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The mixed Wentzel–Kramers–Brillouin-full-wave approach and its application to lower hybrid wave propagation and absorption

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The mixed Wentzel–Kramers–Brillouin (WKB)-full-wave approach for the calculation of the 2D mode structure in tokamak plasmas is further developed based on our previous work [A. Cardinali and F. Zonca, Phys. Plasmas 10, 4199 (2003) and Z. X. Lu et al., Phys. Plasmas 19, 042104 (2012)]. A new scheme for theoretical analysis and numerical implementation of the mixed WKB-full-wave approach is formulated, based on scale separation and asymptotic analysis. Besides its capability to efficiently investigate the initial value problem for 2D mode structures and linear stability, in this work, the mixed WKB-full-wave approach is extended to the investigation of radio frequency wave propagation and absorption, e.g., lower hybrid waves. As a novel method, its comparison with other approaches, e.g., WKB and beam tracing methods, is discussed. Its application to lower hybrid wave propagation in concentric circular tokamak plasmas using typical FTU discharge parameters is also demonstrated. © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4798408]

I. INTRODUCTION AND BACKGROUND

The spatiotemporal evolution of 2D mode structures in tokamak plasmas is important and various methods are used to actually compute that for different purposes. In radio frequency wave heating/current drive problems, one important issue is the wave energy/momentum propagation and wave spectrum evolution, thus the Wentzel–Kramers–Brillouin (WKB) method is widely used in these analyses. The WKB method treats the solution in the form of Eikonal representation and expands the original partial differential equation order by order using a smallness parameter; i.e., the variation of the wave vector in a wavelength. The boundary/initial conditions are fixed by the antenna, which can be also an “internal antenna” when applied to the investigation of plasma stability as, e.g., in Reversed Shear Alfven Eigenmode (RSAE) and Beta induced Alfven Eigenmode (BAE) problems. Besides the traditional WKB method, other methods, e.g., complex WKB and beam tracing methods, are proposed and numerically implemented to investigate the radio frequency wave propagation, taking into account wave propagation properties usually neglected in the standard WKB approach. For example, the extension of WKB method to the complex space is proposed to treat the diffraction effect and the non-Hermitian component of the dielectric tensor. The methods mentioned above have their own limitations, e.g., the failure to describe diffraction in the traditional WKB method and the complications of the continuation of the solution to the physical (real) space in the complex WKB method. More recently, with the great progress of the super computer capabilities, full wave solvers have been developed to investigate the propagation and absorption of lower hybrid wave in tokamak plasmas.

Another option, based on asymptotic analysis, consists in a mixed WKB-full-wave approach, proposed for the 2D mode structure analysis and studied theoretically and numerically. The original motivation was based on the point view of wave-packet propagation, noting that in many problems, the parallel mode structure is generated before a significant propagation of the wave-packet in the radial direction if \( v_{g\perp} \ll v_{g\parallel} \), where \( v_{g\perp} \) and \( v_{g\parallel} \) are the wave-packet group velocities in the perpendicular and parallel directions with respect to the equilibrium magnetic field. Then, it is possible to separate the original two dimensional wave equation into two one dimensional problems, i.e., the parallel wave equation and the radial propagation. This mixed approach can be applied, e.g., to investigate the 2D structure of Alfven eigenmodes and electrostatic drift-waves, and the parallel wave equation reduces to the well known “ballooning representation.” The application of this mixed approach was also proposed to the lower hybrid wave propagation, based on the fact that in the “forced” RF problem, the wave pattern can always be considered as superposition of a band of eigenmodes, with the coefficients determined by the boundary condition. This approach, however, becomes cumbersome in numerical implementation. Thus, in this work, a further step is made to demonstrate that an alternative approach can be adopted to investigate the lower hybrid wave propagation efficiently, and numerical simulations are performed to demonstrate it. The wave equation is analyzed in the electrostatic limit for the sake of simplicity and as elementary illustration of the WKB-full-wave approach, taking into account the wave absorption determined by the anti-Hermitian part of the dielectric tensor in general tokamak geometry. Meanwhile, we note that the present approach is readily extended to the electromagnetic case. The mixed WKB-full-wave approach is formulated in Sec. II, in the framework of asymptotic analysis of electrostatic waves in tokamak plasmas. The breaking down of the mixed method for the analysis of wave propagation near the wave reflection layer is handled by analytic
matching to the full wave solution, which is given explicitly in the case of circular geometry. In Sec. II, the mixed WKB-full-wave approach is also compared with other methods, such as complex WKB and beam tracing, showing that wave phenomena, e.g., diffraction, are tractable in the present analysis. In Sec. III, the mixed WKB-full-wave approach is applied to electrostatic lower hybrid wave propagation in concentric circular geometry, providing numerical simulation results as demonstration of its effective applicability to addressing this problem.

II. THE ASYMPTOTIC ANALYSIS OF THE ELECTROSTATIC WAVE IN A TOROIDAL PLASMA

A. Physics model

To investigate the wave propagation and absorption in the electrostatic limit, we start from the Vlasov-Poisson system

\[ \mathcal{L}(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) = \nabla \cdot (\bar{\varepsilon}(\mathbf{r}) \cdot \nabla \tilde{\psi}(\mathbf{r})) = 0, \]  

where \( \bar{\varepsilon}(\mathbf{r}) = \varepsilon^w_{\perp}(\mathbf{r}) + i \varepsilon^A(\mathbf{r}) \),

\[ \mathcal{L}(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) = \mathcal{L}^H(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) + \mathcal{L}^A(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}). \]  

Here, the form of \( \bar{\varepsilon} \) for electromagnetic wave with thermal effects is given by Ref. 21. In the cold plasma limit, the “cold” dielectric tensor provides a good approximation of \( \varepsilon^w_{\perp} \), with

\[ \bar{\varepsilon}^H(r) = Sf_n + (P - S)bb - iDb \cdot \bar{\varepsilon}, \]  

where \( \bar{\varepsilon} \) is the Levi-Civita symbol and \( S, D, \) and \( P \) are elements of the cold dielectric tensor in Stix notation.\(^{22} \) As to the anti-Hermitian part, finite thermal effects must be kept and, since \( \varepsilon^A \) is wave vector dependent, its representation in configuration space is given as a convolution integral,\(^{15} \) i.e.,

\[ \varepsilon^A(\mathbf{r})E(\mathbf{r}) = \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} \varepsilon^A(\mathbf{k})E(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d\mathbf{k}' \tilde{\varepsilon}^A(\mathbf{r}' - \mathbf{r}, \mathbf{r}')E(\mathbf{r}'), \]  

where \( \tilde{\varepsilon}^A(\mathbf{r}' - \mathbf{r}, \mathbf{r}) \) and \( E(\mathbf{r}') \) are Fourier conjugates of \( \varepsilon^A(\mathbf{k}) \) and \( E(\mathbf{k}) \). In general, the more accurate electromagnetic description of lower hybrid wave propagation and absorption should be adopted, with the thermal effect taken into account. In fact, at the plasma edge near the antenna, the electrostatic approximation is not capable to address the coupling problem. Furthermore, since the electrostatic approximation suppresses the slow-fast wave mode conversion artificially, the electromagnetic description is, again, needed in poor accessibility condition. On the other hand, assuming as boundary conditions a wave-packet with a specific shape at the starting surface, supplied by any electromagnetic solver describing antenna coupling in realistic conditions with good accessibility, we can still adopt the electrostatic analysis for the sake of simplicity, thereby illustrating the characteristic features of the mixed WKB-full-wave approach while keeping technical complications at a minimum. Given boundary conditions and the distribution function, the solution of the integro-differential system, Eq. (1), gives the wave field pattern with absorption included. For cases with \( \varepsilon^A \ll \varepsilon^H \) (weak damping), e.g., lower hybrid wave propagation in fusion plasmas away from the hybrid resonant layer, the wave propagation can be calculated by considering the Hermitian part only and, then, as higher order correction, the anti-Hermitian part can be taken into account to describe the absorption’s effect in the amplitude equation. Thus, for the wave propagation, we have

\[ \mathcal{L}^H(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) = \nabla \cdot (\mathcal{S} \nabla \tilde{\psi}(\mathbf{r})) + \nabla \cdot ((P - S) \nabla \tilde{\psi}(\mathbf{r})). \]  

Consider, now, tokamak magnetic flux coordinates \( (r, \theta, \zeta) \), where \( r, \theta, \) and \( \zeta \) are the radial-like, poloidal-like, and toroidal-like coordinates, respectively, constructed in such a way that the magnetic field lines are straight, as discussed in our previous work and references therein.\(^{19} \) With the Jacobian defined as \( J = (\nabla r \cdot \nabla \zeta \times \nabla \theta)^{-1} \), the equation in this coordinates is

\[ \mathcal{L}^H(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) = \frac{1}{J} \left[ \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right) \frac{P - S}{J B^2} \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right) \tilde{\psi} \right] \right] + \frac{P - S}{J B^2} \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right)^2 \tilde{\psi} + \frac{1}{J} \frac{\partial}{\partial \zeta} \left( S \tilde{g}^{\theta \zeta} \right) \frac{\partial}{\partial \zeta} \tilde{\psi}, \]  

where \( J = (\psi)'^{-1}J \), with \( \psi \) defined as the poloidal magnetic flux and its derivative is with respect to \( r \), and, repeated Greek symbols implicitly mean summation. Considering short wavelength waves, in a simplified case, we can ignore the variation of equilibrium parameters and geometric factors compared with that of the wave field, e.g., \( \nabla_r S / \nabla_r \psi = O(n^{-1}) \), where the toroidal mode number \( n \) is much larger than 1. Also reminding that \( P \) is a flux function, Eq. (6) can be reduced to\(^{16} \)

\[ \mathcal{L}^H(\mathbf{r}, \nabla) \tilde{\psi}(\mathbf{r}) = \frac{P - S}{J B^2} \left( \frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right)^2 \tilde{\psi} + S \tilde{g}^{\theta \zeta} \frac{\partial^2}{\partial \zeta^2} \tilde{\psi}. \]  

In the following analysis, moreover, we keep the variation of equilibrium parameters and geometric factors since it might be important for higher order corrections. Then, one step to simplify the equation is to chose \( \zeta \) as the toroidal angle, such that the metric elements \( g^{\theta \zeta} \) and \( \tilde{g}^{\theta \zeta} \) vanish. Meanwhile, considering only the linear problem choosing \( \tilde{\psi}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \tilde{\psi}(\mathbf{r}, 0) \), the Hermitian part of the wave equation in straight field line coordinates becomes
Equation (8) gives the two-dimensional (2D) structure of the electrostatic wave without absorption, originally described by Eq. (1). Choosing the lower hybrid frequency range, i.e., \( \Omega_e \ll \omega \ll \Omega_c \), Eq. (8) is used to describe the lower hybrid wave propagation for current drive in tokamak plasmas with

\[
\mathcal{L}^H (r, \nabla) \psi (r) = \frac{P - S}{j \beta^2} \left( \frac{\partial}{\partial \theta} + i n q \right)^2 \psi + \frac{1}{j} \left( \frac{\partial}{\partial \theta} p - S \right) \left( \frac{\partial}{\partial \theta} + i n q \right) \psi + S \left( \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial r \partial \theta} \right) \psi,
\]

where \( D_{QL} \) is the quasi-linear diffusion coefficient and \( \partial F_e / \partial \theta \) is the Fokker-Planck collision term. In the electrostatic limit,

\[
D_{QL}(v) = \frac{4}{2v^3} \delta \left( k_1 - \omega \right) |E_\parallel(k_1)|^2,
\]

where \( E_\parallel(k_1) \), the Fourier component of the electric field, can be derived from the scalar field \( \psi (r) \), the solution of Eq. (1). As a closed system, Eqs. (1) and (12) can be calculated together to yield the self-consistent wave pattern, with quasi-linear absorption taken into account. However, for a typical tokamak with size of \( \sim 1 \) m, the calculation of field pattern with short wavelength \( (\lambda_\parallel \sim 1 \text{ cm and } \lambda_\perp \sim 1 \text{ mm}) \) is very demanding of computer resources if the full-wave problem is addressed. On the contrary, the asymptotic approach based on a wave-packet propagation picture works well and efficiently. As in other WKB approaches, the wave is launched by the antenna and propagates in the plasmas and reflects between the periphery and axis region until the energy is absorbed by the resonant particles. The mixed WKB-full-wave approach and its advantages are discussed below.

### B. Asymptotic analysis using the mixed WKB-full-wave approach

The field-aligned structure is typical of many fluctuations in tokamak plasmas. As to the lower hybrid wave, Eq. (1) analysis with plane wave approximation, \( k_\perp / k_\parallel = -P / S \), demonstrates a similar property, \( |k_\parallel| \ll |k_\perp| \). The analogy of these two disparate cases, namely, the linear stability problem and the forced radio frequency wave propagation, can be summarized by a more general condition that the parallel group velocity is much larger than the perpendicular group velocity. When this requirement is satisfied, the wave-packet propagates rapidly along the field line and the parallel mode structure is generated without significant perpendicular (radial) propagation. Then, it is reasonable to solve a full wave equation in parallel direction and trace the radial propagation using the WKB method. For linear stability analysis, the discrete spectrum of the parallel wave equation is usually restricted to the most unstable mode, which makes the solution of the parallel full-wave problem very efficient and rapid. On the other hand, in the radio frequency wave propagation problem, a band of parallel mode structures are excited by the antenna with every eigenmode that can be, in principle, derived from the mixed WKB-full-wave approach. This approach, however, is extremely inefficient, therefore, in this work, we discuss an alternative and efficient approach, based on the same mixed WKB-full-wave approach for the
calculation of the wave propagation and absorption. Its comparison with other asymptotic methods, i.e., conventional WKB, complex WKB and beam tracing, is also discussed.

1. The mixed WKB-full-wave approach in the bulk plasma region

The perturbed scalar potential for the wave equation can be written in the Eikonal form

$$\tilde{\psi}(r) = \tilde{\psi}(r_0) \exp \left\{ \int_{r_0}^r k \cdot dr \right\},$$

where $r_0$ is a reference point, $k = k(r)$ is a complex vector field in the configuration space, and $\tilde{\psi}(r)$ value is independent on the integral path. For a well-defined and non-vanishing field, we have

$$k = -i \nabla \tilde{\psi} / \tilde{\psi}. \quad (15)$$

A given $\tilde{\psi}(r)$ has a one-to-one correspondence with the vector field $k(r)$ and, thus, an equation for $\tilde{\psi}$ can be cast into an equivalent one for $k(r)$. For 2D mode structures with a much shorter wavelength than the variation length of the equilibrium parameters, investigated here, the vector field $k(r)$ is much smoother spatial function than $\tilde{\psi}(r)$, and thus, the resolution required to properly represent $k(r)$ is much lower than that for $\tilde{\psi}(r)$ from the point view of numerical calculation. Now, considering the 2D problem of Eq. (8) with the anti-Hermitian part taken into account, we have the equivalent form as

$$D = \frac{P - S}{J^2 B^2} \left[ \frac{\partial m}{\partial \theta} - (m + nq)^2 \right] + \frac{i(m + nq)}{J} \left( \frac{\partial P - S}{\partial \theta} \right) B^2 + S \left[ g^{tr} \left( \frac{\partial k_r}{\partial r} - \frac{\partial^2}{\partial r^2} \right) + g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} - m_0^2 \right) - 2g^\theta \frac{m_0 k_{r0}}{R^2} \right] + \frac{i k_{r0}}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right) + \frac{i m_0}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right) = 0, \quad (16)$$

where $\tilde{\psi}(r, i) = \tilde{\psi}(r, \theta) e^{i\omega \theta}$ gives the parallel mode structure, with $i$ defined as the coordinate along the magnetic field, i.e., $\nabla_{\|} \equiv \partial_i$; and the identity $\partial m/\partial r = \partial k_r/\partial \theta$ is used (see the discussion later on in this section). When the WKB applicability condition holds in the radial direction, i.e., $\epsilon \sim |\nabla k_r/k_r^2| < 1$, the solution $k_r(\theta)$ and $m_r(\theta)$ can be obtained asymptotically by writing their series expansion order by order in $\epsilon$

$$k_r = k_{r0} + k_{r1} + k_{r2} + \cdots, \quad (17)$$

$$m = m_0 + m_1 + m_2 + \cdots. \quad (18)$$

The lowest order solution can be obtained by ignoring $\partial k_r/\partial r$, $\partial m/\partial r$ and the anti-Hermitian part, i.e.,

$$D_0 = \frac{P - S}{J^2 B^2} \left[ \frac{\partial m_0}{\partial \theta} - (m_0 + nq)^2 \right] + \frac{i(m_0 + nq)}{J} \left( \frac{\partial P - S}{\partial \theta} \right) B^2 + S \left[ g^{tr} \left( \frac{\partial k_{r0}}{\partial r} - \frac{\partial^2}{\partial r^2} \right) + g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} - m_0^2 \right) - 2g^\theta k_{r0}^2 \frac{m_0 k_{r0}}{R^2} \right] + \frac{i k_{r0}}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right) + \frac{i m_0}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right) = 0. \quad (19)$$

Given $m_0(r, \theta)$ on a magnetic flux surface, $k_{r0}(r, \theta)$ is obtained readily

$$k_{r0} = \frac{1}{2A} (-B \pm \sqrt{B^2 - 4AC}),$$

$$A = -g^{tr} S,$$

$$B = -2g^\theta S m_0 \frac{\partial}{\partial \theta} (SJ_{gr}^r) + \frac{i}{J} \frac{\partial}{\partial \theta} (SJ_{gr}^{\theta \theta}),$$

$$C = \frac{P - S}{J^2 B^2} \left[ \frac{\partial m_0}{\partial \theta} - (m_0 + nq)^2 \right] + \frac{i(m_0 + nq)}{J} \left( \frac{\partial P - S}{\partial \theta} \right) B^2 + S \left[ g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} - m_0^2 \right) \frac{n^2}{R^2} \right] + \frac{i m_0}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right) + \frac{i m_0}{J} \left( \frac{\partial m_0}{\partial \theta} \right) g^{\theta \theta} \left( \frac{\partial m_0}{\partial \theta} \right). \quad (20)$$
and thus the equation for the parallel mode structure is satisfied up to $O(\epsilon^0)$. As a further step to complete the equation set for $k_0$ and $m_0$, we note the identity,

$$\nabla \times \nabla \psi = i(\nabla \times k)\psi - k \times k\psi = 0,$$  \hspace{1cm} (21)

from which

$$\nabla \times k = 0.$$  \hspace{1cm} (22)

Noticing Eq. (22), the uniqueness in constructing $\tilde{\psi}$ is guaranteed, since it is equivalent to

$$\exp\left\{\epsilon k_{(r, \theta)} \cdot dr\right\} = 1;$$  \hspace{1cm} (23)

and, thus $\tilde{\psi}$ computed from Eq. (14) via different integral paths gives the same result. At the lowest order, we have

$$\nabla \times k_0 = 0,$$  \hspace{1cm} (24)

which, in the 2D case, readily yields

$$\frac{\partial m_0}{\partial r} = \frac{\partial k_0}{\partial \theta}.$$  \hspace{1cm} (25)

This equation may be used to calculate $m_0$ evolution from one flux surface to a neighboring one, when $k_0$ is known at a given radius, as obtained from the WKB solution in the radial direction. Equations (20) and (25) can be used to describe the lowest order wave-packet propagation. Since $\tilde{\psi}(r_0, \theta)$ is given on the starting surface, e.g., given by antenna boundary conditions, $m_0(r_0, \theta) = m(r_0, \theta)$ is obtained readily from $\psi(r_0, \theta)$ by setting $m_1(r_0, \theta) = m_2(r_0, \theta) = \cdots = 0$, and $k_0(r_0, \theta)$ is derived from the local parallel wave Eq. (20). Equation (25) is then used to propagate $m_0(r, \theta)$ to an adjacent flux surface along the propagation direction and so on, until $k_0$ and $m_0$ are obtained in the region of interest and the corresponding $\psi_0(r, \theta)$ is constructed. Note that, already at this lowest order expansion, the present approach is different from the standard WKB method, due to the presence of $\partial \theta m_0$ in Eq. (20).

The higher order corrections can be obtained noting

$$k_{i1} \sim O(\epsilon), \quad \frac{\partial k_{0}}{\partial r} \sim O(\epsilon), \quad \frac{\partial m_0}{\partial r} \sim O(\epsilon), \quad \frac{\partial m_1}{\partial \theta} \sim O(\epsilon),$$  \hspace{1cm} (26)

where $m$ and $\partial m/\partial \theta$ are treated on the same footing, consistent with a full wave treatment in the parallel direction, although their value might be disparate, as in the case where standard 2D WKB may be adopted. Then, the equation at $O(\epsilon)$ is obtained as

$$D_1 = \frac{P - S}{J^2 B^2} \left[ \frac{\partial m_1}{\partial \theta} - 2m_0 m_1 - 2m_0 m_1 \right] + \frac{im_1}{J} \left( \frac{\partial P - S}{\partial \theta} J B^2 \right)$$

$$+ S \left[ i g^r \frac{\partial k_{0}}{\partial r} - 2g^r k_{0} k_{r1} + g^\theta \left( \frac{\partial m_1}{\partial \theta} - 2m_0 m_1 \right) + \frac{im_1}{J} \left( \frac{\partial J}{\partial \theta} g^\theta \right) + \frac{im_1}{J} \left( \frac{\partial J}{\partial \theta} g^\theta \right) \right]$$

$$+ \frac{ik_{1}}{J} \left( \frac{\partial}{\partial r} S J g^r \right) + \frac{im_1}{J} \left( \frac{\partial}{\partial \theta} S J g^\theta \right) + \frac{im_1}{J} \left( \frac{\partial}{\partial \theta} S J g^\theta \right)$$

$$+ i \frac{1}{2\pi \psi_0(r, \theta)} \frac{\partial}{\partial t} \int \delta' \varepsilon_{F,\parallel}^{A}(t - t') \frac{\partial}{\partial t'} \tilde{\psi}_0(r, t') = 0.$$  \hspace{1cm} (27)

This equation is readily solved for $k_{r1}$ on a given flux surface, where $k_0$ is known by Eq. (20), while $m_0$ is known from the previous radial position using Eq. (25), determining, thus, $\tilde{\psi}_0(r, \theta)$. Meanwhile, similarly to $m_0(r, \theta)$, $m_1(r, \theta)$ is also known from the previous radial position and is obtained from

$$\frac{\partial m_1}{\partial r} = \frac{\partial k_{r1}}{\partial \theta},$$  \hspace{1cm} (28)

noting that $m_1(r, \theta) = 0$ at the initial position $r_0$. The contribution from the anti-Hermitian part is computed by the convolution term without referring to the Poynting theory to calculate the electric field from the power density, as one may do in the standard WKB approach, since the parallel mode structure in the present treatment gives the electric field at the lowest order on the whole flux surface. After the wave field at $O(\epsilon)$ is obtained, higher order corrections can be derived with the same approach. Since the derivation is straightforward but tedious, we omit its explicit derivation.

2. **Local full-wave solution near the reflection layer**

When the wave-packet propagates to a region where WKB approximation in radial direction breaks down, the full-wave solution should be adopted to derive appropriate matching in the connection region and, hence the physical wave reflection condition. The partial differential equation in the reflection region can be solved either analytically, in simplified case, or by one of the known numerical methods, with boundary conditions given in the form of the matching condition set by the mixed WKB-full-wave solution in the neighboring region. In the present case, with the scope of elucidating the solution method, we give an analytical treatment assuming a high aspect ratio tokamak with concentric circular surface. The equation can then be locally expanded.
to obtain the local asymptotic solution of the full-wave equation. In the high density reflection layer near the axis, the lowest order equation is given by the one-dimensional cylindrical approximation, which can be used to obtain an asymptotic solution as in Ref. 25. In this central region, e.g., $r/a < 0.1$, the toroidal correction $\sim r/R_0$ is much smaller than the inverse aspect ratio, $a/R_0$. Ignoring this correction and writing the full wave Eq. (5) at the lowest order, we have

$$\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \left( 1 - \frac{P}{S} \right) \left( \frac{nq + m}{q_0 R_0} \right)^2 - \frac{n^2}{R_0^2} - \frac{m^2}{r^2} \right\} \Psi_m(r) = 0,$$

(29)

where $\Psi_m(r)$ are $\tilde{\psi}(r, \hat{\theta})$’s Fourier components,

$$\tilde{\psi}(r, \hat{\theta}) = \sum_m \Psi_m(r)e^{im\theta},$$

(30)

and the geometric effect $\frac{1}{r} \frac{\partial}{\partial r} (Jg^{rr})$ is maintained for capturing cylindrical effects. This equation is equivalent to that given in Ref. 25. However, here, we should match the field by reconstructing the wave pattern on the whole magnetic surface. The solution of Eq. (29) can be expressed in terms of the Bessel functions of the first and second kind, $J_m(x)$ and $Y_m(x)$, for the Bessel equation

$$\left\{ \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial}{\partial x} \right) + \left( 1 - \frac{m^2}{x^2} \right) \right\} \Psi_m(x) = 0,$$

(31)

which is equivalent to Eq. (29) with

$$x = \alpha r, \quad \alpha = \sqrt{1 - \frac{P}{S} \left( \frac{nq + m}{q_0 R_0} \right)^2 - \frac{n^2}{R_0^2}}.$$

(32)

Noting the singularity of $Y_m(x)$ at $x = 0$, the solution in the 2D configuration space can be written as superposition of Bessel functions of the first kind

$$\tilde{\psi}(x, \hat{\theta}) = \sum A_m e^{im\theta} J_m(x),$$

(33)

which is the combination of incident and reflected wave fields, expressed as superposition of Hankel functions

$$\tilde{\psi}^{in/\text{ref}}(x, \hat{\theta}) = \sum \Psi_m^{in/\text{ref}}(x)e^{im\theta} = \sum A_m H_m^{\pm}(x)e^{im\theta}.$$  

(34)

The $A_m$ coefficients are determined by the incident wave at the connection radius, i.e.,

$$A_m = \frac{\hat{\Psi}_m^{in}(x_c)}{H_m^{+}(x_c)}.$$

(35)

When substituted back into Eq. (33), the local full-wave solution is obtained. The connection formulae between incident and reflected wave at $r_c$ is obtained in terms of the connection factor $\beta_m$,

$$\Psi_m^{\text{ref}}(x_c) = \beta_m \Psi_m^{in}(x_c), \quad \beta_m = \frac{H_m^{+}(x_c)}{H_m^{-}(x_c)}.$$  

(36)

Thus, $\tilde{\psi}^{\text{ref}}(x_c, \hat{\theta})$ can be constructed to continue using the mixed approach after reflection of the wave-packet near $r = 0$. More generally, the value of $\beta_m$ can be calculated numerically, while, in the following, we give its asymptotic solution by referring to Hankel’s asymptotic expansions; i.e.,

$$H_m^\pm(x) = \sqrt{\frac{2}{\pi x}} |P(m, x)\pm iQ(m, x)| e^{\pm i\chi},$$

$$\chi = x - \left( \frac{m}{2} + \frac{1}{4} \right) \pi,$$

$$P(m, x) \sim \sum_0^\infty (-1)^l \left( \frac{m}{2} - l \right) \left( \frac{m}{2} + l \right) \frac{(2x)^{2l+1}}{4l!(8x)^l}$$

(37)

$$Q(m, x) \sim \sum_0^\infty (-1)^l \left( \frac{m}{2} + l + 1 \right) \left( \frac{m}{2} - l \right) \frac{(2x)^{2l+1}}{4l!(8x)^l}$$

$$\mu = 4m^2,$$

which applies under the large argument expansion condition, i.e.,

$$\epsilon_H = |\mu/x| \ll 1.$$  

(38)

At the lowest order, $O(\epsilon_H^0)$, with $P(m, x) = 1$, $Q(m, x) = 0$, Eq. (37) gives the spherical wave approximation

$$H_m^\pm(x) \sim \frac{1}{\sqrt{2\pi x}} e^{\pm(i\mu - \mu/2\pi)}$$

(39)

Furthermore, recalling that for typical parameters of tokamak plasmas,25 Eq. (32) is simplified to (assuming $k_{||} > 0$)

$$\alpha = \sqrt{1 - \frac{P}{S} \left( \frac{nq + m}{q_0 R_0} \right)^2},$$

(40)

and assuming that the large argument expansion condition is satisfied near the center of the spectrum, $m_c$; i.e.,

$$\epsilon_H(m_c) = \frac{\mu(m = m_c)}{2(m = m_c)c_L} \ll 1,$$

(41)

the connection factor can be obtained explicitly at $O(\epsilon_H^0)$, 

$$\beta_m = \frac{\psi^{in}_m}{\psi^{\text{ref}}},$$

$$\delta_0 = \frac{m c_L}{R_0} \sqrt{1 - \frac{P}{S} - \frac{\pi}{2}},$$

$$\delta_1 = \frac{2r_c}{q_0 R_0} \sqrt{1 - \frac{P}{S} - \pi},$$

(42)

while the connection formula gives
\[ \hat{\psi}^{\text{ref}}_0 (x_c, \hat{\theta}) = \hat{\psi}^{\text{in}}_0 (x_c, \hat{\theta} + \delta_1)e^{i\delta_2}. \] (43)

The wave-packet is phase shifted by \( \delta_0 \) because of the finite toroidal mode number, while it is shifted in \( \hat{\theta} \) by \( \delta_1 \) due to the propagation inside \( r_c \). This can also be understood from the point view of 2D WKB. In fact, applying 2D WKB and the cylindrical approximation in this region,

\[ \frac{d\hat{\theta}}{dr} = \frac{1}{k_r} \left[ \left( 1 - \frac{P}{S} \right) \frac{k_1}{q_0 R_0} - \frac{m}{r^2} \right]. \] (44)

Thus, the reflected ray’s shift in \( \hat{\theta} \) is obtained using the large argument condition (38)

\[ \Delta \hat{\theta} = 2r_c \frac{q_0 R_0}{q_0 R_0} \sqrt{1 - \frac{P}{S}} - \pi. \] (45)

This result demonstrates the agreement between the 2D WKB and the lowest order solution of local full-wave equation. Higher order corrections can be obtained by maintaining more terms in Eq. (37) as well as adding effects due to finite toricidity. Here, we only give \( O(\epsilon_B^2) \) corrections in \( Q(m, x) \) to demonstrate the modulation of the amplitude of the wave-packet. With \( P(m, x) \approx 1 \) and \( Q(m, x) \approx (\mu - 1)/8x \), we have the connection factor up to \( O(\epsilon_B^4) \),

\[ \beta_m = \beta_{m, 0} \beta_{m, 1} = e^{i\delta_0 + \delta_1 m + \delta_2 m^2}, \] (46)

\[ \delta_0 = \delta_0 - \frac{1}{4x}, \quad \delta_2 = \frac{1}{x}, \] (47)

and, thus,

\[ \hat{\psi}^{\text{ref}}_{0, 1} (x_c, \hat{\theta}) = \hat{\psi}^{\text{in}}_0 (x_c, \hat{\theta} + \delta_1)e^{i\delta_2}. \] (48)

It can be demonstrated that, in the continuous Fourier transform, the \( e^{i\delta_2 m^2} \) contribution in the connection factor leads to the modulation of the wave-packet amplitude in \( \hat{\theta} \) (the broadening effect), while maintaining the spectrum intensity unchanged. As a result, higher order corrections in the local full-wave solution give more information than the traditional WKB method and might be important to describe the wave behavior at the reflection layer.

The solution at the lower density reflection layer near the plasma edge can also be obtained by the same procedure. By expanding the density profile locally near the cut-off layer, the local full-wave solution gives the Airy function,\(^{25}\) which can be matched with the mixed WKB-full-wave solution as done above. The absorption in the connection regions can also be readily obtained for every poloidal mode, once the distribution function is given. With the corresponding connection formulae, the wave propagation and absorption with multiple reflections can be calculated with the mixed WKB-full-wave approach until the wave energy is fully absorbed.

### 3. Comparison of the mixed WKB-full-wave approach with other methods

As mentioned above, the mixed WKB-full-wave approach was formulated first considering that the actual mode structure on a given magnetic flux surface can be decomposed by projection on a complete set of basis functions (eigenmodes) of the parallel wave equation, obtained from the original problem assuming a WKB Ansatz for the radial wave-packet propagation.\(^{18}\) This method is of great efficiency, when a small number of basis functions are required for an accurate description of the parallel mode structure. This is the typical case encountered in drift-wave stability analyses, where one parallel eigenmode is generally sufficient and, using radial scale separation between equilibrium non-uniformity and fluctuation wavelength, the mixed WKB-full-wave approach reduces to the well-known ballooning formalism.\(^{19}\) Applying the same method to the propagation of radio frequency waves, on the contrary, is inefficient, since the accurate description of the wave patterns on a magnetic flux surface generally requires a band of parallel eigenmodes. The calculation of the parallel mode structure, as proposed in Sec. II, is more direct and does not require projecting the solution on a set of basis functions. This approach assumes the Eikonal Ansatz only in the radial direction and, order by order in the radial envelope WKB asymptotic expansion, is equivalent to a full-wave solution in the parallel direction. In fact, starting from the boundary/initial condition on a reference flux surface, where the parallel wave spectrum is known exactly, the radial wave spectrum can be computed as asymptotic series, \( k_r = k_{r, 0} + k_{r, 1} + \cdots \), at the desired order. Then, noting that Eq. (22) is exact, the parallel wave spectrum is computed with the required accuracy, using Eqs. (25), (28), and so on, order by order. Thus, this method would actually generate the exact full-wave solution only if the WKB series was uniformly convergent. However, noting the asymptotic character of the WKB expansion, we conclude that the parallel wave spectrum is still computed with a full-wave approach at the relevant order, having always treated \( \partial_\parallel m \) on the same footing as \( m^2 \).

In the limit of \( v_{e, \perp} \ll v_{e, \parallel} \), the wave-packet propagates along the field line without significant radial displacement. Thus, within the present approach, the wave-packet rapidly covers the flux surface and yields a parallel mode structure, in which only parallel eigenmodes survive the otherwise rapid phase mixing.\(^{19}\) In this case, the present approach readily reduces to that adopted in Ref. 19 and formulated in Ref. 18, since, for linear stability analyses, only the most unstable/least stable mode structure must be retained.

The mixed WKB-full-wave approach has new features compared with other asymptotic methods, i.e., the traditional WKB method,\(^{1, 2}\) the complex WKB method,\(^{7, 8}\) and the beam tracing method.\(^{10-12}\) In the traditional WKB approach, the ray trajectory is solved at the lowest order, with the local wave vector \( \mathbf{k} \) in real space, and the correction to the amplitude is obtained at the next order. Since the amplitude is determined by the surface element confined within a ray bundle, the amplitude goes to infinity at the caustic layer where different rays intersect. As a result, additional techniques, like asymptotic matching\(^{25}\) and Maslov
theory, are adopted to obtain the solution at the reflection layer. The mixed WKB-full-wave approach also needs to use a similar technique in the region where the WKB method fails. However, since eikonal Ansatz is used in the radial direction only, difficulties to handle the high dimensional WKB are not of concern and the asymptotic matching with the full-wave solution is much easier in the connection region.

When the mixed approach is compared with complex WKB and beam tracing methods, an important issue arising is diffraction. In WKB, diffraction is ignored at the two lowest orders. Since diffraction is related with the spatial width of the beam and, thus, wave spectrum, affecting wave-particle resonant interactions, the complex WKB method and beam tracing method are proposed to take this effect into account by keeping the imaginary part of the wave vector. In complex WKB, at the lowest order, the ray moves in the complex space. In order to obtain the wave field at a given point in the real space, the initial/boundary condition of the rays should be continued to the complex space. Since there is not a one on one correspondence between the real space and the corresponding lifting to the complex space, a number of issues arise in practical calculations. On the contrary, the mixed WKB-full-wave approach solves the full wave equation along the equilibrium magnetic field and, thus, accounts for wave-like features, e.g., diffraction, in this direction. Meanwhile, the complex ray tracing is used in the radial direction, so that diffraction can be taken into account in this direction as well. Furthermore, since the complex ray tracing is used in one direction only, we can always force the ray to propagate in real space, while the wave vector in this direction is a complex number. These features make the mixed WKB-full-wave approach capable of investigating diffraction physics.

As further asymptotic method capable of describing diffraction, beam tracing adopts spatial scale separation, i.e., the ratio between wave length and the beam width. In this way, the information in beam’s transverse direction, e.g., beam width and iso-phase curvature, and the information in the longitudinal direction along the propagation, e.g., the amplitude and the eikonal information, can be treated separately in the real/imaginary part of the original equation order by order. The mixed WKB-full-wave approach, on the contrary, does not need this extra ordering but treats the local wave vector as a complex number. As a result, it describes on the same footing the same physics meaning. The local full-wave solution is adopted to give the physically correct connection formulae, while the beam tracing method gives poor results because the beam width loses its physics meaning.

In summary, the features of the mixed WKB-full-wave approach make the investigation of interesting physics tractable. We also note that this method generally requires solving more complicated equations than other approaches, i.e., the solution of the parallel wave equation rather than Hamiltonian ray equations in WKB or complex WKB methods. However, the mixed WKB-full-wave approach provides a viable option to address the various physics issues mentioned above within a theoretical framework that remains tractable.

III. LOWER HYBRID WAVE (LHW) PROPAGATION AND ABSORPTION IN TOKAMAK PLASMAS WITH CONCENTRIC CIRCULAR FLUX SURFACES

A. Governing equations and FTU reference model equilibrium

The general form of equations governing LHW propagation and absorption in arbitrary magnetic flux coordinates can be written in the framework of wave-packet propagation studies. However, in order to keep technical difficulties at a minimum level and still to demonstrate the mixed-WKB-full-wave approach for calculating wave propagation and absorption in tokamak plasmas, we adopt a simplified model mentioned in the previous work, the concentric circular flux surface coordinates with straight magnetic field lines, \( (r, \theta, \zeta) \), where \( \zeta = \phi \). Then, by solving

\[
q(r) = \frac{B \cdot \nabla \zeta}{B \cdot \nabla \theta},
\]

we have

\[
\dot{\theta} = 2 \arctan \left\{ \frac{1 - \nu_r \tan \frac{\theta}{2}}{1 + \nu_r \tan \frac{\theta}{2}} \right\},
\]

where \( \nu_r = r/R_0 \). The inverse transformation can be obtained from Eq. (50) as follows:

\[
\begin{align*}
\nu_r &= \frac{\cos \theta - \nu_r}{1 - \nu_r \cos \theta}, \\
\cos \theta &= \frac{\sqrt{1 - \nu_r \cos \theta}}{1 - \nu_r \cos \theta}, \\
\sin \theta &= \frac{1 - \nu_r \cos \theta}{1 - \nu_r \cos \theta}. 
\end{align*}
\]

The derivatives with respect to \( \theta \) can be obtained as

\[
\begin{align*}
\frac{\partial \theta}{\partial \theta} &= \frac{1 - \nu_r^2}{1 + \nu_r \cos \theta} = \frac{1 - \nu_r \cos \theta}{\sqrt{1 - \nu_r^2}}, \\
\frac{\partial \theta}{\partial r} &= \frac{-\sin \theta}{R_0 \sqrt{1 - \nu_r^2 (1 + \nu_r \cos \theta)}} = \frac{-\sin \hat{\theta}}{R_0 \sqrt{1 - \nu_r^2}}, \\
\frac{\partial \theta}{\partial \zeta} &= \frac{1 - \nu_r^2}{1 + \nu_r \cos \theta}.
\end{align*}
\]

and the metric elements can be written as follows:

\[
\begin{align*}
\frac{\partial \rho}{\partial \rho} &= \frac{1}{1 + \nu_r \cos \theta}, \\
\frac{\partial \rho}{\partial \theta} &= \frac{-\sin \theta}{R_0 \sqrt{1 - \nu_r^2 (1 + \nu_r \cos \theta)}}, \\
\frac{\partial \rho}{\partial \zeta} &= \frac{1 - \nu_r^2}{1 + \nu_r \cos \theta}.
\end{align*}
\]
It can also be shown that \( J B^2 \) is independent of \( \theta \), which is the general feature of Boozer coordinates like those of the specific model equilibrium chosen here,

\[
J B^2 = \frac{R_0}{r} \sqrt{\frac{1 - \nu_r^2 q^2 \psi^2}{1 - \nu_r^2}} \left[ 1 + \frac{\nu_r^2}{1 - \nu_r^2 q^2} \right].
\]

(54)

With the metric elements above, Eq. (5) can be written explicitly and the wave-packet propagation, computed from the parallel wave equation and the radial propagation, can be calculated starting from a flux surface with the initial/boundary condition determined by the antenna, using the general approach discussed in Sec. II B. As to the wave absorption, we use the linear electron Landau damping, Eq. (11), as an estimate to demonstrate the difference in the absorbed power density profiles predicted by the present approach and conventional 2D WKB. In the following analysis, a poloidal localized wave-packet is given as initial/boundary conditions at the starting radial position \( r_A \) in the form of \( \psi(r_A, \theta) = \exp(\cos(\theta)/\Delta \theta) \), where \( \Delta \theta \) properly chosen, as described below, to take into account the poloidal extent of the lower hybrid wave launcher, \( \Delta \theta_{ant} \). The corresponding complex poloidal wave vector field at \( r_A \) is given as \( m(r_A, \theta) = (0, \sin(\theta)/\Delta \theta) \). The vertical length of the wave launcher \( L_{vert} \) and the launcher’s radial position \( r_{ant} \) determine the value \( \Delta \theta_{ant} = 2 \arcsin(0.5 L_{vert}/r_{ant}) \). The toroidal mode number, \( n \), is derived from the parallel wave number at \( r_A \) by \( n = (\omega n_{ijA}/e)(J B) \mid_{r=r_A, \theta=0} \). For large aspect ratio, \( n \approx \omega q R_0 n_{ijA} / c \). Here, we ignored \( m \)'s contribution, since near the launcher, \( m \sim O(1), n \sim O(10^2) \), and thus the parallel wave number \( n_{ijA} \) is dominated by \( n \). The lower hybrid wave launcher excites a band of toroidal harmonics; for each one, the wave packet due to the poloidal mode structure at the antenna is evolved in \( (r, \theta) \) space using the mixed WKB-full-wave approach. Considering the relation between \( n \) and \( n_{ijA} \) and that the \( n_{ijA} \) spectrum is usually provided as input by an antenna-plasma coupling code, we label the wave-packet by \( n_{ijA} \) instead of \( n \). In the simulation, the parameters of FTU shot "FTU_32555-745" are used, i.e., plasma radius \( a = 0.285 \) m, major radius \( R_0 = 0.935 \) m, inverse aspect ratio \( q_0 \equiv a/R_0 = 0.305 \), lower hybrid frequency \( f = 8.0 \) GHz, radial position of the antenna \( r_{ant} = 0.32 \) m, vertical length of the wave launcher \( L_{vert} = 0.4 \) m, on-axis magnetic field \( B_0 = 5.4 \) T, parabolic density profile with central/edge ion density \( n_0 = 2 \times 10^{14} \text{cm}^{-3}, n_e = 1 \times 10^{13} \text{cm}^{-3} \), electron temperature profile \( T_e(r) = \mu_{Te} (1 - \nu_T e r^2) / \rho_e \), with \( \mu_{Te} = 2.5, \mu_{Te}, \nu_T e \) determined by central/edge electron temperature \( T_{e0} = 6 \) keV and \( T_{e0} = 0.02 \) keV, and exponential safety factor profile with central and edge safety factor values given, respectively, by \( q_0 = 1 \) and \( q_0 = 4.8 \).

The values of \( L_{vert} \) and \( r_{ant} \) give the antenna’s poloidal extent as \( \Delta \theta_{ant} = 1.35 \) and, correspondingly, we choose \( \Delta \theta = 0.16 \), so that the wave-packet has a low amplitude beyond the antenna region in \( \theta \) direction, i.e., \( \psi(r_A, \theta = \Delta \theta_{ant}/2) / \psi(r_A, \theta = 0) = 0.25 \). The starting radial position of the wave-packet propagation is set as \( r_A = 0.95 a \).

B. Numerical results

A number of issues are addressed in the numerical calculation by solving the LH wave-packet propagation equation at the lowest order and estimating the linear electron Landau damping at the next order. Meanwhile, numerical simulation results obtained by the mixed WKB-full-wave approach are compared with those from the traditional WKB method. Wave propagation pattern and power deposition for the single pass propagation are investigated first. In the approach formulated and implemented here, the field is calculated directly and the propagation pattern agrees with the qualitative behavior expected from 2D WKB method as shown in Figure 1, with the parallel wave number at the edge given by \( n_{ijA} = 1.8 \). At the reflection layer near \( r = 0 \), the WKB method fails in the radial direction, which calls for the full wave solution in the connection region, as discussed in Sec. II B 2 and to be demonstrated later. Besides the wave pattern in configuration space, the wave spectrum is also important, since it determines the wave-particle resonance and, thus, the wave absorption; furthermore, when it is combined with a quasi-linear-Fokker-Plank solver self-consistently, it determines the current drive and heating. The parallel wave number is computed, and the corresponding spectrum is shown in Figure 2 versus the radial position. The central parallel wave number calculated from the mixed WKB-full-wave approach agrees with that calculated by the 2D WKB method, implying qualitatively the agreement of wave

\[
\begin{align*}
g^{\prime\prime} &= 1, \\
g^{\prime\prime\prime} &= \left( 1 - \nu_r \cos \theta \right)^2 \left( 1 - \nu_r^2 \right)^2 + \frac{1}{R_0^2} \left( 1 - \nu_r^2 \right)^2, \\
g^{\prime\prime\prime\prime} &= \frac{1}{R_0^2 \nu_r^2}, \\
\psi &= \frac{r_A}{R_0} + \nu_r \cos \theta, \\
\psi^2 &= \frac{1}{R_0^2 \nu_r^2} \left[ 1 + \frac{\nu_r^2}{q^2 (1 - \nu_r^2)} \right], \\
B^2 &= \left( \frac{\psi}{r} \right) \left[ 1 - \frac{\nu_r^2}{1 - \nu_r^2 q^2} \right].
\end{align*}
\]

(53)
absorption using these two methods. However, since the spectrum is broadened when propagating inward, the mixed WKB-full-wave approach gives a more realistic and accurate description of the enhanced wave damping, as will be demonstrated later on. The mixed WKB-full-wave approach captures more information than the traditional WKB method, e.g., the spectrum shape and its evolution versus \( r \). In summary, as a general remark, we conclude that the novel method, proposed here, provides an accurate and efficient numerical simulation tool for computing LH propagation and absorption, which could be compared with numerical simulation results from the beam tracing and full-wave approaches. Further investigations about this issue will be reported in future work.

To estimate the wave spectrum evolution and energy absorption, we launch a bundle of wave-packets labeled by \( n|A \), or equivalently, the toroidal mode number \( n \), each of which carries power \( P_{nA}(r_A) \) at \( r_A \) according to the parallel spectrum determined by the antenna, fitted by the Gaussian distribution

\[
P_{nA}(r_A) = \frac{P_A}{\sqrt{\pi}} \exp \left(-\frac{(n - n|A|)^2}{\sigma^2} \right).
\]  

Here, \( P_A \) is a normalization factor determined by the total injected power, the central parallel wave number \( n|C| = 1.8 \), and the spectrum width \( \Delta n|A| = 0.33 \). For every wave-packet, its propagation is calculated and the corresponding power spectrum evolution \( \tilde{P}_{nA}(n, r) \) is obtained. Note that, at the antenna position, each wave-packet of the parallel spectrum of Eq. (55) has its own “finer scale” \( n \) spectrum, denoted here as \( \tilde{P}_{nA}(n, r_A) \), which accounts for the initial poloidal mode structure at the antenna mouth. More precisely,

\[
\tilde{P}_{nA}(n, r_A) = K_{nA} \int_{-\infty}^{\infty} \exp[i(\omega R_0/c)(n|A| - n)q_d \theta] \\
\times \exp[2\cos \theta/\Delta \theta] d\theta,
\]

with \( K_{nA} \) a normalization constant. While the parallel spectrum varies on a \(~C/(\omega R_0)\) \(~10^{-2}~\) scale. However, as each wave-packet in the parallel spectrum propagates inward, as described by \( \tilde{P}_{nA}(n, r) \), the initial finer scale structures rapidly broaden, with important consequences on wave absorption that are shown below. When the propagation of all the wave-packets is calculated, the total power spectrum at different radial position, \( P(n|A|, r) \), is obtained by summing up \( \tilde{P}_{nA}(n, r) \) with different \( n|A| \). The \( n|A| \) spectrum at various \( r \) is plotted in Figure 3, with the normalized power spectrum defined as \( P_N(n|A|, r) = \tilde{P}(n|A|, r) / \tilde{P}(n|A|, r_A) \). The parallel wave number is upshifted and broadened during the inward propagation of the LH wave-packet, due to toroidal geometry effects, which contributes to the enhanced absorption of the wave energy. In order to estimate the power deposition, the spatial damping rate for every wave-packet,

\[
\gamma_{n|A|}(r) = \frac{d}{2\tilde{P}_n(r)|\tilde{P}_n|} \tilde{P}_n(r),
\]

is calculated along the propagation of every wave-packet labeled by different \( n|A| \). For a wave-packet labeled by \( n|A| \), since the corresponding \( n \) spectrum is broadened during propagation, as shown in Figure 2, the damping rate is higher than that of the reference ray with \( n|A| \) as the initial parallel wave number calculated by traditional WKB method, as shown in the upper frame of Figure 4. The corresponding normalized power, \( P_{n|A|}(r) = P_{n}(r)/P_{n}(r_A) = \exp(2f_{nA} \gamma_{n|A|}(r) dr) \), is shown in the lower frame of Figure 4 to demonstrate the power depletion of the wave-packet labeled by different \( n|A| \). Here, we note that launching a bundle of rays from different injection angles for one single \( n|A| \) helps describing the spectrum broadening in the framework of traditional WKB, as demonstrated in Ref. 23 and 31. Since the damping rate \( \gamma_{n|A|}(r) \) is obtained for every wave-packet, the total damping rate, \( \gamma_{tot}(r) \), is simply estimated by summing up the damping rates of all wave-packets with the normalized launched power spectrum as weight factor. The total power, \( P_{tot}(r) = P_{tot}(r_A) \exp\{2\int_{r_A}^{r} \gamma_{tot}(\tilde{r}) d\tilde{r} \} \), is

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**FIG. 2.** The parallel wave number spectrum at different radii \( r \) (color contour plot) and \( n_I \)'s evolution versus \( r \) using 2D WKB method (yellow circle line).

**FIG. 3.** The \( n_I \) spectrum at different radial positions, \( r/a = 0.25 \) (red squares), \( 0.60 \) (green triangles), and \( 0.95 \) (blue circles). \( P_0(n_I, r) \) is normalized as \( \sum_{n} P_N(n|A|, r) dn_A = 1.0 \). 140 wave-packets are launched with different \( n|A| \) centered initially at \( n|C| = 1.8 \). The spectrum is upshifted to larger \( n_I \) and broadened when propagating inward.
calculated along radial direction, as shown in the upper frame of Figure 5, with the normalized total power defined as 
\[ PN_{\text{tot}}(r) = \frac{P_{\text{tot}}(r)}{P_{\text{tot}}(r_A)} \]. The normalized total deposited power, 
\[ dP_{\text{tot}}(r) = \frac{\int r \, d\rho}{r_A} = \frac{2 \gamma_{r,0} \exp \left\{ 2 \int_{r_A}^r \gamma_{r,0}(\rho) \, d\rho \right\} }{r_A} \], is also calculated as shown in the lower frame of Figure 5. As consequence of spectrum broadening, the power deposition profile is shifted outward compared with that of the reference ray calculated from the 2D WKB method.

The results above demonstrate the capabilities of the mixed WKB-full-wave approach compared with the 2D WKB method in a single pass propagation calculation, where the wave deposits most of its energy during its first pass towards the magnetic axis. This is the usual case for tokamak plasmas with relatively high temperature, especially when parametric decay takes place at the plasma edge.\(^{30}\) For tokamak scenarios with relatively low temperature, the wave propagates for several passes between the low and high density reflection layers until \( n_e \) is upshifted to a value where enough resonant electrons damp the wave.\(^{32}\) To demonstrate the capabilities of the mixed WKB-full-wave approach for multi-pass numerical simulations, we discuss the case of a two pass propagation and its comparison with the traditional WKB calculation in Figure 6. The wave-packet evolution agrees with the ray trajectory calculated by 2D WKB method during the whole round trip. Near the inner reflection radius, in the connection region, the regular structure of the wave field is calculated and reconstructed, in contrast to the asymptotic matching procedure of singular with regular mode structures adopted in the traditional 2D WKB method. Meanwhile, the focusing/defocusing of the wave-packet near the caustic region is well described by the full-wave solution in the form of the superposition of Bessel functions, formulated in Sec. II B 2. After reflection, the wave-packet propagation is calculated using the mixed WKB-full-wave approach again, starting from the connection radius with the initial/boundary conditions derived from the connection formula, Eq. (36). The superposition of incident and reflected wave gives the field pattern and interference structures are visible where two rays intersect. Beyond the two passes demonstrated here, for the subsequent passes, the connection procedure can be carried out as many times as needed, depending on the deposition of the wave energy, following the analysis of Sec. II B 2. In a...
self-consistent calculation with quasi-linear diffusion and collisional relaxation taken into account, the wave-packet might propagate for several passes before delivering energy to the plasma and significantly contributing to modify the plasma distribution function along the trajectory. In turn, the distribution function determines the wave-packet damping rate, as described by Eqs. (1) and (12). This calculation needs additional work on the interface between the wave field solver using the mixed WKB-full-wave approach and the Fokker-Planck equation solver, e.g., introducing iterative numerical schemes to solve the governing integro-differential equations self-consistently. For this reason, this topic is left out of the scope of the present work. We note, however, that the intrinsic difficulty of such an approach remains of a lesser level than that of the full-wave treatment.\textsuperscript{15} self-consistently coupled with the non-linear equation system in the quasi-linear formulation of the problem.\textsuperscript{33,34}

IV. CONCLUSIONS AND DISCUSSIONS

In this work, analytical and numerical work is carried out to investigate the propagation and absorption of short wavelength quasi-electrostatic lower hybrid waves, treated as wave packets in 2D plasma equilibria. The mixed WKB-full-wave approach is formulated by analyzing the radial propagation using traditional WKB, but computing the actual wave-packet structure along magnetic field lines at the relevant order, consistent with the radial WKB Ansatz. The wave-packet evolution is determined by this approach, with the initial/boundary conditions defined by the antenna. The asymptotic series expansion, based on the WKB approximation in the radial direction, is formulated with the Hermitian and anti-Hermitian parts of the dielectric tensor taken into account to describe wave-packet propagation and absorption. Appropriate connection formulae are derived and analyzed based on the local full-wave solution at the reflection layer, where the WKB approximation in radial direction breaks down. As a novel method, its connection with our previous work and comparisons with other asymptotic analyses are discussed. The analysis of lower hybrid wave propagation in concentric circular tokamak plasmas is performed as application, using typical FTU discharge parameters, and its comparison with traditional WKB method is given to address a number of issues; i.e., the wave propagation pattern, spectrum evolution, and wave damping in single pass propagation. Meanwhile, the capability of this approach to address multiple pass propagation is also demonstrated. Further applications of this method and more in depth discussions of, e.g., its comparison with complex WKB, beam tracing, and full wave methods are beyond the scope of the present work and will be subject of future investigations in specific cases such as diffraction physics, wave energy absorption in the linear case and current drive, and heating in the framework of quasi-linear theory. As intermediate approach between 2D full wave method and traditional 2D WKB, the mixed WKB-full-wave provides a viable option to investigate various physics, while maintaining an acceptable computational efficiency. For example, the numerical simulations reported in this work have been performed on a laptop with Intel i3-2350 M (2.3 GHz) CPU, using typically 140 s of CPU time in the cases reported in Figure 6.

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31. The detailed discussion is beyond our scope in this paper.