Analysis of the validity of the asymptotic techniques in the lower hybrid wave equation solution for reactor applications

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Knowing that the lower hybrid (LH) wave propagation in tokamak plasmas can be correctly described with a full wave approach only, based on fully numerical techniques or on semianalytical approaches, in this paper, the LH wave equation is asymptotically solved via the Wentzel-Kramers-Brillouin (WKB) method for the first two orders of the expansion parameter, obtaining governing equations for the phase at the lowest and for the amplitude at the next order. The nonlinear partial differential equation (PDE) for the phase is solved in a pseudotoroidal geometry (circular and concentric magnetic surfaces) by the method of characteristics. The associated system of ordinary differential equations for the position and the wavenumber is obtained and analytically solved by choosing an appropriate expansion parameter. The quasilinear PDE for the WKB amplitude is also solved analytically, allowing us to reconstruct the wave electric field inside the plasma. The solution is also obtained numerically and compared with the analytical solution. A discussion of the validity limits of the WKB method is also given on the basis of the obtained results. © 2007 American Institute of Physics. [DOI: 10.1063/1.2805435]

I. INTRODUCTION

In a burning plasma such the International Thermonuclear Experimental Reactor (ITER), as well as in future tokamak reactors, the use of the lower hybrid current drive (LHCD) is a highly desirable tool and it is particularly indicated to form and maintain a specific q-profile, in especially for the steady state and hybrid scenarios, where the magnetic shear is very low or negative (q-profile inversion). Modeling of these experiments by ray-tracing techniques coupled with Fokker-Planck codes have also been performed, leading to a new approach in the description of the lower hybrid (LH) wave-plasma interaction. Ray tracing techniques, in general, are based on the solution of ray tracing equations for position and wavevector, which stem from a Wentzel-Kramers-Brillouin (WKB) analysis of the wave equation. The solution of these equations essentially describes the evolution of the parallel wavenumber (i.e., of the power spectrum) inside the plasma, starting from the antenna position. The parallel wavenumber is, in fact, responsible of the wave absorption via electron Landau damping and its accurate evaluation is essential to correctly describe the wave absorption. The electric field amplitude of the propagating wave remains unsolved within this approximation and is calculated via the energy conservation theorem (Poynting theorem).

Most of the ray-tracing codes use a multireflection scheme in describing the wave propagation. This scheme consists of following the ray trajectories indefinitely from the antenna to the plasma centre (where the ray reflects) and from the center to the cutoff (where another reflection point is located), until the launched wave spectrum is fully absorbed by the plasma. This likely happens when the parallel wavenumber is sufficiently high to satisfy the Landau resonance condition for wave absorption. In other words, this means that the ray trajectory is integrated until the parallel wavenumber reaches a sufficiently high value to satisfy the Landau resonance condition.

A different approach that seems to very well reproduce JET experiments assumes (supported by experimental evidence) that the wave spectrum at the antenna is broadened as much as needed to fill in the so-called spectral gap nonlinear interactions with the plasma edge (parametric decay instabilities), without invoking multireflection. The wave, in this case, should be absorbed by the plasma in a single pass of the ray trajectory propagating from the antenna toward the centre. Meanwhile, wave propagation modeling should be simpler than in the multireflection approach. Moreover, some intrinsic critical issues of the multireflection (multipass) approach, due to the breaking of the WKB approximation near the reflection points, are avoided in this case.

Knowing that the LH wave propagation in tokamak plasmas can be correctly described with a full wave approach only, based on fully numerical techniques or on semianalytical approaches, in this paper we asymptotically solve the LH wave equation via the WKB method for the first two orders of the expansion parameter, obtaining governing equations for the phase at the lowest and for the amplitude at the next order. The nonlinear partial differential equation (PDE) for the phase is solved in a pseudotoroidal geometry (circular and concentric magnetic surfaces) by the method of characteristics as described in Ref. 12. The associated system of ordinary differential equations (ODE) for the position and the wavenumber is obtained and analytically solved by choosing an appropriate expansion parameter. The quasilinear PDE for the WKB amplitude is also solved analytically,
allowing us to reconstruct the wave electric field inside the plasma. The solution is also obtained numerically and compared with the analytical solution. A discussion of the validity limits of the WKB method is also given on the basis of the obtained results.

II. RELEVANT EQUATIONS

The LH wave plasma interaction is described by the Maxwell-Vlasov system since the wave propagation involves time scales that are much shorter than the collision time scale. This system consists of nonlinear partial integro-differential equations. In order to derive reduced and tractable equations, we assume that in most situations of practical interest the field amplitude is sufficiently small to justify the linearization of the kinetic equation. Moreover, for the LH wave, the cold plasma and electrostatic approximations are well satisfied and the most important propagation and absorption physics is preserved in this limit. Indeed, the wave absorption, related with the anti-Hermitian part of the dielectric tensor, is weak and spatially broad and it can be singled out in the wave equation as an imaginary term proportional to the scalar potential. The wave equation must in principle include this term as well, but here, we are mainly interested in discussing the validity of the WKB method when applied to LH wave propagation; for this reason, we solve the cold plasma wave equation that formally neglects the absorption physics. The system thus reduces to a simple scalar second-order differential equation for the electrostatic potential, related to the electric field of the wave. It can be written as

\[ \nabla \cdot \left[ \varepsilon_0 \nabla \Phi(\vec{r}) \right] = \gamma \Phi(\vec{r}), \]  

(1)

where \( \Phi(\vec{r}) \) is the scalar potential. \( \varepsilon(\vec{r}) \) is the cold dielectric tensor, which can be written in Stix notation as

\[ \varepsilon(\vec{r}) = S_L + (P - S) b \cdot b - i D \varepsilon_{LC} \cdot \vec{b}, \]

and \( \varepsilon_{LC} \) is the Levi-Civita tensor of rank 3. Moreover, \( \vec{b} \) is the unit vector along the magnetic field, i.e., \( \vec{b} = \vec{B} / |\vec{B}| \), and \( \gamma \) is the damping factor

\[ \gamma = 2i \sqrt{\pi} \omega_{pe} \left( \frac{\omega}{k_B t_{th}} \right)^2 e^{i(\omega k_B t_{th})^2} \]

with \( \omega_{pe} \), \( t_{th} \), \( k_B \) the plasma frequency, the thermal velocity, and parallel wavevector, respectively. Equation (1) can be solved by the WKB asymptotic expansion method. This means that the scalar potential can be written as

\[ \Phi(\vec{r}) = \Phi_0(\vec{r}) e^{ikS_{-1}(\vec{r})}, \]  

(2)

introducing the so-called Eikonal ansatz, where \( \Phi_0(\vec{r}) \) is the slowly varying amplitude, \( S_{-1}(\vec{r}) \) is the phase (or the Eikonal function), which varies on a faster scale length than \( \Phi_0(\vec{r}) \), and

\[ \kappa^{-1} = \frac{L_S}{L_A} = [\nabla S_{-1}(\vec{r})]^{-1} \times \left[ \begin{array}{c} \nabla \Phi_0(\vec{r}) \\ \Phi_0(\vec{r}) \end{array} \right] \ll 1 \]

(3)

is the expansion parameter that accounts for the different scale variation. This inequality states that \( L_S \ll L_A \sim L_{pp} \), i.e., the wavelength associated with the LH wave must be much smaller than the typical amplitude scale length, which is of the same order of magnitude of the scale of variation of the plasma parameters \( L_{pp} \). A generally more stringent condition for the validity of the WKB approximation (or Eikonal ansatz) is based on the fact that we must be allowed to rigorously identify the gradient of the Eikonal function with the wavevector; i.e., \( \nabla S_{-1}(\vec{r}) = \vec{k} \).

In principle, at a generic point \( \vec{r} \) near a point \( \vec{r}_0 \) the following relation holds:

\[ S_{-1}(\vec{r}) = S_{-1}(\vec{r}_0) + (\vec{r} - \vec{r}_0) \cdot \nabla S_{-1}(\vec{r})|_{\vec{r} = \vec{r}_0} + \frac{(\vec{r} - \vec{r}_0)^2}{2} \nabla \cdot \nabla S_{-1}(\vec{r})|_{\vec{r} = \vec{r}_0} + \cdots. \]

(4)

It is possible to define \( \nabla S_{-1}(\vec{r}) = \vec{k} \) only if the second term on the right hand side (RHS) of Eq. (4) is much smaller then the first; this is equivalent to require

\[ \left| \frac{\nabla \cdot \vec{k}}{k^2} \right| = \left| \frac{\nabla \cdot \nabla S_{-1}(\vec{r})}{k^2} \right| \ll 1. \]  

(5)

This last inequality is connected with the curvature of the phase surface, and it states that the wavelength must be smaller than the curvature radius of the phase surface.

Finally, using Eq. (2), at the lowest order in \( \kappa^{-1} \), the wave equation becomes

\[ S((\nabla S_{-1}(\vec{r}))^2) + P((\nabla S_{-1}(\vec{r}))^2) = 0. \]

(6)

This equation is a nonlinear PDE for the phase integral \( S_{-1}(\vec{r}) \) and it can be solved by the method of characteristics. The operators \( \nabla_\parallel \) and \( \nabla_\perp \) are, respectively, the parallel and perpendicular gradients. When substituting \( \nabla S_{-1}(\vec{r}) = \vec{k} \), Eq. (6) reduces to the usual cold electrostatic dispersion relation

\[ \bm{S}k_\perp + Pk_\parallel^2 = 0. \]

(7)

Assuming that \( \gamma = O(\kappa^{-1}) \), at the first order in \( \kappa^{-1} \), we obtain

\[ \bm{S}[2 \nabla A_0(\vec{r}) \cdot \nabla S_{-1}(\vec{r}) + A_0(\vec{r}) \nabla^2 S_{-1}(\vec{r})] + -[(S - P)] \]

\[ \times [2 \nabla \cdot A_0(\vec{r}) \cdot \nabla S_{-1}(\vec{r}) + A_0(\vec{r}) \nabla^2 S_{-1}(\vec{r})] = \gamma A_0(\vec{r}). \]

(8)

This is a quasilinear PDE for the slow variation of the amplitude of the scalar potential. In addition, in this case the solution can be obtained by the method of characteristics.

The solution of both Eqs. (6) and (8) enables us to reconstruct the wave field.

III. ANALYTICAL SOLUTION OF THE CHARACTERISTIC EQUATIONS IN TOROIDAL GEOMETRY

We can specialize the WKB equations for phase and amplitude in pseudotoroidal geometry \( (r, \theta, \phi) \). The metric coefficients are
If we assign an equilibrium magnetic field structure of the form
\[
\vec{B} = [B_r, B_\theta, B_\phi] = [B_r = 0, \hat{B}_\phi(r) \left(1 + \frac{r}{R_0} \cos \theta \right)^{-1}, \hat{B}_\phi(r)]
\]
the phase equation becomes
\[
S \left[ \frac{\partial S_{-1}(r, \theta, \phi)}{\partial r} \right]^2 + S \frac{\hat{B}_\phi \partial S_{-1}(r, \theta, \phi)}{r} \frac{\partial \theta}{\partial r} + S \hat{B}_r \partial S_{-1}(r, \theta, \phi) \frac{\partial \phi}{\partial r} + \frac{\hat{B}_\phi}{R} \left( \frac{\partial S_{-1}(r, \theta, \phi)}{\partial \phi} \right)^2 = 0.
\]

Using \( \nabla S_{-1}(\vec{r}) = \vec{k} \), which means \( \partial S_{-1}(\vec{r})/\partial r = k_r \), \( \partial S_{-1}(\vec{r})/\partial \theta = m_\theta \), \( \partial S_{-1}(\vec{r})/\partial \phi = n_\phi \), and defining the wavevector in parallel and perpendicular directions as
\[
\frac{d m_\phi}{dx} = \pm \frac{n_\phi \hat{B}_\phi}{s} \left( \frac{P}{S} \right) + n_\phi \hat{B}_\phi m_\phi - \delta_0 n_\phi \hat{B}_\phi \left( \frac{P}{S^2} \frac{\partial S_{-1}}{\partial B} \right) h_r \sin \theta,
\]
where we have used normalized quantities \( x = r/a \), \( h_r = 1 + \varepsilon x \cos \theta \), \( n_\phi = (c/\omega \kappa) \), \( \delta_0 = (c/\omega a) \approx 1 \), and \( n_r \), in Eqs. (12) and (13), is obtained using the dispersion relation Eq. (7) and the third equation of Eqs. (11); i.e.,
\[
n_r = \pm \sqrt{-\frac{P}{S} n_i^2 - \left( \frac{\hat{B}_\phi}{\hat{B}_\phi} - \frac{\delta_0 n_\phi}{h_r \hat{B}_\phi} \right)^2}.
\]
The solution at high density is
\[
x_{\text{ref}}^{\text{hd}} \frac{\partial \omega}{a} = \left( \delta_0^{-1/2} \frac{\partial \omega}{a} \right)_{n_\phi = n_\text{ref}} 1 - \frac{n_{\phi}}{n_\text{ref}}.
\]
direction the antenna radiates the same parallel wavenumber; this means that the toroidal wavenumber is not constant when changing the poloidal injection angle, as can be noticed in Eq. (17).

The quantity \( \delta = 1 - n_{i0}/n_{i\text{ref}} \), where \( n_{i\text{ref}} \) is the parallel wavenumber at the reflection point, i.e., \( n_{i\text{ref}} = (\delta_0 e q_{i0}^2 m_{	ext{ref}} + n_{i0}) \), is undetermined and will be specified after solving the equation for \( m_\theta \).

The second solution for the reflection radius is obtained at low density, i.e., \( x_{\text{ref}} = l_{r\text{ref}}/a \), near the cutoff density

\[
x_{\text{ref}}^2 = x_{\text{cutoff}} + \frac{\delta}{2 c_{\text{cutoff}}^2 \partial \omega_{p}(x)} \frac{\epsilon^2}{q_{\theta}^2} \bigg|_{x=x_{\text{cutoff}}}
\]

These are the reflection points for the trajectory. In principle, the LH wave can be reflected between these points until it is absorbed by the plasma (the multireflection scheme).

In this geometry, the phase integral, derived from \( \nabla S_{-1}(\tau) = \hat{k} \), can be written as

\[
S_{-1}(x, \theta, \phi) = S_{-1}(x_0, \theta_0, \phi_0) + \delta_0^2 \int_{x_0}^{x} n_{i}(x) \, dx
+ \int_{1}^{x} m_{\theta}(x) \, dx + n_{\phi}(\phi(x, \theta_0) - \phi_0).
\]

Once the trajectory equation system [Eqs. (12) and (13)] is solved, the phase can be computed explicitly from Eq. (19), by making a simple integration. Equations (12) and (13) can be solved analytically in terms of simple functions after considering (i) that the density and the \( q \)-profile are sufficiently flat from the separatrix to the plasma center; (ii) that the equation system can be expanded by choosing \( \delta=(m_i/m_e) \), the electron/ion mass ratio, as expansion parameter; and (iii) that \( x_{\text{ref}}^2 \) is a constant and would depend only on the launching angle \( \theta_0 \). Equations (12) and (13) can then be written as

\[
\frac{d\theta}{dx} = \pm \text{sign}(n_i) \delta^{1/2} e^2 \hat{\omega}_{p0}(x) q^{-1}(x) \frac{\left[ x^2 - (x_{\text{ref}}^2)^2 \right]^{1/2}}{x \sqrt{x^2 - (x_{\text{ref}}^2)^2}}
\]

\[
\frac{d\phi}{dx} = \pm \text{sign}(n_i) \delta^{1/2} e^2 \hat{\omega}_{p0}(x) \left( 1 + e x \cos \theta \right) \frac{\left[ x^2 + e^2 (x_{\text{ref}}^2)^2 \right]^{1/2}}{q^{-1}(x) \left[ x \sqrt{x^2 - (x_{\text{ref}}^2)^2} \right]}
\]

\[
\frac{dm_{\theta}}{dx} = \pm \text{sign}(n_i) \delta^{1/2} e^2 \hat{\omega}_{p0}(x) q^{-1}(x) \frac{\left[ x^2 - (x_{\text{ref}}^2)^2 \right]^{1/2}}{x \sqrt{x^2 - (x_{\text{ref}}^2)^2}} \times \left[ q^{-1}(x) m_{\theta} + \delta_0^2 e^{-1} n_{i0} \right],
\]

where \( \lambda = (\omega/\Omega_e)^2 = (\omega_{pi}/\Omega_e)^2 = 10^4 n_{e0} a^4 / B_{10}^2 = 10^{-4} \) is the cutoff density. We can consider it constant.

The safety factor and density profiles are chosen of the following form:

\[
\hat{\omega}_{p0}(x) = \hat{\omega}_{p0} e^{-a_x x},
\]

\[
q(x) = q_0 e^{-a_x x},
\]

where \( a_x = \ln(n_{i0}/q_0) \), and \( a_y = \ln(q_{i0}/q_0) \). This choice guarantees the flatness of the profiles. In general, the LH wave is coupled with the plasma far from the cutoff density for \( x \ll x_{\text{ref}} \). For this reason, the use of the approximation \( (-P/S) \approx -\delta^{1/2} \hat{\omega}_{p0}^{-1} \) is justified in the ray equations (12) and (13). On the contrary, in order to correctly describe the wave which is reflected at the cutoff reflection point \( x = x_{\text{ref}}^2 \), we must consider the term \( (-P/S)(-1) - \delta^{1/2} \hat{\omega}_{p0}^{-1} \) in Eqs. (12) and (13), without approximations. In this case, in fact, owing to the very low density near the cutoff, the quantity \( \hat{\omega}_{p0}^{-1}(x_{\text{ref}}^2) = O(\delta) \) and the term \( (-P/S) \) tends to vanish.

In this way, we can solve Eqs. (20) and (21) and obtain for \( \theta \) and \( \phi \)

\[
\phi_0(x) = \theta_0 + C_1 \sqrt{1 - (x_{\text{ref}}^2)^2} - \sqrt{x^2 - (x_{\text{ref}}^2)^2}
- C_2 \left( \arccos(x_{\text{ref}}^{1/2}) - \arccos(x_{\text{ref}}^{1/2}) \right),
\]

\[
\theta = \phi_0 + q_{i0} \left( \phi_0(x) - \phi_0 \right),
\]

where the constants are

\[
C_1 = - \pm \text{sign}(n_i) \delta^{1/2} q_{i0}^{-1} \hat{\omega}_{p0}^{-1},
\]

\[
C_2 = - \pm \text{sign}(n_i) \left( 1 - n_{i0}^2 / n_{i\text{ref}}^2 \right),
\]

Meanwhile, the solution for \( m_{\theta} \) is

\[
m_{\theta}(x, \theta_0) = C_3 \left( e^{C_4 \left[ \sin \theta_0 (x) \cos \theta_0 (x) - \cos \theta_0 (x) \sin \theta_0 (x) \right]} - 1 \right),
\]

where

\[
C_3 = \delta_0^2 e^{-1} q_{i0} n_{i0} \hat{\omega}_{p0}^{-2} \lambda^{-1} = \delta_0^2 e^{-1} \hat{\omega}_{p0}^{-2} \lambda^{-1} \left( 1 + e \cos \theta_0 \right),
\]

\[
C_4 = \pm \text{sign}(n_i) \delta^{1/2} e^2 q_{i0}^{-1} \hat{\omega}_{p0}^{-1} \lambda
\]

and
Finally, the evolution of the parallel wavenumber along \( x \) is given by

\[
n_{1}(x, \theta_{0}) = \frac{\delta_{0} \rho_{m}(x, \theta_{0})}{q(x)} + \frac{n_{0}}{1 + \varepsilon \cos \theta(x, \theta_{0})}.
\]  

(28)

Note that the constant \( C_{3} \) does not depend on \( \theta_{0} \) if we consider \( n_{\theta} \) constant for each value of \( \theta_{0} \); it does depend on \( \theta_{0} \), instead, if we consider \( n_{\theta} \) constant for each value of \( \theta_{0} \).

When \( x \) is approaching the reflection point \( x_{\text{ref}} \), the poloidal angle at the reflection is

\[
\theta_{\text{ref}}(x) \approx C_{1}(1 - \sqrt{1 - (x_{\text{ref}})} - C_{2}(\arccos|x_{\text{ref}}| - 2\kappa \pi)
\]

\[
= C_{1} - C_{2} \frac{\pi}{2}(1 - 2\kappa),
\]  

(29)

while both functions in Eqs. (27) become

\[
I_{1}(x = x_{\text{ref}}) = I_{\text{ref}}
\]

\[
= \frac{1}{C_{1}} \left[ \sin[C_{1}(1 - \sqrt{1 - (x_{\text{ref}})}^2)] - x_{\text{ref}}^b \right.
\]

\[
\times \left[ (\arccos[C_{1} + [2\kappa \pi - \arccos(x_{\text{ref}})])] \right]
\]

\[
+ \frac{1}{C_{1}} \left[ \sin[C_{1}(1 - \sqrt{1 - (x_{\text{ref}})}^2)]
\]

\[
+ \arccos(x_{\text{ref}})] \sin[C_{1} + \arccos(x_{\text{ref}})],
\]

(30)

\[
I_{2}(x = x_{\text{ref}}) = I_{\text{ref}}
\]

\[
= -\frac{1}{C_{1}} \left[ \cos[C_{1}(1 - \sqrt{1 - (x_{\text{ref}})}^2)] - x_{\text{ref}}^b \right.
\]

\[
\times \left[ (\cos[C_{1} + [2\kappa \pi - \arccos(x_{\text{ref}})])] \right]
\]

\[
- \frac{1}{C_{1}} \left[ \cos[C_{1}(1 - \sqrt{1 - (x_{\text{ref}})}^2)]
\]

\[
+ \arccos(x_{\text{ref}})] \cos[C_{1} + \arccos(x_{\text{ref}})],
\]

(31)

and its value and “sign” depend on the poloidal injection angle and the macroscopic plasma parameters: the electron mass ratio \( \delta \), the density \( \rho_{\text{in}} \), the safety factor \( q_{\text{d}} \), the inverse aspect ratio \( \varepsilon \), \( \delta_{\theta} \), and \( n_{0} \). In particular, by solving the inequality

\[
e^{C_{3}[\sin \theta_{\text{inj}} \cos \theta_{\text{d}}]} - 1 \geq 0,
\]

we can establish the range of the poloidal injection angles for which \( m_{\theta} \) is positive, negative, or equal to zero; e.g., for poloidal injection angle

\[
\theta_{0} = -\arctan \left( \frac{I_{\text{ref}}}{I_{\text{ref}}} \right)
\]

(32)

we have \( m_{\text{ref}} \geq 0 \). Using now the definition of the parallel wavenumber, we have its value at the reflection point

\[
n_{\text{ref}} = \delta_{\theta} q_{\text{d}}^{-1} m_{\text{ref}} + n_{0}
\]

\[
= n_{0} e^{C_{3}[\sin \theta_{\text{inj}} \cos \theta_{\text{d}}]} - 1
\]

\[
= n_{\text{inj}}(1 + \varepsilon \cos \theta_{0}) e^{C_{3}[\sin \theta_{\text{inj}} \cos \theta_{\text{d}}]} - 1,
\]

(33)

and we can specify the value of the reflection radius as a function of the poloidal injection angle:

\[
x_{\text{ref}}^b = \frac{\delta^{1/2} q_{0}}{\omega_{\text{inj}}} \left[ 1 - e^{-C_{3}[\sin \theta_{\text{inj}} \cos \theta_{\text{d}}]} \right],
\]

(34)

The position of the reflection radius depends on the injection angle; Eq. (34) also states that for some values of the injection angle the reflection point can be very close to the geometric axis of the torus. In particular \( x_{\text{ref}}^b \equiv 0 \) when

\[
\theta_{0} = -\arctan \left( \frac{I_{\text{ref}}}{I_{\text{ref}}} \right) = \pm \frac{\pi}{2} \pm 2\kappa \pi, \quad \kappa = 0, 1, 2 \ldots
\]

Note that we have taken advantage of the fact that \( I_{\text{ref}} \) is very close to zero.

The radial wavenumber can be rewritten as

\[
I_{\text{ref}} = -\frac{\pi}{2} \pm 2\kappa \pi, \quad \kappa = 0, 1, 2 \ldots
\]
\begin{equation}
\begin{aligned}
    n_r &= \pm \sqrt{\left(\frac{P}{S}\right)^{\frac{1}{2}} + \left(\frac{\dot{B}_\theta n_\theta - \delta \varepsilon n_\theta}{\hbar B_\theta}\right)^{\frac{1}{2}}} \\
    &= \pm \text{sign}(n_i)[1 + \delta^2 \omega^2_{pol}(x)]^{\frac{1}{2}} n_i(x, \theta_0) \frac{\sqrt{x^2 - (x_{ref})^2}}{x}.
\end{aligned}
\end{equation}

When \( x \) is approaching the reflection point at high density, the radial wavenumber also tends to zero, because \([-1 + \delta^2 \omega^2_{pol}(x)] \to 0\).

The phase surface is a strap of the toroidal magnetic surface. Because the plasma is axisymmetric, we are interested in the projection of the phase on a poloidal section. The poloidal phase can be obtained by simple integration of Eq. (19), without the toroidal part

\begin{equation}
S_{-1}(x, \theta) = S_{-1}(x_0, \theta_0) + \delta_0^{-1} \int_{x_0}^x n_i dx + \int_{x_0}^x m_\theta \frac{d\theta}{dx} dx
\end{equation}

Using the expressions for the radial and poloidal wavenumbers [Eqs. (25) and (35), respectively], we are able to express the integral kernel in Eq. (36) as

\begin{equation}
\begin{aligned}
    \left[ \delta_0^{-1} n_r + m_\theta \frac{d\theta}{dx} \right]
    &= \pm \text{sign}(n_i) \delta_0^{-1} \delta^{1/2}_{pol} \omega_{pol} n_{\text{ref}} \left[ 1 - \frac{\sqrt{x_{\text{ref}}^2}}{x \sqrt{x^2 - (x_{\text{ref}})^2}} \right]
    \times \left[ \frac{x - (x_{\text{ref}})^2}{\sqrt{x^2 - (x_{\text{ref}})^2}} \right]
    \times \left[ \frac{\text{sign}(x_{\text{ref}}) \delta_0^{-1} \delta^{1/2}_{pol} \omega_{pol} n_{\text{ref}}}{n_{\text{ref}} - n_{\text{ref}}^2} \right] e^{C_{4} \sin \theta_{J_1(x)} \cos \theta_{J_2(x)}}
    \times \frac{1}{x \sqrt{x^2 - (x_{\text{ref}})^2}}.
\end{aligned}
\end{equation}

Expanding the exponential function using the fact that

\( \sin \theta_{J_1(x)} + \cos \theta_{J_2(x)} = O(\delta^{1/2}) \ll 1 \)

and performing the integral, we finally obtain the quantity

\begin{equation}
\begin{aligned}
    S_{-1}(x, \theta) &= S_{-1}(x_0, \theta_0) + \delta_0^{-1} \int_{x_0}^x n_i dx + \int_{x_0}^x m_\theta \frac{d\theta}{dx} dx
    + \text{sign}(n_i) \delta_0^{-1} \delta^{1/2}_{pol} \omega_{pol} n_{\text{ref}} \left[ 1 - \frac{\sqrt{x_{\text{ref}}^2}}{x \sqrt{x^2 - (x_{\text{ref}})^2}} \right]
    \times \left[ \frac{x - (x_{\text{ref}})^2}{\sqrt{x^2 - (x_{\text{ref}})^2}} \right]
    \times \left[ \frac{\text{sign}(x_{\text{ref}}) \delta_0^{-1} \delta^{1/2}_{pol} \omega_{pol} n_{\text{ref}}}{n_{\text{ref}} - n_{\text{ref}}^2} \right] e^{C_{4} \sin \theta_{J_1(x)} \cos \theta_{J_2(x)}}
    \times \frac{1}{x \sqrt{x^2 - (x_{\text{ref}})^2}}.
\end{aligned}
\end{equation}
where \( \alpha = \theta_0 + C_1 - \arccos(x_{\text{ref}}) = \theta_0 + C_1 - (\pi/2 + \kappa) \), with \( \kappa = 0, 1, 2, \ldots \).

This is a complicated expression for the phase, but the dominant term is the first one on the right-hand side (RHS), which is ordered as \( O(\delta^{-1/2}) \), while other terms are ordered as \( O(1) \) and \( O(\delta) \); for this reason, they are negligible. Using this ordering, we can consider a simplified expression for the phase

\[
S_{-1}(x, \theta) = S_{-1}(x_0, \theta_0) \pm \text{sign}(n_0) \delta_{01} \delta^{1/2} \hat{\alpha}_{\phi 0} n_{10} \\
\times \sqrt{\left[ 1 - x_{\text{ref}}^{-2} - \sqrt{x_{\text{ref}}^{-2}} \right]^2 - \left[ 1 - x_{\text{ref}}^{-2} - \sqrt{x_{\text{ref}}^{-2}} \right]^2}
\]

(38)

Equation (38) shows that when we are far from the plasma center \( x \gg x_{\text{ref}} \), the surfaces of constant phase are strap of magnetic surfaces \( (x = \text{const}) \), while the distortion due to the modulation term \( \cos \theta_0 \) becomes more important near the magnetic axis \( x = x_{\text{ref}} \).

It is possible to calculate the curvature radius of the constant-phase surface \( \rho = |\nabla_r S_{-1}/|\nabla_S_{-1}| \); in polar coordinates, it reads

\[
\rho = \left[ x^2 + \left( \frac{\partial x}{\partial \theta_0} \right)^2 \right]^{3/2} \\
\times \left[ x^2 + 2 \frac{\partial x}{\partial \theta_0} - x \frac{\partial^2 x}{\partial \theta_0^2} \right]^{3/2} \\
\approx \frac{(x^2 + \varepsilon^2 \sin^2 \theta_0)^{3/2}}{|x^2 + 2 \varepsilon^2 \sin^2 \theta_0 - x \varepsilon \cos \theta_0|},
\]

(39)

where \( \varepsilon \) is the inverse aspect ratio. Far from the reflection point \( x \gg x_{\text{ref}} \), the curvature radius of the wave front is essentially \( x \) (the radial coordinate), with a small modulation due to the “sinus” term ordered as \( O(\varepsilon) \). In this case, the wave front is strap of magnetic surfaces (circles) weakly distorted in \( \theta_0 \). Approaching the reflection point, the expression for the curvature radius becomes more complicated. In this case, the dependence of \( \theta_0 \) on \( x_{\text{ref}} \) cannot be neglected and the surfaces are always circles with a bigger distortion; the curvature radius is \( \rho = |x_{\text{ref}} + (\partial x_{\text{ref}}/\partial \theta_0)^2|^{1/2} \) and it is ordered as \( O(\varepsilon^{1/2}) \).

As mentioned in Sec. II, the curvature radius of the wave front, together with the local wavelength (which can be deduced from the dispersion relation), is an important quantity to establish the validity limits of the WKB expansion. A discussion about the violation of the WKB approximation when the inequality Eq. (5) is no longer valid (near the reflection points) is given in Sec. V.

IV. SOLUTION OF THE AMPLITUDE EQUATION IN TOROIDAL GEOMETRY

Equation (8) is the general equation for the amplitude, which, in this geometry, reads

\[
2S A_{0}(r, \theta) \frac{\partial A_{0}(r, \theta)}{\partial r} + 2 \left[ \frac{PK_0}{R_0(r)} + \frac{SK_0}{r} \right] \frac{\partial A_{0}(r, \theta)}{\partial \theta} + \frac{A_{0}(r, \theta)}{rR} \left[ S \frac{\partial}{\partial r} [rR k_0(r, \theta)] + \frac{R_0}{r} (S^2 + P^2 \hat{\alpha}_0^2) \frac{\partial}{\partial \theta} [h m_{10}(r, \theta)] \right] = 0.
\]

(40)

The solution can be given in terms of trajectories. The trajectory in real space is exactly the first equation of the system Eq. (12), while the evolution of the amplitude along the characteristics (in normalized coordinates) is governed by

\[
\frac{dA}{dx} = \frac{A}{2xh_{n_1}} \left[ \frac{\partial}{\partial x} [x h_{n_1}(x, \theta)] \right] \\
+ \frac{e^{-1}}{x} \left( \hat{\alpha}_0^2 + \frac{P}{S} \hat{\alpha}_0^2 \right) \frac{\partial}{\partial \theta} [h m_{10}(x, \theta)].
\]

(41)

In order to solve this equation, it is necessary to know the evolution along the trajectory of the radial and poloidal wavenumbers; in other words, it is necessary to have already solved the problem at lowest order for the phase. In the above equation, in fact, the derivatives of the wavenumber appears explicitly. In general, the wavenumbers \( n_r(x, \theta) \) and \( m_{10}(x, \theta) \) depend on \( x \) and \( \theta \), and in turn on \( \theta_0 \), after solving the equation for \( \theta \). In order to calculate the derivatives, it is necessary to evolve a bundle of rays starting at several \( \theta_0 \), and \( x_0 = 1 \), to built the implicit functions \( n_r(x, \theta) \) and \( m_{10}(x, \theta) \); then, the derivatives \( \partial n_r(x, \theta)/\partial x \big|_{\theta=\text{const}} \) must be calculated at \( \theta = \text{const} \), and \( \partial m_{10}(x, \theta)/\partial \theta \big|_{\theta=\text{const}} \) at \( x = \text{const} \). Using the analytical solution given in the previous section, Eq. (25) for the poloidal wavenumber and Eq. (35) for the radial wavenumber, we have that, with very good approximation

\[
\frac{\partial}{\partial x} [x n_r(x, \theta_0)] = \frac{d}{dx} \ln [x n_r(x, \theta_0)]
\]

(42)

and

\[
\frac{\partial m_{10}(x, \theta_0)}{\partial \theta} = \frac{\partial m_{10}(x, \theta_0)}{\partial \theta_0} = C_3 C_4 \left\{ \frac{\partial^2 (\theta_0 x)}{\partial \theta_0^2} \frac{\partial n_{10}}{\partial \theta_0} + \frac{\partial^2 (\theta_0 x)}{\partial \theta_0^2} \right\},
\]

(43)
\[ \mathcal{J}(x, \theta_0) = \sin \theta_0 J_1(x) + \cos \theta_0 J_2(x), \]  

\[ \frac{\partial \mathcal{J}(x, \theta_0)}{\partial \theta_0} = \cos \theta_0 J_1(x) - \sin \theta_0 J_2(x). \]  

Equation (43) has been obtained by expanding the exponential term in the \( m_p \) formula, as was done in the phase equation. Now, Eq. (41) can be rewritten by considering the same approximation used in the previous section for the solution of the characteristics equations; we obtain

\[ \frac{dA}{dx} = \frac{A}{2xn_\ell(x, \theta_0)} \left[ \frac{\partial}{\partial x} \left[ x n_\ell(x, \theta_0) \right] \right]  
+ e^{-1} \frac{2}{x} (1 - \delta^{-1} e^{2} \omega_p^2 \alpha_x^2 x^2) \frac{\partial n_\ell(x, \theta_0)}{\partial \theta_0} \right]. \]  

The solution can be obtained analytically by substituting Eq. (43) into Eq. (45); we have

\( A(x) = A_0 \sqrt{\frac{n_\ell(x \equiv 1, \theta_0)}{xn_\ell(x, \theta_0)}} \exp \left( \pm \frac{1}{2} \frac{\delta \ell_0}{(x_\text{ref})^2} \int_0^x \left( \frac{(x_\text{ref})^2 - x^2}{x_\text{ref}^2 - (x_\text{ref})^2} \right)^2 dx \right) \left[ \mathcal{J}(x, \theta_0) \frac{\partial n_\ell}{\partial \theta_0} + \frac{\partial \mathcal{J}(x, \theta_0)}{\partial \theta_0} \right] \right]. \]  

where the radial wavenumber has been analytically calculated in Eq. (35). Near the reflection point \((x_\text{ref})\), the amplitude evolution is dominated by the square root and it is sensitive to the singularity for \( x \rightarrow x_\text{ref} \). This is the cylindrical limit of the amplitude equation. Far from the reflection point the exponential term in Eq. (46) becomes important, and the integral can be calculated analytically. We have

\[ A(x) = A_0 \sqrt{\frac{n_\ell(x \equiv 1, \theta_0)}{xn_\ell(x, \theta_0)}} \times \exp \left[ \pm \frac{1}{2} \delta \ell_0 \frac{\delta^{1/2} e^{2} \omega_p^2 \alpha_x}{n_\ell} \frac{\partial n_\ell}{\partial \theta_0} \Xi(x, \theta_0) \right. \]  

where we have, at the lowest order in \( \delta \),

\( \Xi(x, \theta_0) = \int_1^x \frac{\mathcal{J}(x, \theta_0) x dx}{\sqrt{x^2 - (x_\text{ref})^2}} \) \[ = \left[ \sqrt{x^2 - (x_\text{ref})^2} - \sqrt{1 - (x_\text{ref})^2} \right] \]  

\[ \times \cos \left[ \theta_0 + C_1 \left( 1 - \sqrt{1 - (x_\text{ref})^2} \right) \right] \]  

and

\[ Y(x, \theta_0) = \int_1^x \frac{\partial \mathcal{J}(x, \theta_0)}{\partial \theta_0} x dx \]  

\[ = \left[ \sqrt{x^2 - (x_\text{ref})^2} - \sqrt{1 - (x_\text{ref})^2} \right] \]  

\[ \times \sin \left[ \theta_0 + C_1 \left( 1 - \sqrt{1 - (x_\text{ref})^2} \right) \right]. \]  

The evolution of the amplitude along the trajectory in toroidal geometry is similar to the one-dimensional cylindrical case with a correction due to toroidal effects. When the ray reaches the reflection point the amplitude diverges, as can be seen in Eq. (46), and a local full wave analysis must be applied to have the correct value of the amplitude at this point. For this purpose, the full wave equation near the reflection point in the cylindrical approximation can be written as

\[ x^2 \frac{\partial^2 \Phi(x)}{\partial x^2} + x \frac{\partial \Phi(x)}{\partial x} + \beta^2 [x^2 - (x_\text{ref})^2] \Phi(x) = 0, \]  

where

\[ \beta^2 = \delta^{1/2} \frac{\delta \ell_0}{\omega_p^2 \alpha_x^2}. \]  

This is the Bessel equation and its solution is given in terms of Bessel functions of the first and of the second kind (Weber functions)

\[ \Phi(x) = aJ_\nu(\beta x) + bY_\nu(\beta x), \]  

where

\[ \nu = \delta^{1/2} \frac{\delta \ell_0}{\omega_p^2 \alpha_x^2} \]  

and \( a \) and \( b \) are two arbitrary constants.

If we impose the regularity conditions on the axis of the torus, we have

\[ \Phi(x \rightarrow 0) = aJ_\nu(\beta x) + bY_\nu(\beta x) = 0. \]  

Expanding the Bessel functions in the small argument

\[ J_\nu(\beta x \rightarrow 0) = \left( \frac{\beta x}{2} \right)^\nu \frac{\Gamma(\nu + 1)}{\Gamma(\nu)}, \]  

\[ Y_\nu(\beta x \rightarrow 0) = -\left( \frac{1}{\pi} \right)^\nu \Gamma(\nu) \left( \frac{\beta x}{2} \right)^{-\nu}. \]  

Thus, in order to avoid divergences of the field on axis, we must take \( b = 0 \) and

\[ \Phi(x) = aJ_\nu(\beta x). \]  

To match the solution above [Eq. (55)] with the WKB solution obtained so far [Eq. (47)], we must expand Eq. (55) for \( x \rightarrow \infty \); we have
\[ \Phi(x \to \infty) = a J_n(\beta x \to \infty): a \sqrt{\frac{2}{\pi \beta x}} \times \cos \left( \beta x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right). \]  

(56)

Comparing Eq. (56) with Eq. (47), we obtain the arbitrary constant \( a \):

\[ a = A_0 \sqrt{\frac{\pi \delta_0}{\omega_0}} \frac{1}{\omega_{pl0} r_{ref}}. \]  

(57)

At the reflection point, we can now calculate the field and find

\[ \Phi(x) = a J_n(\nu) = \frac{2^{1/3} a}{3^{2/3} \Gamma(\frac{2}{3}) \nu^{1/3}} \left( \frac{2}{3} \right) \beta^{1/3} \delta_0 \frac{\hat{\omega}_{pl}(x_{ref}) \hat{\omega}_{pl}(x) \hat{\omega}_{pl}(x_{ref})}{\hat{\omega}_{pl}(x_{ref})^{1/3} \hat{\omega}_{pl}(x_{ref})^{1/3}}. \]  

(58)

Obviously, the amplitude diverges also when the ray, reflected at plasma center, reaches the plasma periphery near the cutoff layer. To complete this section, a full wave equation is derived near the low-density reflection point, and the solution matched with the WKB solution. In this case the wave equation is

\[ \frac{\partial^2 \Phi(x)}{\partial x^2} + \beta(x - x_{ld}) \Phi(x) = 0, \]  

(59)

where

\[ \beta = 2 \delta_0 \hat{\omega}_{pl}(x_{ref}) \hat{\omega}_{pl}(x) \hat{\omega}_{pl}(x_{ref}) \bigg|_{x=x_{cutoff}} < 0. \]  

(60)

Solutions are

\[ \Phi(x) = \mu \hat{A}_1 [B^{1/3}(x - x_{ld})] + \eta \hat{B}_1 [B^{1/2}(x - x_{ld})], \]  

(61)

with

\[ A_i = \frac{1}{\pi} \left( \frac{1}{3} B^{1/3}(x - x_{ld}) \right) \hat{K}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right], \]

\[ B_i = \frac{1}{\pi} \left( \frac{1}{3} B^{1/3}(x - x_{ld}) \right) \hat{L}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right] \]

\[ + \hat{I}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right]. \]

for \( x > x_{ld} \) and

\[ A_i = \frac{1}{3} \left( \frac{1}{3} B^{1/3}(x - x_{ld}) \right) \hat{J}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right] \]

\[ + \hat{J}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right], \]

\[ B_i = \frac{1}{3} \left( \frac{1}{3} B^{1/3}(x - x_{ld}) \right) \hat{I}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right] \]

\[ - \hat{J}_{1/3} \left[ \frac{2}{3} B^{1/2}(x - x_{ld})^{3/2} \right]. \]

(62)

Expanding Eq. (62) above for \( x < x_{ld} \) we have

\[ \Phi(x) = \mu \hat{A}_1 [B^{1/3}(x - x_{ld})] + \eta \hat{B}_1 [B^{1/2}(x - x_{ld})] = \mu' \left( \frac{\sin \left[ \frac{2}{3} \beta^{1/3}(x - x_{ld}) + \frac{\pi}{4} \right]}{\pi \beta^{1/3}(x - x_{ld})^{1/4}} \right). \]  

(63)

Comparing the above expression with the WKB asymptotic solution Eq. (47) far from the low-density reflection point enables us to obtain the value of the arbitrary constant

\[ \mu' = A_0 \sqrt{\frac{\pi}{2}}. \]

The calculation performed in this section allows us to derive the correct value of the scalar potential everywhere in the plasma, included at the low- and high-density reflection points, where the WKB approximation is no longer valid.

In next section, a comparison between the analytical and numerical solutions is given, choosing plasma parameters typical of an advanced tokamak or a tokamak-like reactor, for which the lower hybrid is used to control the profiles or to be used for steady state operations, respectively.

V. COMPARISON BETWEEN ANALYTICAL AND NUMERICAL SOLUTIONS AND DISCUSSION OF THE WKB APPROXIMATION

In Figs. 1–4, a comparison between analytical and full numerical solutions is shown for the poloidal and toroidal angles \( \theta \) and \( \phi \), and for the radial \( n_r \), poloidal \( m_p \), and parallel \( n_t \) wavenumbers. The plasma parameters used in the simulations are as follows: plasma radius \( a = 60.7 \text{ cm} \), major radius \( R_0 = 186.7 \text{ cm} \), lower hybrid frequency \( f = 3.7 \text{ GHz} \), on-axis magnetic field \( B_0 = 6.7 \text{ T} \), central density \( n_0 = 2 \times 10^{14} \text{ cm}^{-3} \), \( T_{e0} = T_{i0} = 10 \text{ keV} \), safety factor \( q_0 = 2 \), and flat profiles of density and safety factor as indicated in Eqs. (22).

In particular, in Fig. 1 the poloidal and toroidal angles are plotted versus the normalized radial variable (the plasma separatrix is at \( x = 1 \)) in the case of analytical (solid line) and numerical (circles dashed) solution. The formula in Eqs. (23) and (24) seems to be very accurate in describing the evolution of the angles along the trajectory. In Fig. 2 the values of the poloidal wavenumber \( m_p \) is plotted versus \( x \) (the radial
variable). Figures 3 and 4 show, respectively, the radial and parallel wavenumbers versus $x$. The circles dashed line refers to the numerical solution, and the solid line to the analytical formulae. The agreement is very good.

At the reflection points the quantities characterizing the propagation (i.e., the position and wavenumbers) are plotted versus the poloidal launching angle $\theta_0$. This is important to establish also the form of the equiphase surface at the reflection point, and how much it is distorted with respect to a magnetic surface strap. For this purpose, in Fig. 5(a) a bundle of rays launched from $-180^\circ < \theta_0 < +180^\circ$ are plotted in the poloidal section of the torus up to the internal reflection point. In principle, these rays reflect back at the reflection points and the trajectory will come back towards the cutoff layer at another poloidal angle, where they eventually experience another forward reflection. The same bundle of rays starting from $-90^\circ < \theta_0 < +90^\circ$ is shown in Figs. 5(b)–5(d). In Fig. 5(c), in particular, a magnification of

FIG. 1. (Color online) Comparison between the analytical (solid line) and the full numerical solutions (circles dashed) of the ray equations for the poloidal and toroidal angles $\theta$ and $\phi$ vs the normalized radial variable $x$. The plasma parameters used in the simulations are: plasma radius $a=60.7$ cm, major radius $R_0=186.7$ cm, lower hybrid frequency $f=3.7$ GHz, on-axis magnetic field $B_0=6.7$ T, central density $n_0=2\times10^{14}$ cm$^{-3}$, $T_{e0}=T_{i0}=10$ KeV, safety factor $q_0=2$, flat profiles of density and safety factor.

FIG. 2. (Color online) Analytical (solid line) and the full numerical solutions (circles dashed) comparison for the poloidal wavenumber $m_\theta$ vs the normalized radial variable for the same plasma parameters of Fig. 1.

FIG. 3. (Color online) Analytical (solid line) and the full numerical solutions (circles dashed) comparison for the radial wavenumber $n_r$ vs the normalized radial variable for the same plasma parameters of Fig. 1.

FIG. 4. (Color online) Analytical (solid line) and the full numerical solutions (circles dashed) comparison for the parallel wavenumber $n_\parallel$ vs the normalized radial variable for the same plasma parameters of Fig. 1.
FIG. 5. (Color online) (a) Poloidal cross section of the LH trajectories for starting poloidal angles covering all the poloidal extension $0 < \theta_0 < 2\pi$, and for the same plasma parameters of Fig. 1. The rays are stopped at the first reflection point near the magnetic axis. (b) Poloidal cross section of the LH trajectories for starting poloidal angles covering the poloidal extension $-\pi/2 < \theta_0 < +\pi/2$, and for the same plasma parameters of Fig. 1. The rays are stopped at the first reflection point near the magnetic axis. (c) Magnification of the LH trajectories as in Fig. 5(b) near the high density reflection point. (d) 3D plot of the LH trajectories for starting poloidal angles $-\pi/2 < \theta_0 < +\pi/2$, as in (b). (e) Poloidal cross section of the equiphase surfaces of the LH trajectories Fig. 5(a). The equiphase surfaces are highly distorted near the plasma center.
Fig. 5(b) near the high density reflection point is shown, whereas in Fig. 5(d) the same bundle is plotted in three-dimensional (3D) space. The equiphase surfaces (perpendicular to the direction of the wavevector), coincide with the magnetic surfaces, as it is possible to see in Fig. 5(e), when the ray is far from the plasma center. Near the magnetic axis, there is a distortion of the surfaces that are almost circles. In Fig. 6, finally, the reflection point line $\bar{x}_{\text{ref}}(\theta_0)$ is shown in 3D space. This behavior was predicted in Eq. (34) of the previous section. In particular, the curvature radius of the wave front is $\lambda_{\text{ref}}^{bd}$ at the reflection point, which is very close to the magnetic axis $x_{\text{ref}}^{bd}=O(\delta^{1/2})$. Considering that the normalized (to the plasma radius) LH wavelength at the plasma center is $\tilde{\lambda}=\delta_0 \delta^{1/2}$ and $\delta_0 \ll 1$, it is necessary to pay attention that the curvature radius be much greater than the wavelength $\lambda^{bd}$ at the reflection point. In general, for big tokamaks (e.g., JET, ITER) and high operation frequencies this condition is satisfied because $\delta_0=O(\delta^{1/2})$. However, what happens at the reflected ray at the low-density reflection point $\lambda_{\text{ref}}^{bd}$, near the cutoff radius? In this case, the curvature radius of the equiphase surface is $x_{\text{ref}}^{bd}=O(1)$, but the wavelength is going to infinity because the wavenumber is zero at the cutoff. The wavelength in this case is

\[
\lambda \approx \frac{\delta_0}{\Delta x \left. \frac{\partial n}{\partial x} \right|_{\text{cutoff}}} = \delta_0 \delta^{-1}.
\]

Even if $\delta_0=O(\delta^{1/2})$, the inequality $\lambda \ll \lambda_{\text{ref}}^{bd}$ is never satisfied at the low-density (cutoff) reflection point. This means that the inequality Eq. (5), concerning the validity of the WKB approximation, is violated at the low-density reflection point, where the curvature radius of the wave front is smaller than the wavelength and the diffraction is dominating the physics of the wave propagation.

To show the dependence of the rays on the poloidal launching angle and to check the reliability of the full analytical integration of the ray equations, we have plotted the quantities $x_{\text{ref}}$ (Fig. 7), $\theta_{\text{ref}}$ (Fig. 8), $\delta_n=\left|1-n_{\parallel}/n_{\text{ref}}\right|$ (Fig. 9), $m_{\text{ref}}$ (Fig. 10), and $n_{\text{ref}}$ (Fig. 11), versus the poloidal launching angle $\theta_0$, both in the case of full numerical integration of the ray equations (circles, dashed line) and for the analytical solution (solid line). As it is possible to notice on the figures the agreement between the analytical formulae Eqs. (31), (33), and (34) and the full numerical solution is very good.
In Fig. 7, in particular, is worth to note that $x_{\text{ref}} = 0$ when $\theta_0 = \pm \pi/2$, as established in Sec. III. In Fig. 8, it is possible to note the jump of $1-n_{i0}/n_{\text{ref}}$ in the constant $C_2$, as it appears in Eq. (29). This “sign” change is indicated in the second of the formulae in Eqs. (24). In Fig. 12, finally, the amplitude Eqs. (47) and (48), is plotted versus $x$. When $x \to x_{\text{ref}}$, the square root term on the RHS of the equation will dominate on the exponential, leading to a singularity in the WKB amplitude at the point $x=x_{\text{ref}}$, as it can be observed in the figure. The $x_{\text{ref}}=\text{const}$ surface is a caustic surface for the field. On that surface the scalar potential presents a singularity, reflecting the fact that the radial group velocity of the wave vanishes, and locally a full wave treatment is necessary to assure a correct description of the wave propagation, as given in Sec. V.

VI. CONCLUSIONS

A detailed analysis of the WKB method applied to the solution of the LH wave equation in the cold electrostatic limit has been presented. Two first-order PDE for the phase and the amplitude are deduced that can be solved by the method of characteristics. Once the ODE system for the characteristics curves is solved, the phase integral and the amplitude evolution can be determined. The knowledge of
the phase and the amplitude enables us to reconstruct the scalar potential of the electric field associated with the LH wave propagating inside the plasma. This allows us to develop a detailed analysis of the validity limits of the WKB method applied to this kind of problems. In particular, the main inequalities involved in the WKB approximation can be monitored point by point inside the plasma, and the breakdown of the WKB approximation can be established. An analytical solution of the ODE system for the characteristics curves, based on the choice of an appropriate expansion parameter, is obtained in the present work in a pseudotoroidal geometry characterized by a quasicylindrical equilibrium magnetic field. This assumption allows the analytical determination of the phase integral and the wave amplitude. The results are compared with the full numerical solution and show the existence of a caustic surface that is the locus of the reflection points for the trajectories, very close to the magnetic axis. Although the phase integral on that surface is completely determined, the field amplitude presents a singularity and the main inequality for the validity of the WKB method is violated. A local full wave treatment of the wave equation must be provided in order to determine the field at the caustic. Moreover, in the multireflection application of the WKB method, a serious concern on the validity of the method itself arises at the cutoff reflection point. In this case, a violation of the second inequality involving the phase-surface curvature radius as compared with the wavelength is produced, which invalidates the approximation.