Lecture 1

The physics of beam-plasma system (I):

wave-particle interactions and wave spectrum

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Lecture Plan and Teaching Material

Abstract: Prerequisite of the course is a preliminary knowledge of general plasma physics, fluid description of magnetized plasmas and kinetic theory. Most of the relevant introductory material can be found on the Lecture Notes from the IFTS Intensive Course on Advanced Plasma Physics-Spring 2009 *Kinetic theory of meso- and micro-scale fluctuations in magnetic fusion plasmas.*

Main areas that will be explored are:

i) The physics of beam-plasma system (I): wave-particle interactions and wave spectrum (Lecture 1)

ii) The physics of beam-plasma system (II): nonlinear behaviors (Lecture 2)

iii) Nonlinear Alfvén Eigenmode dynamics near marginal stability (Lecture 3)

iv) Beam modes in tokamaks: the fishbone burst cycle (Lecture 4)

v) Energetic Particle Modes and the notion of avalanche (Lecture 5)

vi) Nonlinear equilibria and long time scale behaviors (Lecture 6)

Sources: Large portions of these lecture notes (Lectures 1, 2, 3) are based on:

T.M. O'Neil, Phys. Fluids **8** 2255 (1965).
T.M. O'Neil and J.H. Malmberg, Phys. Fluids **11** 1754 (1968).
T.M. O'Neil, J.H. Winfrey and J.H. Malmberg, Phys. Fluids **14** 1204 (1971).
T.M. O'Neil and J.H. Winfrey, Phys. Fluids **15** 1514 (1972).

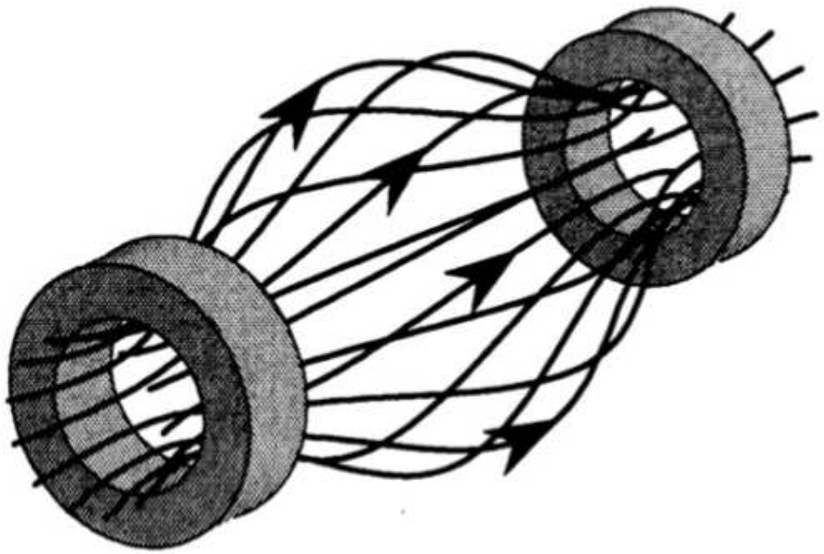
H.L. Berk, B.N. Breizman and M. Pekker, Phys. Rev. Lett. **76** 1256 (1996).
H.L. Berk, B.N. Breizman and N.V. Petiashvili, Phys. Lett. A **234** 213 (1997).

Important reading material: Prof. Liu Chen Lecture notes given at the Hefei 2008 Summer School and at Beijing University, September 2008. The Lecture Notes from the IFTS Intensive Course on Advanced Plasma Physics-Spring 2009 *Kinetic theory of meso- and micro-scale fluctuations in magnetic fusion plasmas*

Lecture Notes: Available in electronic form. At the end of each lecture, a list of specific reading material is given explicitly

Exercises and Research Projects: In the lectures, both Exercises (E) and Research Projects (RP) of various difficulty levels are suggested.

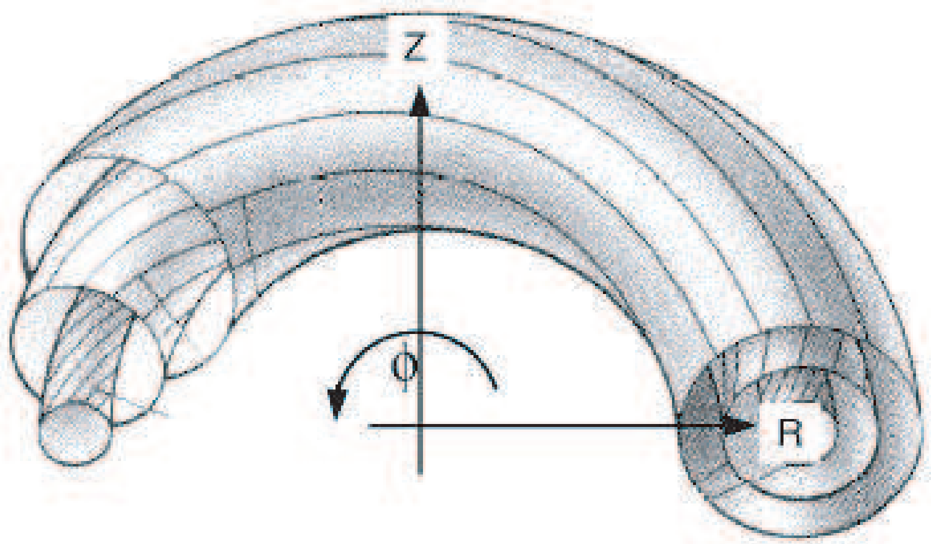
The beam-plasma system



The interaction of a supra-thermal electron beam with a plasma in a strong axial magnetic field was investigated by many authors in the early 60's, notably O'Neil, Malmberg, Mazitov, Ichimaru, Drummond, Shapiro ...

This simple system is very important since it was the problem on which various processes were understood:

- Mode dispersion relations and nonlinear behaviors
- Wave-particle interactions and Nonlinear Landau damping



The interest for the beam-plasma system has been revived in the 90's by Berk, Breizman and coworkers, who proposed it as paradigm for interpreting experimental observation of Alfvén Eigenmode excitation by energetic particles and related nonlinear dynamics and transport processes.

In Lectures 1,2,3 we will address the beam-plasma system physics and try to assess to which extent we may use it for interpreting burning plasma behaviors in toroidal systems:

- Advantages of using a simple 1-D system for complex dynamics studies
- Roles of mode structures and geometry in determining nonlinear behaviors

Brief review of basic concepts

- Electric displacement $\boldsymbol{D}$ and dielectric tensor (uniform plasma, $\boldsymbol{B} = \hat{\boldsymbol{z}}B$, Fourier representation for fields)

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E} = \boldsymbol{E} + \frac{4\pi i}{\omega}\boldsymbol{J}$$

- Susceptibility $\boldsymbol{\chi}$: $\boldsymbol{\epsilon} = \mathbf{1} + \boldsymbol{\chi}$ (sometimes $\mathbf{1} + 4\pi\boldsymbol{\chi}$) and conductivity tensor $\boldsymbol{\sigma}$: $\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$ are connected

$$\boldsymbol{\epsilon} = \mathbf{1} + \frac{4\pi i}{\omega}\boldsymbol{\sigma} \Rightarrow \boldsymbol{\chi} = \frac{4\pi i}{\omega}\boldsymbol{\sigma}$$

- "Cold" dielectric tensor (see, e.g., T.H. Stix *Waves in Plasmas* AIP 1992, New York NY)

$$\boldsymbol{\epsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

- With $\omega_{ps}^2 = 4\pi n_s e_s^2/m_s$ and $\omega_{cs} = e_s B/(m_s c)$

$$S = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2} \qquad D = \sum_s \frac{\omega_{cs}\omega_{ps}^2}{\omega(\omega^2 - \omega_{cs}^2)} \qquad P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

- Relevant case for the beam-plasma system studies is $B \to \infty, \omega_{cs} \to \infty$, for which the perpendicular dynamics is suppressed and, given $\omega_p^2 = \sum_s \omega_{ps}^2$

$$\boldsymbol{\epsilon} = \mathbf{1} - \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}(\omega_p^2/\omega^2)$$

- From Ampère's and Faraday's laws one derives

$$\boldsymbol{\nabla\nabla}\cdot\boldsymbol{E}-\nabla^2\boldsymbol{E}=\frac{\omega^2}{c^2}\boldsymbol{E}-\frac{\omega_p^2}{c^2}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\cdot\boldsymbol{E}$$

- Use $k^2c^2 \gg \omega^2$ and Poisson's equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\frac{\omega_p^2}{\omega^2}\frac{\partial E_z}{\partial z}$$

- ... to reduce the above equation to the following form:

$$\frac{\partial}{\partial z}\left(\frac{\omega_p^2}{\omega^2}\frac{\partial E_z}{\partial z}\right)-\nabla^2 E_z=0$$

E: Is the "electrostatic" approximation valid for the waves described by this last equation? Can you prove it and compute the magnetic perturbation?

E: Show that you can use the gyrokinetic equation to derive the result above. What allows you to do so? What additional information can you get?

Beam-plasma dispersion relation

- Follow O'Neil and Malmberg 1968. Assume ions as fixed neutralizing background and write the kinetic equation in one dimension (strong equilibrium $\boldsymbol{B}$) for the electron fluctuating δf

$$\partial_t \delta f + v \partial_z \delta f - (e/m) E_z \partial_v f = 0$$

- With $E_z = -\partial_z \delta\phi = E_{z0}(r,\theta)\exp(ikz - i\omega t)$ and $\omega_p^2 = 4\pi n e^2/m$, the Poisson equation becomes

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\omega_p^2}{k^2}\left(\frac{\partial}{\partial z} E_z\right) \frac{1}{n} \int_{-\infty}^{\infty} \frac{k \partial_v f}{kv - \omega} dv$$

- Assume that electrons are made by a thermal plus a supra-thermal component: $f/n = f_T + (n_B/n) f_B$, n, n_B the thermal and beam electron densities

- Governing equation: non-uniform case

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial z}E_z\right)\frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{k\partial_v(f_T+(n_B/n)f_B)}{kv-\omega}dv\right]-\nabla^2 E_z=0$$

E: Can you recover the equation for Langmuir wave in non-uniform plasmas (Lecture 2 - IFTS Intensive Course on Advanced Plasma Physics - Spring 2009)? What is the meaning of the continuous spectrum?

- Study first the uniform case, $E_{z0}(r,\theta)=E_{z0}$ as in O'Neil and Malmberg 1968, and take $f_T=(2\pi v_T^2)^{-1/2}\exp(-v^2/2v_T^2)$, with a "warm" beam (important for nonlinear saturation mechanism; see later)

$$f_B=\frac{v_B}{\pi}\frac{1}{(v-v_D)^2+v_B^2}$$

- Introduce the "warm" plasma dielectric constant

$$\epsilon(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2v_T^2}{\omega^2} + \dots\right) + i\sqrt{\frac{\pi}{2}}\frac{\omega\omega_p^2}{|k|^3v_T^3}\exp\left(-\frac{\omega^2}{2k^2v_T^2}\right)$$

E: Fill in the details using the definition of f_T to derive the expression of the "warm" plasma dielectric constant.

- The beam-plasma dispersion relation reads

$$\epsilon(k,\omega) = \frac{\omega_p^2}{k^2}\frac{n_B}{n}\int_{-\infty}^{\infty}\frac{k\partial_v f_B}{kv-\omega}dv$$

- And now some algebra ...

- Introduce $\Omega = (\omega - kv_D)/kv_B$

$$
\begin{aligned}
\frac{1}{k^2}\int_{-\infty}^{\infty}\frac{k\partial_v f_B}{kv-\omega}dv &= \frac{1}{\pi k^2 v_B^2}\int_{-\infty}^{\infty}\frac{-2x}{(x^2+1)^2(x-\Omega)}dx \\
&= \frac{1}{\pi k^2 v_B^2}\int_{-\infty}^{\infty}\frac{-2}{(x^2+1)^2}\left(1+\frac{\Omega}{(x-\Omega)}\right)dx \\
&= \frac{1}{k^2 v_B^2}\left[-1+\frac{2\Omega}{\pi}\frac{d}{dy}\int_{-\infty}^{\infty}\frac{dx}{(x^2+y)(x-\Omega)}\Bigg|_{y=1}\right] \\
&= \frac{1}{k^2 v_B^2}\left[-1+2\Omega\frac{d}{dy}\left(i\frac{k/|k|}{\Omega^2+y}-\frac{\Omega/\sqrt{y}}{\Omega^2+y}\right)\Bigg|_{y=1}\right] \\
&= \frac{1}{(\omega - kv_D + i|k|v_B)^2}
\end{aligned}
$$

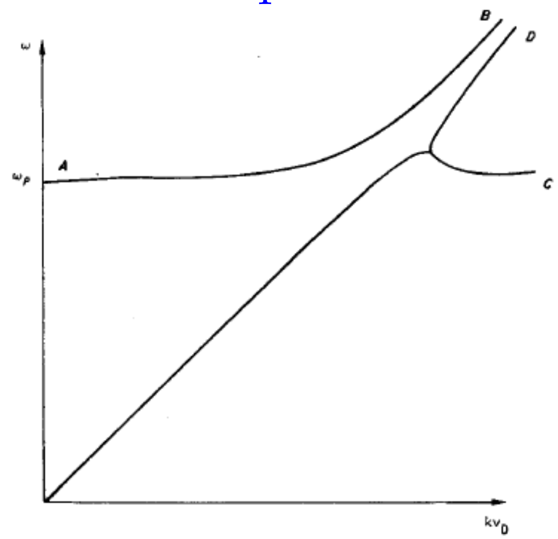
E: Fill in the few missing details. Note that $|k|v_B$ appears: why?

- The "warm" beam-plasma dispersion relation becomes (O'Neil and Malmberg 1968)

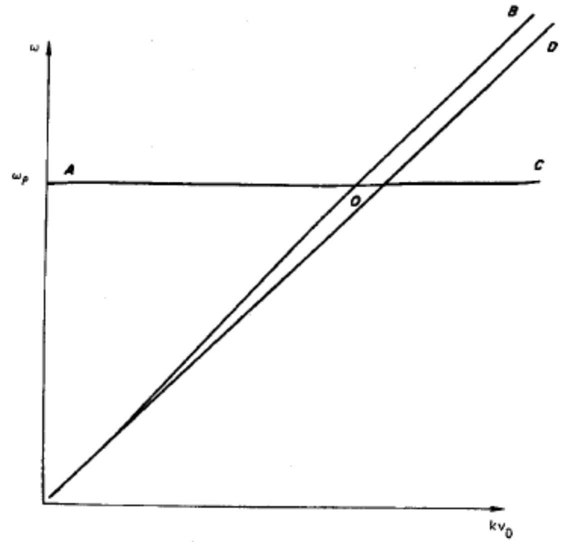
$$\epsilon(k,\omega) = \frac{n_B}{n}\frac{\omega_p^2}{(\omega - kv_D + i|k|v_B)^2}$$

- That $n_B/n \ll 1$ suggests that possible roots occur at $\omega^2 \simeq \omega_p^2$ (Langmuir wave) and $\omega \simeq kv_D$ (beam mode).

cold beam dispersion relation



warm beam dispersion relation



- Behaviors of a cold vs. warm beam are important for nonlinear dynamic properties of the system (Shapiro 1963), since the nonlinear evolution takes place in two stages:

  - first, beam-plasma interaction heats the beam
  - second, the beam distribution is flattened in velocity space by nonlinear interactions

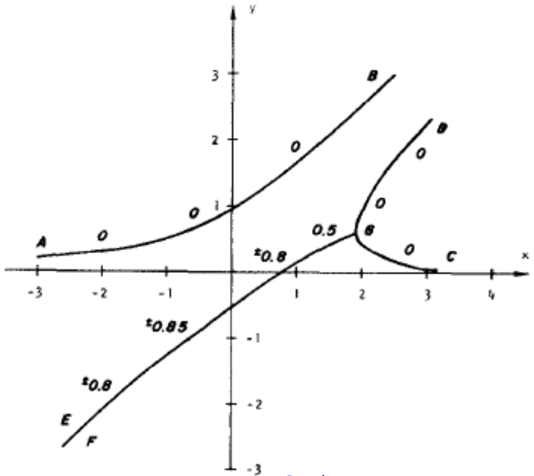
- Assume $v_D \gg v_T$ so that thermal electron warm effects are negligible. Solve for the unstable beam mode at $\omega \lesssim \omega_p$

$$\omega = kv_D - i|k|v_B + i|k|v_D \left(\frac{n_B}{n}\right)^{1/2} \frac{\omega_p}{\sqrt{\omega_p^2 - k^2 v_D^2}}$$

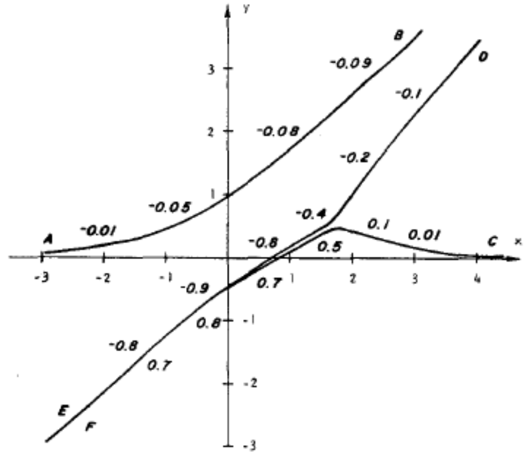
- Most unstable beam mode when $kv_D \to \omega_p$ and $\delta\omega = \omega_p - kv_D \sim (n_B/n)^{1/3}\omega_p \Rightarrow$ Important consequences on nonlinear dynamics

- Introduce $s = (v_B/v_D)(2n/n_B)^{1/3}$ (O'Neil and Malmberg 1968)

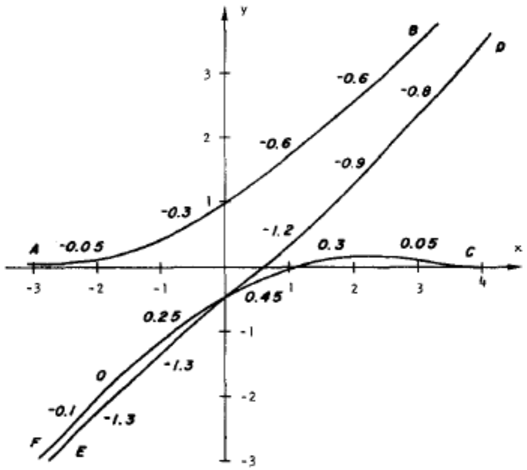
$s = 0$



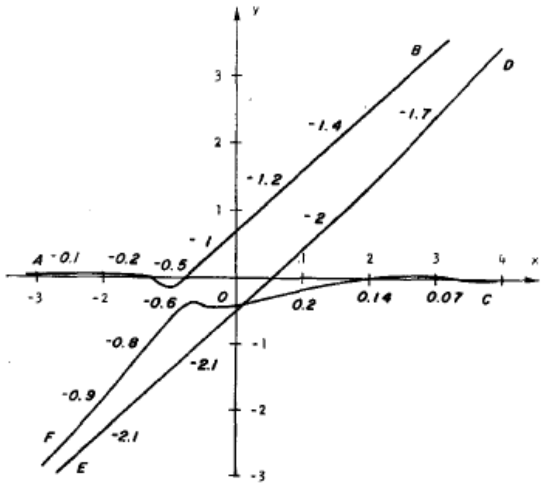
$s = 0.1$



$s = 0.7$



$s = 1.6$

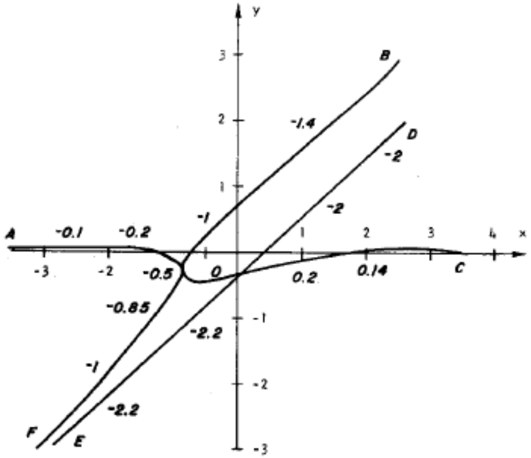


- Nonlinear dynamics is heavily influenced by the fact that most unstable root is obtained for $\omega_p = \omega_0 = k_0 v_D$ ($k_0 = \omega_p / v_D$, O'Neil and Malmberg 1968)

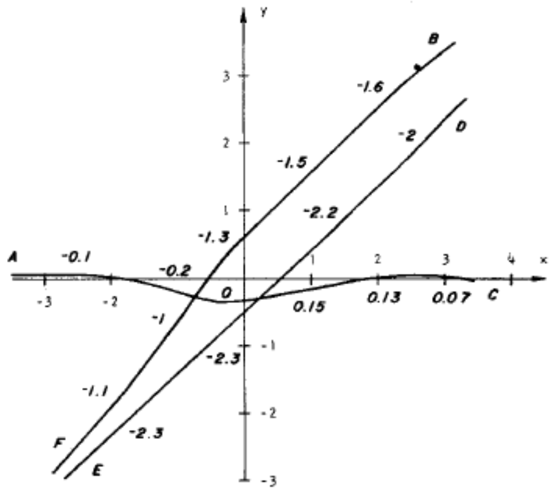
- Introduce $x = (\delta k / k_0)(2n/n_B)^{1/3}$, $y = (\delta \omega / \omega_0)(2n/n_B)^{1/3}$ (O'Neil and Malmberg 1968); the beam plasma system dispersion relation becomes

$$y(y - x + is)^2 = 1$$

$s = 1.63$



$s = 1.8$



- Some estimates for the cold beam case: $s = 0$

$$\frac{\partial\omega}{\partial k} = \frac{2v_D y}{3y - x}$$

$$\frac{\partial^2\omega}{\partial k^2} = \frac{2}{k_0}\left(\frac{2n}{n_B}\right)^{1/3}\left(\frac{-x(\partial\omega/\partial k) + v_D y}{(3y - x)^2}\right)$$

- The most unstable mode has $x = 0$, $y = -(1/2) + i\sqrt{3}/2$ and group velocity $\partial\omega/\partial k = (2/3)v_D$

- The half-width of the linear growth rate spectrum is

$$\Delta k = \left|\gamma_{max}^{-1}\frac{\partial^2\gamma}{\partial k^2}\right|^{-1/2} = \frac{3}{2}k_0\left(\frac{n_B}{2n}\right)^{1/3}$$

E: Derive these results step by step. What happens for $s \neq 0$?

Extension to finite size systems

- Governing equation in non-uniform case

$$\left[\epsilon(k,\omega) - \frac{n_B}{n}\frac{\omega_p^2}{(\omega - kv_D + i|k|v_B)^2}\right] E_{z0}(r,\theta) - k^{-2}\nabla_\perp^2 E_{z0}(r,\theta) = 0$$

- Follow O'Neil and Malmberg 1968 and assume that the plasma column is surrounded by a conducting cylinder of radius R and that both thermal plasma and beam component are uniform

- Then the solution is $E_{z0}(r,\theta) = E_0 e^{i\ell\theta} J_\ell(\lambda_{\ell,m} r/R)$, with $\lambda_{\ell,m}$ the m-th zero of the Bessel function $J_\ell(x)$. The beam-plasma dispersion relation is

$$\epsilon(k,\omega) + \frac{\lambda_{\ell,m}^2}{k^2R^2} = \frac{n_B}{n}\frac{\omega_p^2}{(\omega - kv_D + i|k|v_B)^2}$$

E: Show this result and demonstrate that $\ell = 0$ state is the most unstable one.

- Extend the O'Neil and Malmberg 1968 problem to a nonuniform beam with $n_B = n_{B0} \exp(-r^2/L^2)$, $L \ll R$. Look for solutions (ground state) in the form $E_{z0}(r,\theta) = E_0 e^{i\ell\theta}(r/R)^\alpha \exp(-\beta r^2)$

E: Show that, provided $\beta \gg 1/L^2$, ground state solutions are found for $\alpha = \ell$ and obtain the equation below for β.

- Solutions that are "radially trapped" by the beam ($\beta \gg 1/L^2$) can be found, which satisfy the beam-plasma dispersion relation is

$$\epsilon(k,\omega) + \frac{4\beta}{k^2}(\ell+1) = \frac{n_B}{n}\frac{\omega_p^2}{(\omega - kv_D + i|k|v_B)^2}$$

and have β set by the condition

$$\beta = \frac{(\ell+1)}{2L^2} + \sqrt{\frac{(\ell+1)^2}{4L^4} + \frac{k^2}{L^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \simeq \frac{|k|}{L}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- These results show that there are no unstable radially trapped solutions within a localized beam, i.e. no beam mode exists that can yield further radial modulation of the beam itself
- This is a further element that strongly impacts the nonlinear evolution of the system (Lecture 2)

E: Can you demonstrate the two statements above? Can you relate this fact with the invariants of particle motions?

RP: Solve numerically the equation governing the beam-plasma system in the nonuniform case, assuming Dirichlet boundary conditions and constant thermal plasma density, with a given $n_B(r)$. Describe the beam-plasma dispersion relation and mode structures, including "warm" thermal plasma effects. Discuss your results in the light of the statements above and draw your own conclusions about the impact of your results on the nonlinear system behavior.

Plasma equilibrium and particle invariants

- The equilibrium magnetic field is given by $\boldsymbol{B} = \hat{\boldsymbol{z}}B(r)$ with the equilibrium condition

$$\frac{\partial}{\partial r}\left(B^2(r) + 8\pi P(r)\right) = 0$$

- The $B \to \infty$ limit implies $B(r) = \text{const}$, as well as $\omega_c \to \infty$ and $\rho_L \to 0$.

- In the presence of a scalar potential fluctuation in the form $\delta\phi = \delta\phi_\ell(r)\exp(i\ell\theta - i\omega t + ikz)$, the particle invariants are:

  - Perpendicular kinetic energy (from magnetic moment conservation and $B(r)$ =const)

  - Extended phase space "Hamiltonian": $\ell E - \omega P_\theta$ (E: Show it!)

E: Show that, for particles with charge e and mass m, the canonical momentum in an e.m. field is $\boldsymbol{p} = m\boldsymbol{v} + (e/c)\boldsymbol{A}$. Hint: recall that a general force can be given by the expression $\boldsymbol{F} = -\partial_{\boldsymbol{q}} U(\boldsymbol{q}, \dot{\boldsymbol{q}}) + (d/dt)\partial_{\dot{\boldsymbol{q}}} U(\boldsymbol{q}, \dot{\boldsymbol{q}})$

E: Show that $\dot{E} = (\omega/k)\dot{p}_z$

E: Show that, after averaging on the fast cyclotron motion, the poloidal canonical angular momentum is $P_\theta = (-e/c)Br^2/2$.

- Using the conservation of extended phase space Hamiltonian, $\ell\dot{E} = \omega\dot{P}_\theta$, and $\dot{E} = (\omega/k)\dot{p}_z$, it is possible to demonstrate that

$$\frac{\ell/r}{kr}\frac{\Delta v_z}{\omega_c} = \frac{\Delta r}{r} \simeq \frac{\gamma}{\omega}\frac{\ell/r}{kr}\frac{v_D}{\omega_c}$$

- This shows that beam particles are essentially tight to move at constant r, anchored to the local field lines $\boldsymbol{B} = \hat{\boldsymbol{z}}B$.

Anticipating some nonlinear dynamics issues

- The beam-plasma dispersion relation shows that:
  - The beam mode is most unstable when it is degenerate with the Langmuir wave.
  - In the nonuniform case, the beam finite size cannot effectively trap a radially varying beam mode structure
  - Particle invariants show that beam particles essentially move at r =const
- These properties delineate an analogy between single Alfvén Eigenmode behaviors in burning plasmas with those of Langmuir waves. However they show a strong contrast between dynamics of beam-mode in linear systems and Energetic Particle Modes in toroidal devices.

References and reading material

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