limit

- Assume that $\hat{F}_0(\omega)$ spectrum is sufficiently narrow and consider again the equation

$$i|s|\Lambda\xi = \delta\hat{W}_{f,MHD}\xi + \left(\delta\hat{W}_{f,EP} + \delta\hat{W}_{k,EP}\right)\xi$$

- The equation of $\delta\hat{W}_{f,EP}\xi$ is readily reduced without additional assumptions, since integration is carried out by shifting integration variable $\omega \to \omega + \omega(\tau)$

$$\delta\hat{W}_{f,EP}\xi = \mathcal{I} \circ \int_{-\infty}^{\infty} \frac{(1/r)}{\omega_c} \frac{\partial}{\partial r} \hat{F}_0(\omega - \omega(\tau)) e^{-i\omega t} \xi_{0k}(\tau) d\omega$$

$$\delta\hat{W}_{f,EP}\xi = \mathcal{I} \circ \frac{(\xi/r)}{\omega_c} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \hat{F}_0(\omega) e^{-i\omega t} d\omega = \mathcal{I} \circ \frac{(\xi/r)}{\omega_c} \frac{\partial F_0(t)}{\partial r}$$

- With the same procedure

$$\delta\hat{W}_{k,EP}\xi = \mathcal{I}\circ\int_{-\infty}^{\infty}\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}\hat{F}_0(\omega-\omega(\tau))}{\bar{\omega}_{dk}-\omega}e^{-i\omega t}\xi_{0k}(\tau)d\omega$$

$$\delta\hat{W}_{k,EP}\xi = \mathcal{I}\circ\int_{-\infty}^{\infty}\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}\hat{F}_0(\omega)}{\bar{\omega}_{dk}-\omega(\tau)-\omega}e^{-i\omega t}\xi d\omega = \mathcal{I}\circ\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}F_0(t)}{\bar{\omega}_{dk}-\omega(t)}\xi$$

  provided that the width of $\hat{F}(\omega)$ is small compared with respect to the resonance width. This is our crucial assumption.

- We need to carefully estimate the self-consistency of our assumption. This leads to the conclusion that $\gamma_{F_0} \ll \gamma$, so that the characteristic width of $\hat{F}_0$ is much less than γ.

- Consider the equation for $\hat{F}_0$

$$\hat{F}_0(\omega) = \frac{i}{\omega}\mathrm{St}\hat{F}_0(\omega) + \frac{i}{\omega}\hat{S}(\omega) + \frac{i}{2\pi\omega}F_0(0) + \frac{\omega_c}{\omega}\frac{1}{r}\frac{\partial}{\partial r}\left\{\left[\omega(\tau)Q^*_{k,\omega(\tau)}\frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega - \omega(\tau) + \bar{\omega}_{dk}}\right.\right.$$

$$\left.\left. + \omega^*(\tau)Q_{k,\omega(\tau)}\frac{\hat{F}_0(\omega - 2i\gamma(\tau))}{\omega + \omega^*(\tau) - \bar{\omega}_{dk}}\right]\hat{\omega}_{dk}r^2\left|\xi_{0k}(\tau)\right|^2\right\}$$

- Simple estimates give $\omega_B^2 \sim \omega^2|\xi|/r$ and $\Delta r/r \sim \gamma/\omega$ for a particle to de-correlate from resonance. This calculation, substituted above, readily yields

$$\gamma_{F_0} \sim \gamma(\omega_B^4/\gamma^4)$$

suggesting that near saturation ($\omega_B \sim \gamma$) the present approach is inadequate, yielding $\gamma_{F_0} \sim \gamma$ with very small particle displacement $\Delta r/r \sim \gamma/\omega \sim \omega_B/\omega$.

- The peculiarity of the fishbone is frequency chirping, connected with the loss of particle energy as the wave is amplified. The radial profile of the mode drive is peaked near the $q = 1$ surface (trapped particle population is strongest and the resonant energy largest). Using $n\dot{E} + \omega\dot{P}_\phi$

$$\Delta(\bar{\omega}_d - \omega) \simeq (\Delta E/E) \simeq -(\Delta r/r)\omega q(r/R_0)$$

- As a consequence, particles can move a distance Δr before de-correlation takes place (by finite interaction length, see Lecture 5), with $\Delta r/r \sim (\gamma/\omega)(R_0/rq)$. This yields

$$\gamma_{F_0} \sim \gamma(\omega_B^4/\gamma^4)q^2r^2/R_0^2 \sim \gamma q^2r^2/R_0^2$$

describing the radial scale due to nonlinear distortions of F_0.

- This last estimate is qualitatively correct but too simple minded, since it presumes that $\omega_B \sim \omega$ at saturation. A more precise ordering is obtained considering that, for typical strong fishbones, $\Delta\omega/\omega \sim 1$; i.e. considering $\Delta\omega \sim \dot{\omega}/\gamma_{F_0} \sim \omega_B^2/\gamma_{F_0} \sim \omega$, from which $\omega_B^2 \sim \gamma_{F_0}\omega$.

- Substitute $\omega_B^2 \sim \gamma_{F_0}\omega$ back in the estimate of γ_{F_0} of previous page and obtain

$$\gamma_{F_0} \sim \gamma\gamma^2/(q^2\omega^2)R_0^2/r^2 \qquad \omega_B^2 \sim \gamma^2\gamma/(q^2\omega)R_0^2/r^2$$

- This means that stronger modes have higher normalized saturation amplitude (ω_B/γ) and faster normalized nonlinear rate (γ_{F_0}/γ). This was to be expected for strong fishbone excitation and correspond to normal practice. The characteristic change of mode frequency in one burst is $\Delta\omega/\omega \sim 1$ and the typical radial displacement of a particle before it decorrelated from resonance condition is $\Delta r/r \sim (\gamma/\omega)(R_0/rq)$.

- Note that the "quasilinear approach" looses validity for $\gamma_{F_0} \sim \gamma$, corresponding to $\Delta r/r \sim (\gamma/\omega)(R_0/rq) \sim 1$ and $\omega_B^2/\gamma^2 \sim R_0/(rq^2)$. In that case, all time scales will be set by γ^{-1} and the initial distribution is destroyed, to the point that only numerical treatment is possible.

- Resonant excitation mechanism and self-consistent mode chirping is the key element for consistently introducing the time scale separation for rewriting the equation for the renormalized F_0 in the form of a quasi-linear equation.

E: Review the equation for $\hat{F}_0$ on p. 29 and derive the self-consistent description of wave-particle trapping for $\gamma \to 0$. Hint: in this case the appropriate estimate for the radial scale of $\hat{F}_0$ is Δr_B (see p.7) while time scale is $\omega^{-1} \sim \omega_B^{-1}$. If you leave them undetermined, you can find them consistently from the dimensionless form of the equation.

E: In the light of Lecture 2 and discussion on p. 25, explain which physics is included in the equation for $\hat{F}_0$ with trapping effects.

- Transforming to the t space the $\hat{F}_0$ equation (multiplying by $(-i\omega)e^{-i\omega t}$ and integrating in $d\omega$), this is rewritten readily as

$$\frac{\partial}{\partial t}F_0 = \mathrm{St}F_0 + S - i\frac{\omega_c}{r}\frac{\partial}{\partial r}\left\{\left[\omega(t)Q^*_{k,\omega(t)}\frac{F_0(t)}{2i\gamma(t)-\omega(t)+\bar{\omega}_{dk}}\right.\right.$$
$$\left.\left.+\omega^*(t)Q_{k,\omega(t)}\frac{F_0(t)}{2i\gamma(t)+\omega^*(t)-\bar{\omega}_{dk}}\right]\hat{\omega}_{dk}r^2\left|\xi(t)\right|^2\right\}$$

- The final fishbone burst nonlinear equations are closed by

$$i|s|\Lambda\xi = \delta\hat{W}_{f,MHD}\xi + \left(\delta\hat{W}_{f,EP} + \delta\hat{W}_{k,EP}\right)\xi$$

$$\delta\hat{W}_{f,EP}\xi = \mathcal{I}\circ\frac{(\xi/r)}{\omega_c}\frac{\partial}{\partial r}\int_{-\infty}^{\infty}\hat{F}_0(\omega)e^{-i\omega t}d\omega = \mathcal{I}\circ\frac{(\xi/r)}{\omega_c}\frac{\partial F_0(t)}{\partial r}$$

$$\delta\hat{W}_{k,EP}\xi = \mathcal{I}\circ\int_{-\infty}^{\infty}\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}\hat{F}_0(\omega)}{\bar{\omega}_{dk}-\omega(\tau)-\omega}e^{-i\omega t}\xi d\omega = \mathcal{I}\circ\hat{\omega}_{dk}\frac{Q_{k,\omega(\tau)}F_0(t)}{\bar{\omega}_{dk}-\omega(t)}\xi d\omega$$

Other considerations

RP: Use the quasilinear formulation above to derive the nonlinear fishbone equations in the model form proposed by Zonca et al 07.

E: Can you explain why, given the form of the nonlinear equation for the evolution of $\beta - \beta_c$, consistently accounting for the frequency chirping is not crucial for obtaining the dynamics of the mode burst? Can you say the same for the particle transport?

RP: Write a code that solves the equations at p. 31 and study the nonlinear fishbone cycle with various sources and sinks. Always remember the model assumptions.

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