Nonlinear gyrokinetic theory of Alfvénic fluctuations and energetic particle transport

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Motivation I

- Need to develop the nonlinear physics in order to properly construct the SAW turbulent spectrum and assess the associated heating/acceleration and transports.
- In terms of EP-driven SAW instabilities, there are two possible nonlinear routes. [C&Z POP13, RMP16]
  - One route corresponds to nonlinear wave-wave interactions and the resultant spectral energy transfer.
  - The other route corresponds to nonlinear SAW-EP interactions
- Need a comprehensive theoretical framework for the self-consistent description of drift Alfvén wave excitation and phase space energetic particle transport in fusion plasmas [C&Z RMP16]
  - This theory is based on nonlinear gyrokinetic formulation of fluctuation spectrum evolution and energetic particle transport, treated on the same footing.

Motivation II

- Theoretical framework based on two pillars [C&Z RMP16]
    - The so-called General Fishbone-Like Dispersion Relation [GFLDR; Z&C POP 21 072120 and 072121] (Ruirui Ma's talk)
    - The derivation of the renormalized energetic particle response [Z&C NJP 17 013052)].
    - It is fully deployed and explained in the 2016 Rev. Mod. Phys. article
- Outline of this talk
    - Schematic derivation of the theoretical framework
    - Simple applications of two routes to NL physics:
        - nonlinear wave-wave interactions
        - nonlinear SAW-EP interactions
    - Extensions to broader scope of plasma physics:
        - Earth's chorus chirping rate

Schematic derivation

- General Fishbone-Like Dispersion Relation
  - The GFLDR can be seen as a gyrokinetic energy principle, valid in a wide frequency interval, from the low MHD up to the Alfvén wave frequencies
  - The GFLDR has been systematically verified numerically and validated experimentally in AUG, DIIID, FTU, HL-2A, JET, KSTAR, Tore Supra, etc.
  - Ruirui Ma's presentation for application to DIIID
- Renormalized energetic particle response
  - The phase space zonal structures (PSZS) as the proper definition of nonlinear energetic particle equilibrium evolving on the spatiotemporal meso-scales [Z&C NJP 17 013052]
  - PSZS evolution equation is cast in the form of a Dyson equation

The GFLDR - I

- Evolution of the fluctuation spectrum

$$\left(\delta\phi, \delta A_\parallel \equiv -c\partial_t^{-1}\nabla_\parallel\delta\psi, \delta B_\parallel\right)$$

- Perpendicular pressure balance allows to explicitly solve for $\delta B_\parallel$ [C&Z RMP16] $\Rightarrow$ standard NLGK ordering

$$\nabla_\perp\left(B_0\delta B_\parallel + 4\pi\delta P_\perp\right) \simeq 0\,,$$

- Introducing PSZS, $\bar{F}_0$ (nonlinear equilibrium) [C&Z RMP16], and nonadiabatic particle response, δg, (to be discussed later), other two closing field equations are:

$\Rightarrow$ quasineutrality condition

$$\sum\left\langle\frac{e^2}{m}\frac{\partial\bar{F}_0}{\partial\mathcal{E}}\right\rangle_v\delta\phi + \nabla\cdot\sum\left\langle\frac{e^2}{m}\frac{2\mu}{\Omega^2}\frac{\partial\bar{F}_0}{\partial\mu}\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v\nabla_\perp\delta\phi + \sum\left\langle eJ_0(\lambda)\delta g\right\rangle_v = 0$$

The GFLDR - II

$\Rightarrow$ gyrokinetic vorticity equation [C&Z RMP16]

$$
\begin{aligned}
&B_0\left(\nabla_\parallel + \frac{\delta \boldsymbol{B}_\perp}{B_0}\cdot\boldsymbol{\nabla}\right)\left(\frac{\delta J_\parallel}{B_0}\right) - \boldsymbol{\nabla}\cdot\sum\left\langle \frac{e^2}{m}\frac{2\mu}{\Omega^2}\left(B_0\frac{\partial \bar{F}_0}{\partial \mathcal{E}} + \frac{\partial \bar{F}_0}{\partial \mu}\right)\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v \boldsymbol{\nabla}_\perp \frac{\partial}{\partial t}\delta\phi \\
&+\sum ec\boldsymbol{b}_0\times\boldsymbol{\nabla}\left\langle \frac{2\mu}{\Omega^2}\bar{F}_0\left(\frac{J_0^2-1}{\lambda^2}\right)\right\rangle_v\cdot\boldsymbol{\nabla}\nabla_\perp^2\delta\phi + \frac{c}{B_0}\boldsymbol{b}_0\times\boldsymbol{\kappa}\cdot\boldsymbol{\nabla}\sum\left\langle m\left(\mu B_0 + v_\parallel^2\right)J_0\delta g\right\rangle_v \\
&+\delta\boldsymbol{B}_\perp\cdot\boldsymbol{\nabla}\left(\frac{J_{\parallel 0}}{B_0}\right) + \sum e\left\langle J_0\left[\frac{c}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\left(J_0\delta\phi\right)\cdot\boldsymbol{\nabla}\delta g\right] - \frac{c}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\left(J_0\delta g\right)\right\rangle_v \\
&+\frac{c}{B_0}\boldsymbol{b}_0\times\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\sum\left\langle \frac{e^2}{m}\frac{2\mu}{\Omega^2}\frac{\partial \bar{F}_0}{\partial \mu}\left(\frac{1-J_0^2}{\lambda^2}\right)\right\rangle_v \boldsymbol{\nabla}_\perp\delta\phi\right] = 0 \; .
\end{aligned}
$$

The GFLDR - III

- Construct GK "energy" functional [Chen&Hasegawa 1991]
  - Use the mode structure decomposition (MSD) approach [Z. Lu 2012] (reducing to ballooning formalism at short wavelength)

$$\mathcal{F}[\delta\phi,\delta A_\parallel] = (2\pi)^3\sum_{j\in\mathbb{Z}}\int_0^a dr\frac{d\psi/dr}{2}e^{i2\pi nqj}\int_{-\infty}^{\infty}\mathcal{J}d\vartheta$$

$$\left[-\boldsymbol{\nabla}\times\left(\boldsymbol{b}_0 c\partial_t^{-1}\nabla_\parallel\delta\hat{\phi}^\dagger(r,\vartheta+2\pi j)\right)\cdot\frac{\delta\hat{\boldsymbol{B}}_\perp(r,\vartheta)}{4\pi}\right.$$

$$\left.+\partial_t^{-1}\delta\hat{\phi}^\dagger(r,\vartheta+2\pi j)\boldsymbol{\nabla}\cdot\delta\hat{\boldsymbol{J}}_\perp(r,\vartheta)\right] = \delta W - \delta I\ .$$

- GFLDR generally cast as

"fluid" potential energy/MHD

$$i|s|\Lambda = \boxed{} + \delta\hat{W}_k$$

generalized GK inertia

GK potential energy/compression

The GFLDR - short wavelength I

CNPS Center for Nonlinear Plasma Science Since 2020

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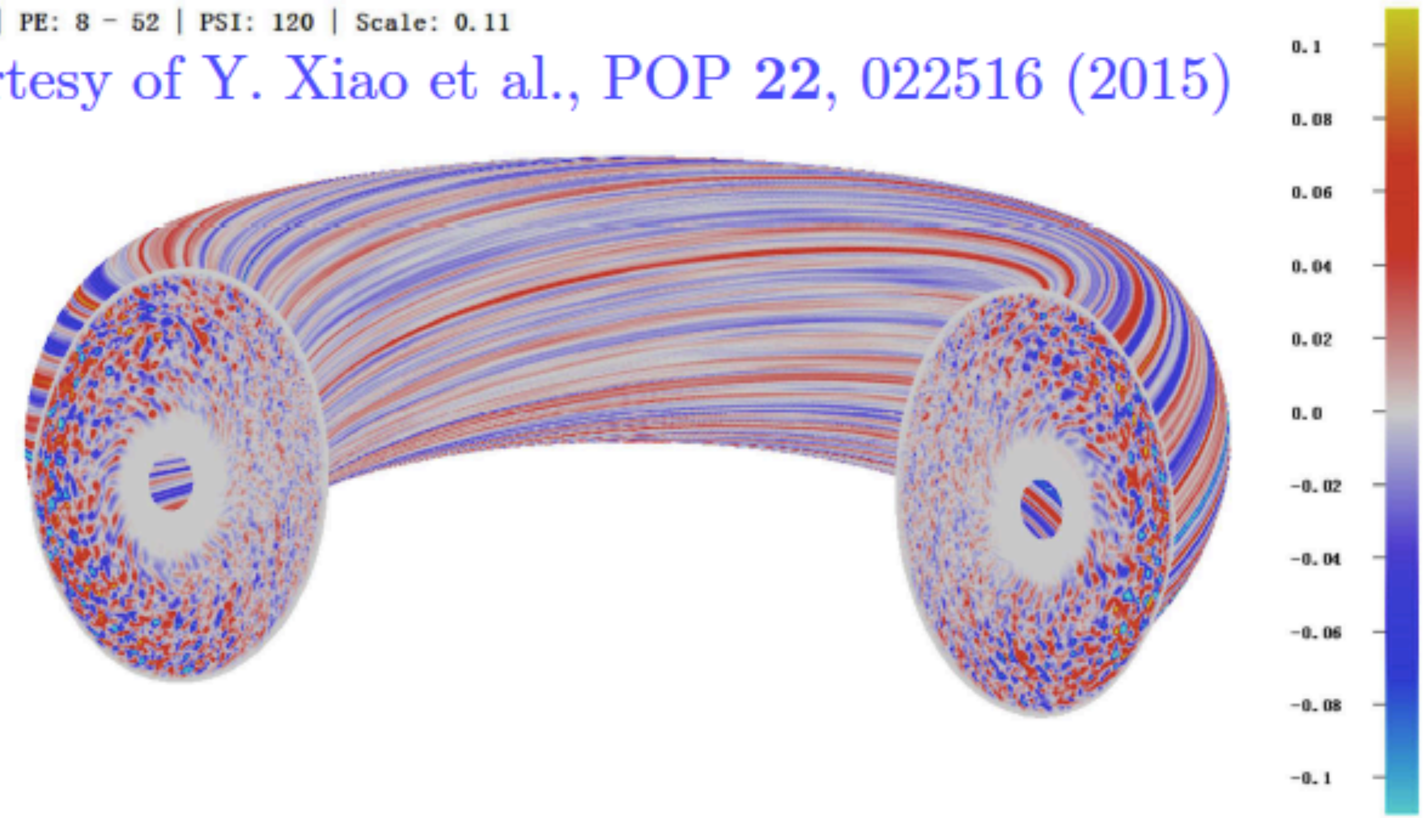
Courtesy of Y. Xiao et al., POP **22**, 022516 (2015)



- ψ = poloidal magnetic flux

$$\boldsymbol{B}_0 = F(\psi)\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi$$

$$\equiv \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\psi$$

$$q \equiv \frac{\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\zeta}{\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}\theta} = q(\psi)$$

- Filaments $\Rightarrow$ Parallel mode structures $\Rightarrow$ weighing of linear and nonlinear interactions [Zonca et al, PPCF15], [C&Z RMP16]

- Correspondingly, nonlinear interactions can take the forms of mode coupling between two toroidal mode numbers, and modulation of the radial envelope.

- Proper description of fluctuation spectrum in toroidal geometry requires extending standard wave-kinetic equation to a non-linear Schrödinger-like equation (NLSE) [C&Z RMP16].

The GFLDR - short wavelength II

- Consistent with the ordering $|\gamma_{Ln}| \sim \tau_{NLn}^{-1} \ll |\omega_n|$ [C&Z RMP16], we can average quasineutrality and nonlinear vorticity equations over linear parallel mode structures, yielding for $\boldsymbol{A}_n(r,t) \equiv \hat{\boldsymbol{e}}_n A_n(r,t)$

$$\underbrace{\hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{D}\left(r,t,k_{nr},\omega_n\right) \cdot \boldsymbol{A}_n(r,t) e^{iS_n(r,t)}}_{\text{linear response}} = \underbrace{\hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{F}(r,t)}_{\text{nonlinear response}} \; ,$$

- Introduce $\delta\hat{\phi}_{\|n} \equiv \delta\hat{\phi}_n - \delta\hat{\psi}_n$, radial envelope, A_n and polarization vector $\hat{\boldsymbol{e}}_n$

$$\begin{pmatrix} e\delta\hat{\psi}_n(r,\vartheta;t)/T_{0i} \\ e\delta\hat{\phi}_{\|n}(r,\vartheta;t)/T_{0i} \end{pmatrix} \equiv A_n(r,t) e^{iS_n(r,t)} \begin{pmatrix} e_1(r,t) y_1(r,\vartheta) \\ e_2(r,t) y_2(r,\vartheta) \end{pmatrix} .$$

with eikonal representation and normalizations

$$\int_{-\infty}^{\infty} |y_{1,2}(r,\vartheta)|^2 \, d\vartheta = 1 \; . \qquad \hat{\boldsymbol{e}}_n^+ \cdot \hat{\boldsymbol{e}}_n = 1 \; .$$

The GFLDR - short wavelength III

- The nonlinear term (including ext. forcing) can be formally written as

$$e^{-iS_n}\hat{\boldsymbol{e}}_n^+ \cdot (\boldsymbol{F} - \boldsymbol{F}_{\rm ext}) = (C_{n,0} + C_{0,n}) \circ A_n(r,t)A_z(r,t) + \sum_{n'+n''=n}^{n',n''\neq n} C_{n',n''} \circ A_{n'}(r,t)A_{n''}(r,t)$$

where $C_{n',n''}$ imply nonlocal interactions in the n toroidal mode number space and whose composition with $A_n(r,t)$ and/or $A_z(r,t)$ is denoted by $\circ$.

- Consistent with the GFLDR theory [C&Z RMP16],

$$e^{-iS_n}\hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{F}/A_n = i\Lambda^{NL} - \delta\bar{W}_f^{NL} - \delta\bar{W}_k^{NL} + e^{-iS_n}\hat{\boldsymbol{e}}_n^+ \cdot \boldsymbol{F}_{\rm ext}/A_n \ .$$

Λ describes the generalized inertia due to the e.m. field behaviors on short length scales, while $\delta\bar{W}_f$ and $\delta\bar{W}_k$ account for "fluid" and "kinetic" potential energy fluctuations due to meso- and macro-scale responses.

The GFLDR - short wavelength IV

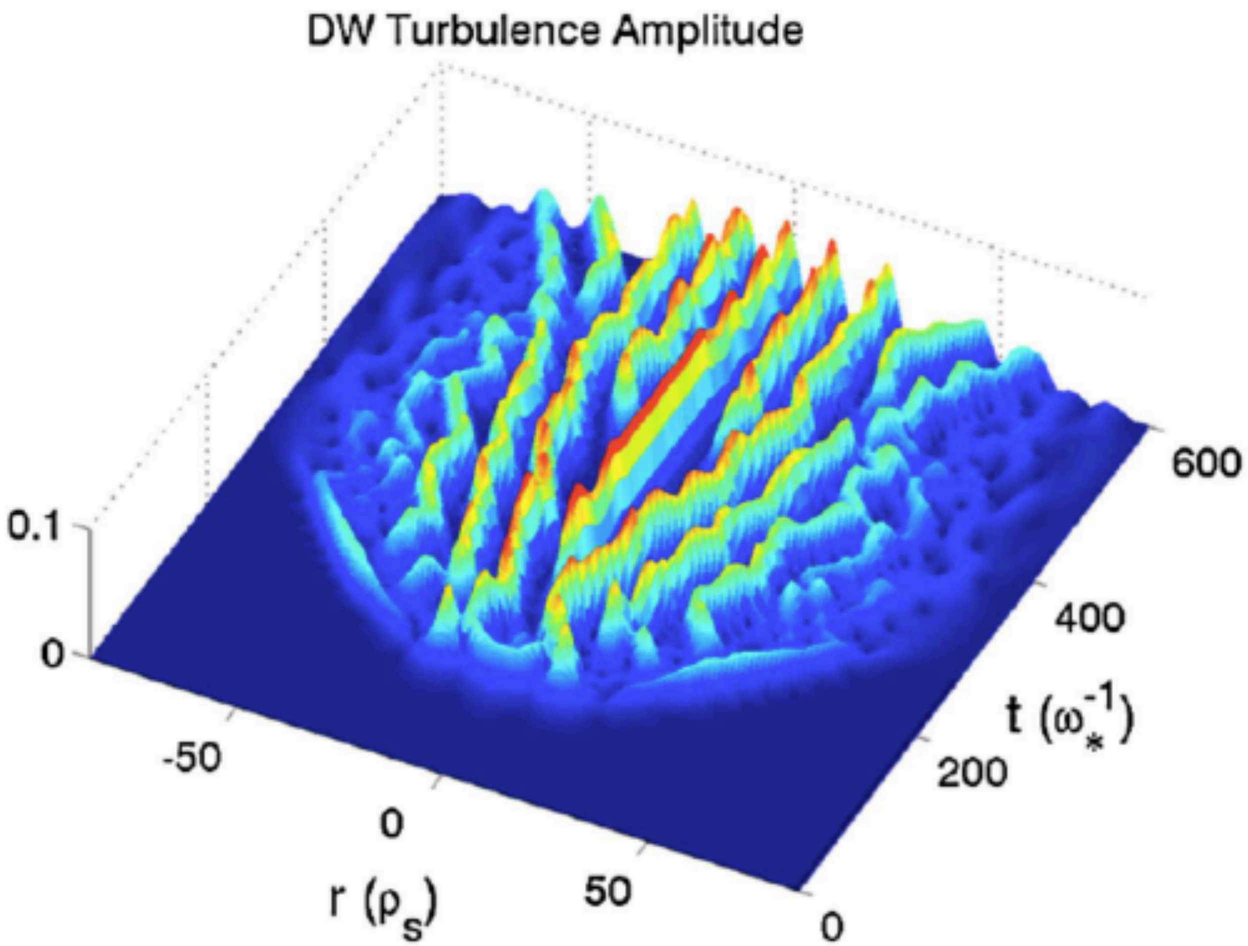
- Nonlinear envelope equation can be written in NLSE-like form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial D^0_{Rn}}{\partial \omega_n}A_n^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D^0_{Rn}}{\partial k_{nr}}A_n^2\right) + 2D^1_{An}A_n^2 - 2iD^1_{Rn}A_n^2 \\
&+iA_n\left(\frac{\partial^2 D^0_{Rn}}{\partial k_{nr}^2} + 2\frac{\partial \hat{e}_n^+}{\partial k_{nr}}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial k_{nr}}\right)\frac{\partial^2 A_n}{\partial r^2} = -2ie^{-iS_n}A_n\hat{e}_n^+\cdot\boldsymbol{F} \\
&-\left(\hat{e}_n^+\cdot\frac{d}{dt}\hat{e}_n - \frac{d}{dt}\hat{e}_n^+\cdot\hat{e}_n\right)\frac{\partial D^0_{Rn}}{\partial\omega_n}A_n^2 + \left(\frac{\partial \hat{e}_n^+}{\partial\omega_n}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial\hat{e}_n}{\partial t} - \frac{\partial \hat{e}_n^+}{\partial t}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial\hat{e}_n}{\partial\omega_n}\right)A_n^2 \\
&-\left(\frac{\partial \hat{e}_n^+}{\partial k_{nr}}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial\hat{e}_n}{\partial r} - \frac{\partial \hat{e}_n^+}{\partial r}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial\hat{e}_n}{\partial k_{nr}}\right)A_n^2 \,.
\end{aligned}
$$

- The NLSE-like structure is of crucial importance for proper analysis of structure formation in strongly magnetized toroidal plasmas, where wave packets can be focused/defocused and back scattered by both nonlinearities as well as by radial nonuniformities [C&Z RMP16].

The GFLDR - short wavelength V

- This goes beyond the standard wave kinetic equation that is typically adopted in literature, is of fundamental importance not only in the description of EP induced avalanches, such as in the case of energetic particle modes (EPM) [Zonca et al 05] but also for the interaction of zonal fields and drift wave turbulence [Guo et al 09].



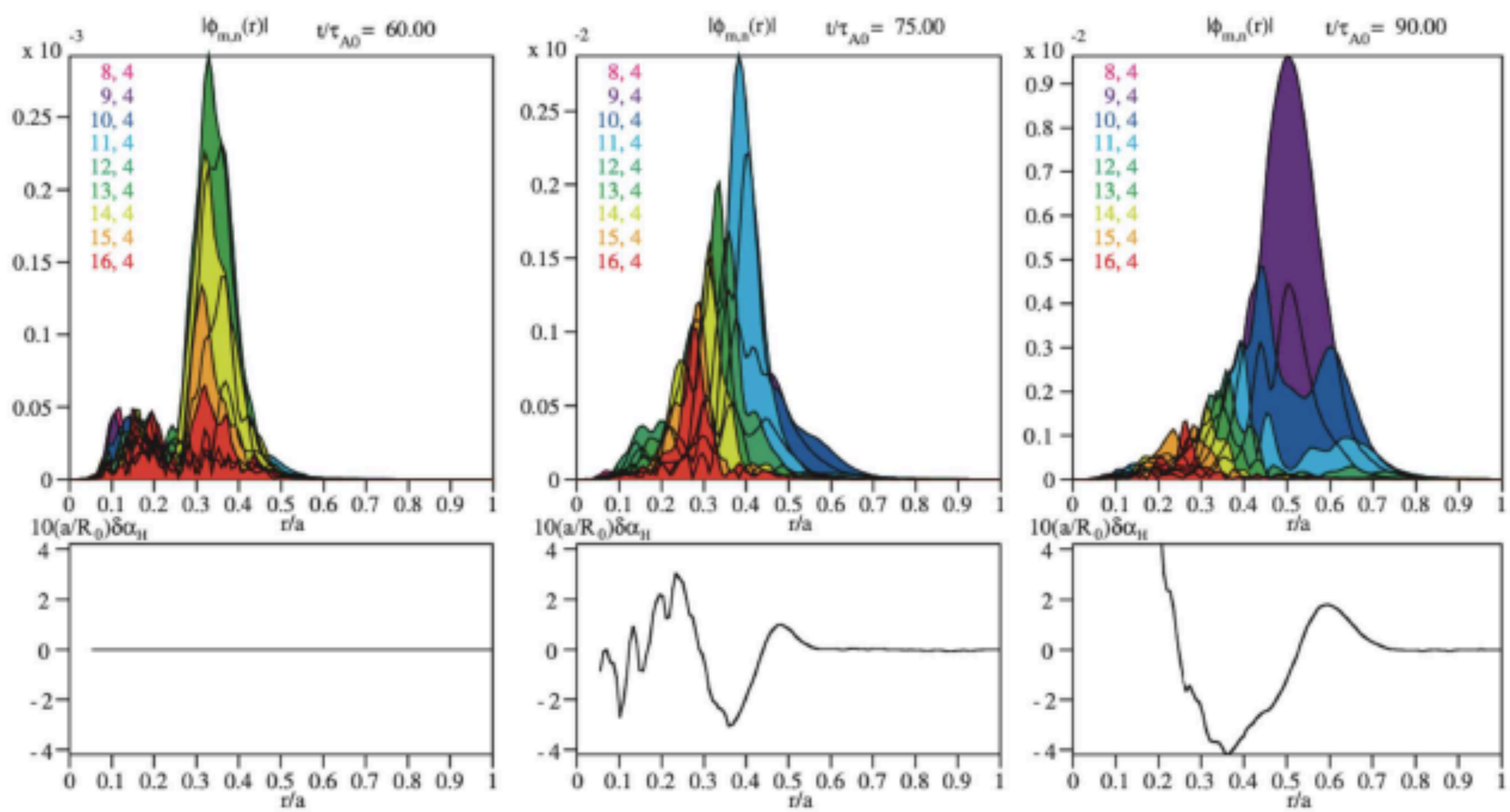
DW turbulence spreading in the presence of growth or damping, dissipation and finite system size effects [courtesy of Z. Guo PRL 103, 055002 (2009)]

The GFLDR - short wavelength VI

- The NLSE-like equation for symmetry breaking fluctuations is closed by the nonlinear evolution equation for $\delta\phi_z$ (quasineutrality; cf. above) and $\delta A_{\parallel z}$ [C&Z RMP16] [M.V. Falessi et al. I-14 Varenna 2020; NJP to be sub.]

$$\frac{\partial}{\partial t}\delta A_{\parallel z} = \left(\frac{c}{B_0}\boldsymbol{b}_0 \times \nabla \delta A_\parallel \cdot \nabla \delta\psi \right)_z \ .$$

Evidence of EP avalanche [Z. et al Nucl. Fusion 45 (2005) 477 – 484]



Relevant limits - I

- **The wave kinetic equation**
  - Dispersive term is dropped
  - Nonlinear term is simplified (k-space diffusion)

□ Nonlinear envelope equation can be written in NLSE-like form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial D^0_{Rn}}{\partial \omega_n} A_n^2\right) - \frac{\partial}{\partial r}\left(\frac{\partial D^0_{Rn}}{\partial k_{nr}} A_n^2\right) + 2D^1_{An} A_n^2 - 2iD^1_{Rn} A_n^2 \\
&+ iA_n \left(\frac{\partial^2 D^0_{Rn}}{\partial k_{nr}^2} + 2\frac{\partial \hat{e}_n^+}{\partial k_{nr}} \cdot \boldsymbol{D}^0_{Rn} \cdot \frac{\partial \hat{e}_n}{\partial k_{nr}}\right) \frac{\partial^2 A_n}{\partial r^2} = -2ie^{-iS_n} A_n \hat{e}_n^+ \cdot \boldsymbol{F} \\
&- \left(\hat{e}_n^+ \cdot \frac{d}{dt}\hat{e}_n - \frac{d}{dt}\hat{e}_n^+ \cdot \hat{e}_n\right) \frac{\partial D^0_{Rn}}{\partial \omega_n} A_n^2 + \left(\frac{\partial \hat{e}_n^+}{\partial \omega_n} \cdot \boldsymbol{D}^0_{Rn} \cdot \frac{\partial \hat{e}_n}{\partial t} - \frac{\partial \hat{e}_n^+}{\partial t} \cdot \boldsymbol{D}^0_{Rn} \cdot \frac{\partial \hat{e}_n}{\partial \omega_n}\right) A_n^2 \\
&- \left(\frac{\partial \hat{e}_n^+}{\partial k_{nr}} \cdot \boldsymbol{D}^0_{Rn} \cdot \frac{\partial \hat{e}_n}{\partial r} - \frac{\partial \hat{e}_n^+}{\partial r} \cdot \boldsymbol{D}^0_{Rn} \cdot \frac{\partial \hat{e}_n}{\partial k_{nr}}\right) A_n^2 .
\end{aligned}
$$

Relevant limits - II

The QL field amplitude equation

- Radial local limit
- Nonlinear term is simplified (QL diffusion equation)

□ Nonlinear envelope equation can be written in NLSE-like form

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial D^0_{Rn}}{\partial \omega_n}A_n^2\right)-\frac{\partial}{\partial r}\left(\frac{\partial D^0_{Rn}}{\partial k_{nr}}A_n^2\right)+2D^1_{An}A_n^2-2iD^1_{Rn}A_n^2\\
&+iA_n\left(\frac{\partial^2 D^0_{Rn}}{\partial k_{nr}^2}+2\frac{\partial \hat{e}_n^+}{\partial k_{nr}}\cdot \boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial k_{nr}}\right)\frac{\partial^2 A_n}{\partial r^2}=-2ie^{-iS_n}A_n\hat{e}_n^+\cdot\boldsymbol{F}\\
&-\left(\hat{e}_n^+\cdot\frac{d}{dt}\hat{e}_n-\frac{d}{dt}\hat{e}_n^+\cdot\hat{e}_n\right)\frac{\partial D^0_{Rn}}{\partial \omega_n}A_n^2+\left(\frac{\partial \hat{e}_n^+}{\partial \omega_n}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial t}-\frac{\partial \hat{e}_n^+}{\partial t}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial \omega_n}\right)A_n^2\\
&-\left(\frac{\partial \hat{e}_n^+}{\partial k_{nr}}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial r}-\frac{\partial \hat{e}_n^+}{\partial r}\cdot\boldsymbol{D}^0_{Rn}\cdot\frac{\partial \hat{e}_n}{\partial k_{nr}}\right)A_n^2\,.
\end{aligned}
$$

Schematic derivation

- General Fishbone-Like Dispersion Relation
  - The GFLDR can be seen as a gyrokinetic energy principle, valid in a wide frequency interval, from the low MHD up to the Alfvén wave frequencies
  - The GFLDR has been systematically verified numerically and validated experimentally in AUG, DIIID, FTU, HL-2A, JET, KSTAR, Tore Supra, etc.
  - Ruirui Ma's presentation for application to DIIID
- Renormalized energetic particle response
  - The phase space zonal structures (PSZS) as the proper definition of nonlinear energetic particle equilibrium evolving on the spatiotemporal meso-scales [Z&C NJP 17 013052]
  - PSZS evolution equation is cast in the form of a Dyson equation

Phase space zonal structures - I

- Since PSZS are undamped by (fast) collisionless dissipation mechanisms, they are naturally expressed as functions of invariants of motion (nearly integrable Hamiltonian system). Separating fast ($[\ldots]_F$) from slow variations,

$$F_z \equiv \bar{F}_0 + e^{-iQ_z}\left(\overline{e^{iQ_z}\delta F_z}\Big|_F + \delta\tilde{F}_{Bz}\right) ,$$

macro- $\oplus$ meso-scale (CGL) micro-scale collisionless damped

Phase Space Zonal Structures [Z&C NJP15]

where $\overline{[\ldots]} = \oint d\ell/v_\| [\ldots]/\oint d\ell/v_\|$, $\overline{[\tilde{\ldots}]} = 0$, F_z is the $n = 0$ gyrocenter particle distribution function. $e^{-iQ_z} \Rightarrow$ nonlocal (integral) particle response

$$\begin{aligned}
\partial_t \overline{e^{iQ_z}\bar{F}_0} = & -\overline{e^{iQ_z}\frac{F(\psi)}{B_0}\partial_t\left\langle\delta A_{\|g}\right\rangle_z\frac{\partial}{\partial\bar{\psi}}\bar{F}_0}\Big|_S - \frac{1}{\tau_b}\frac{\partial}{\partial\psi}\left[\tau_b\overline{e^{iQ_z}\delta\dot{\psi}_z\delta F_z}\right]_S \\
& -\frac{1}{\tau_b}\frac{\partial}{\partial\mathcal{E}}\left[\tau_b\overline{e^{iQ_z}\delta\dot{\mathcal{E}}_z\delta F_z}\right]_S - \frac{1}{\tau_b}\frac{\partial}{\partial\psi}\left[\tau_b\overline{e^{iQ_z}\delta\dot{\psi}\delta F}\right]_{zS} - \frac{1}{\tau_b}\frac{\partial}{\partial\mathcal{E}}\left[\tau_b\overline{e^{iQ_z}\delta\dot{\mathcal{E}}\delta F}\right]_{zS} \\
& + \overline{e^{iQ_z}\left[C_g + \mathcal{S}\right]}\Big|_{zS} .
\end{aligned}$$

[M.V. Falessi I-14 Varenna 2020]

- **3D evolution equation**

Phase space zonal structures - II

- Fast spatiotemporal dependence

- Introducing

$$\overline{\delta g_{Bz}}\big|_S + \overline{\delta g_{Bz}}\big|_F$$

$$\equiv \overline{e^{iQ_z}\delta F_z}\Big|_F - \frac{e}{m}\overline{e^{iQ_z}\left\langle \delta L_g \right\rangle_z \frac{\partial}{\partial \mathcal{E}}\Big|_{\bar\psi}} \bar F_0 + \overline{e^{iQ_z}\frac{F(\psi)}{B_0}\left\langle \delta A_{\parallel g}\right\rangle_z \frac{\partial}{\partial \bar\psi}}\bar F_0 ,$$

one can derive the evolution equation for $\overline{\delta g_{Bz}}\big|_F$ as well [JPCS 2021]

$$\partial_t\, \overline{\delta g_{Bz}}\big|_F = -\, e^{iQ_z}\frac{e}{m}\partial_t \left[\left\langle \delta L_g \right\rangle_z \frac{\partial}{\partial \mathcal{E}}\Big|_{\bar\psi} \bar F_0\right]\Bigg|_F + e^{iQ_z}\frac{F(\psi)}{B_0}\left\langle \delta A_{\parallel g}\right\rangle_z \frac{\partial}{\partial \bar\psi}\partial_t \bar F_0\Bigg|_F$$

$$+ \overline{e^{iQ_z}\left[C_g + \mathcal{S}\right]}\Big|_{zF} - \frac{1}{\tau_b}\frac{\partial}{\partial \psi}\left[\tau_b \overline{e^{iQ_z}\delta\dot\psi_z \delta F_z}\right]_F - \frac{1}{\tau_b}\frac{\partial}{\partial \mathcal{E}}\left[\tau_b \overline{e^{iQ_z}\delta\dot{\mathcal{E}}_z \delta F_z}\right]_F$$

$$-\frac{1}{\tau_b}\frac{\partial}{\partial \psi}\left[\tau_b \overline{e^{iQ_z}\delta\dot\psi \delta F}\right]_{zF} - \frac{1}{\tau_b}\frac{\partial}{\partial \mathcal{E}}\left[\tau_b \overline{e^{iQ_z}\delta\dot{\mathcal{E}} \delta F}\right]_{zF} .$$

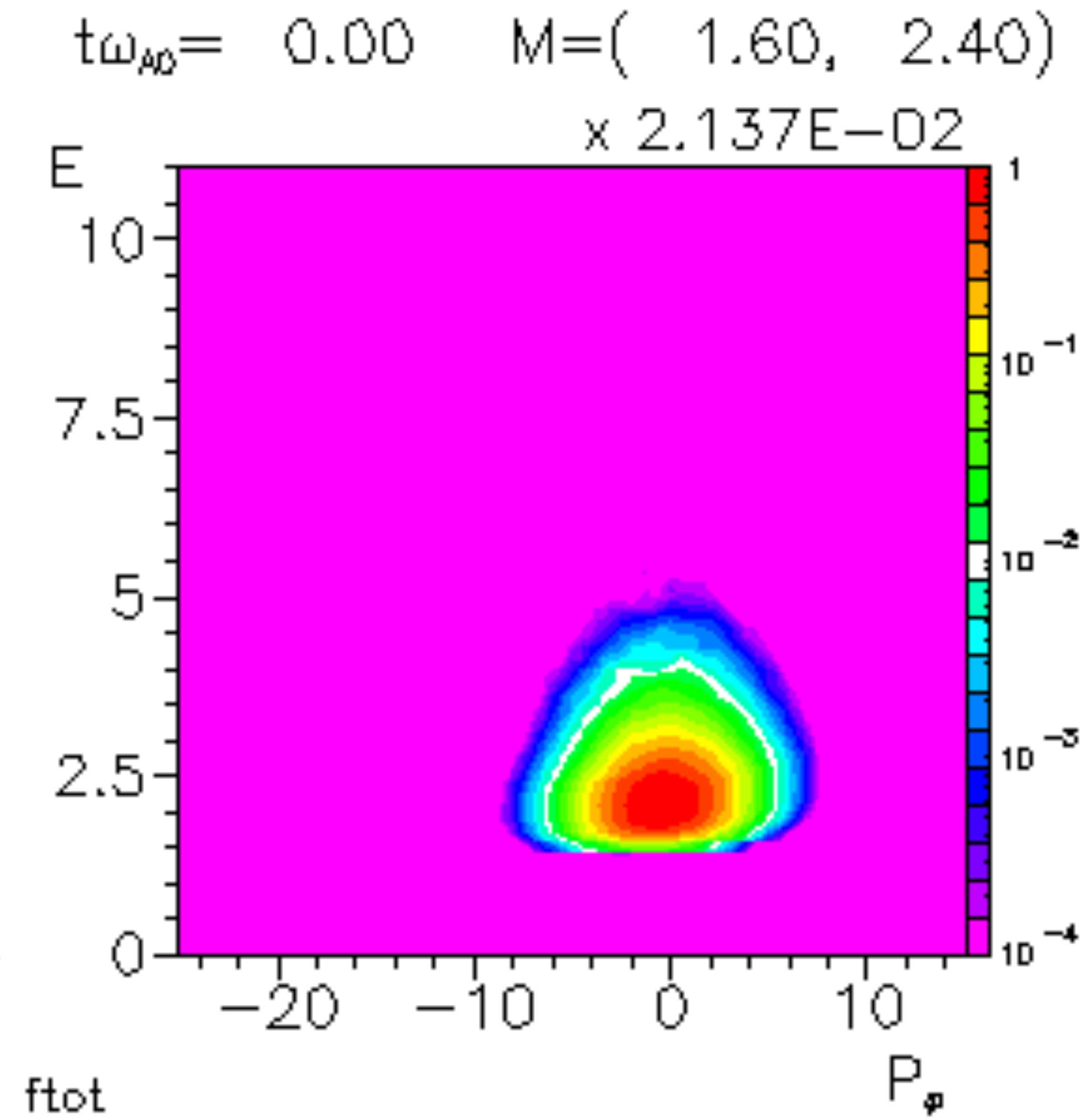
- Analogously, one can derive the evolution equations for $\delta\tilde g_{Bz}$ (and, thus, $\delta\tilde F_{Bz}$), omitted here for brevity [M.V. Falessi I-14 Varenna 2020].

Phase space zonal structures - III

- Numerical verification by HMGC + HYMAGYC codes [S. Briguglio and G. Vlad, 2020 - EUROfusion project MET] - ongoing work within EUROfusion projects ATEP (Lauber/Falessi) & TSVV#10 (Mishchenko)

- Slow and fast spatiotemporal responses

$$\partial_t \left(\overline{e^{iQ_z} \bar{F}_0} + \overline{e^{iQ_z} \delta F_z} \right) = -\frac{1}{\tau_b} \frac{\partial}{\partial \psi} \left[\tau_b \overline{e^{iQ_z} \delta \dot{\psi} \delta F} \right]_z - \frac{1}{\tau_b} \frac{\partial}{\partial \mathcal{E}} \left[\tau_b \overline{e^{iQ_z} \delta \dot{\mathcal{E}} \delta F} \right]_z$$



Phase space zonal structures - IV

- **Emerging key physics issues**
  - Analysis coincides with what is contained in most global GK codes - however it is reduced to 3D space for spatiotemporal mesoscales
  - Mixed full-F/delta-F formulation suitable for long time scale simulation [EUROfusion projects ATEP/TSVV#10 cit]
  - Clear path for a multi-fidelity hierarchical approach

    → **Dyson Schrödinger Model (DSM) [ZCFQ JPCS21]**
  - Works not only for **SAW excited by EP but also for DW turbulent transport**
    - Chen et al. POP2000, 2001, etc. (see [PPCF2015]) and examples on pp. 12-13

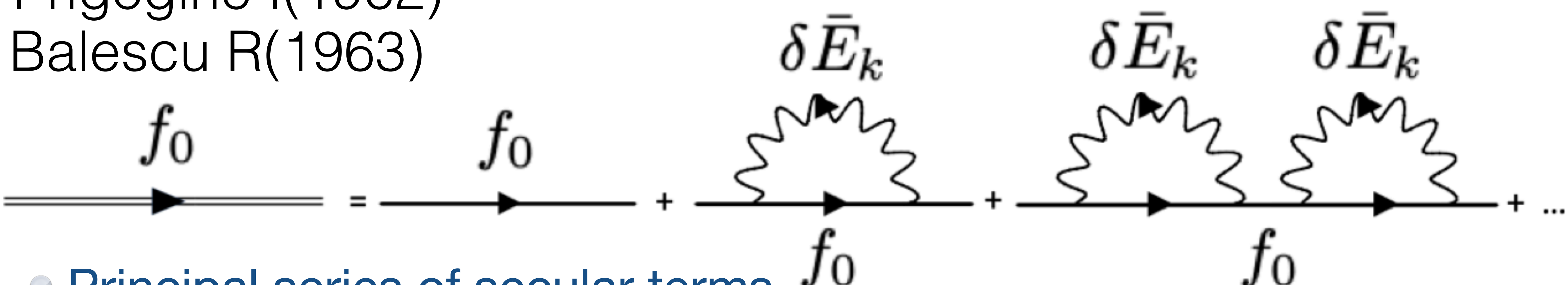
Phase space zonal structures - V

- **Conceptual steps**
  - Both “slow” (CGL equilibrium evolution) and “fast” (NL spatiotemporal scale; e.g. avalanches) evolution of PSZS are dominated by emission and absorption of same k

$$\partial_t \left(\overline{e^{iQ_z}\bar{F}_0} + \overline{e^{iQ_z}\delta F_z}\right) = \boxed{-\frac{1}{\tau_b}\frac{\partial}{\partial\psi}\left[\tau_b \overline{e^{iQ_z}\delta\dot{\psi}\delta F}\right]_z} - \boxed{\frac{1}{\tau_b}\frac{\partial}{\partial\mathcal{E}}\left[\tau_b \overline{e^{iQ_z}\delta\dot{\mathcal{E}}\delta F}\right]_z}$$

Van Hove L (1955)
Prigogine I(1962)
Balescu R(1963)

- **Slow terms generate “secularities”**
- **Renormalization & Dyson like equation**



- Principal series of secular terms

Resonance broadening

- **Conceptual steps… continued**
  - Also finite n (symmetry breaking) fluctuations are dominated by emission and absorption of same k [Dupree 1966; Aamodt 1967; Weinstock 1969; Mima 1973]
  - Nonlinearity competes with the resonance structure, rather than the linear drive
  - Resonance broadening is generally an integral operator, treated assuming broad turbulent spectrum in the original works

Dyson Schrödinger Model I

- Having adopted action angle variables, the solution of **Dyson Schrödinger Model (DSM)** is technically challenging but straightforward
  - Present application refers to the fishbone paradigm (precession resonance) for simplicity. General result discussed in [FCQZ JPCS21,NJP21,PPCF21]
  - Perturbation expansion of particle response in the weak field intensity limit, yields secular terms dominated by emission and absorption of the same k-fluctuation (called diagonal interactions).
  - Extension to stellarators [Zocco et al, ATEP/TSVV#10 cit]

Dyson Schrödinger Model II

- Particle response to test k_0 fluctuation $\Rightarrow$ renormalization [Dupree 66], [Mima 73], [Laval & Pesme 84,99].

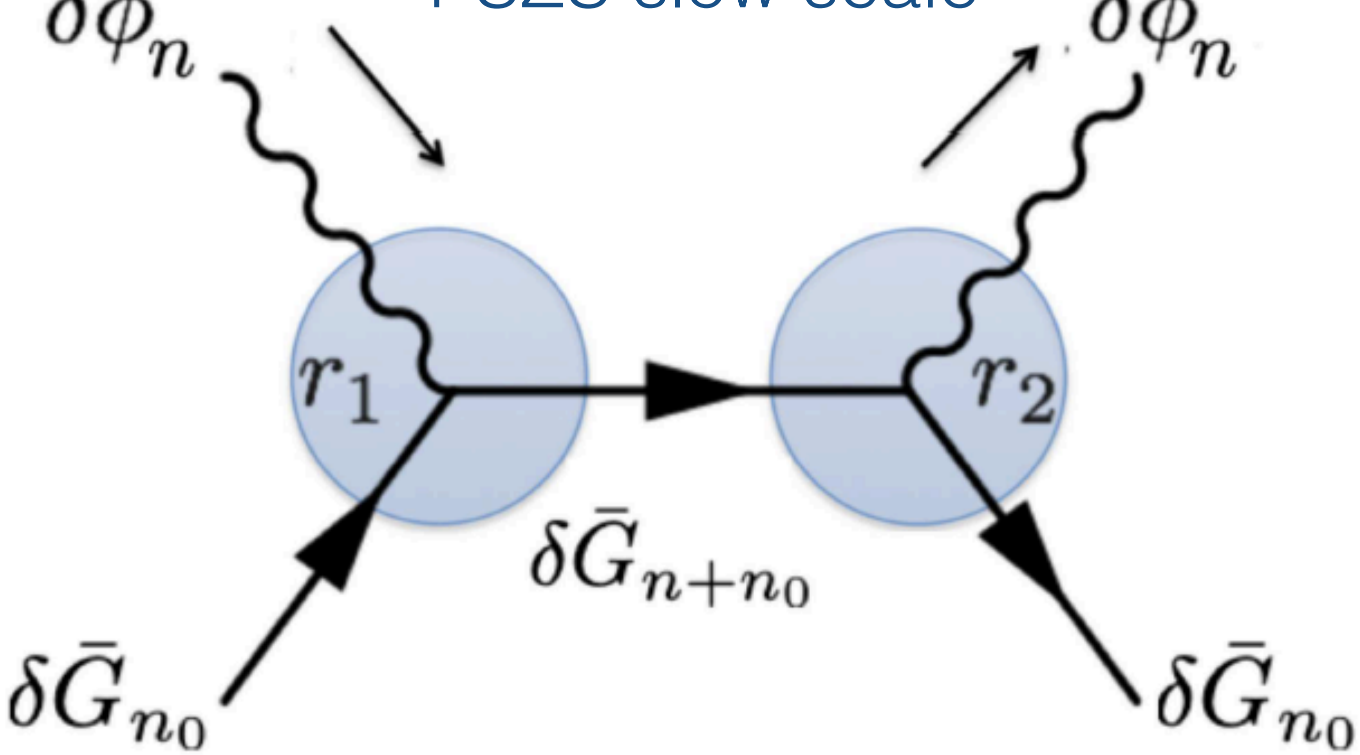
ZS detuning Broadening

$$\left[i\left(\bar{\omega}_{d0}(r) - \omega_{k_0} - i\partial_t\right) - i\frac{cn_0}{d\psi/dr}\overline{e^{iQ_{k_0}}\partial_r \left\langle \delta L_g \right\rangle_z e^{-iQ_{k_0}}} - \Delta - F\frac{\partial}{\partial r} - \frac{\partial}{\partial r}D\frac{\partial}{\partial r}\right]\delta\bar{G}_{k_0}$$

$$= i\overline{\left[e^{iQ_{k_0}}\left(\frac{e}{m}Q\bar{F}_0\left\langle \delta L_g \right\rangle_{k_0}\right)\right]} - i\frac{cn_0}{d\psi/dr}\overline{e^{iQ_{k_0}}\left\langle \delta L_g \right\rangle_{k_0} e^{-iQ_z}}\frac{\partial \delta\bar{G}_z}{\partial r} + [\text{NL OFF DIAG.}]$$

PSZS fast scale

PSZS slow scale



- $Q\bar{F}_0 = i\partial_{\mathcal{E}}\bar{F}_0\partial_t - i\Omega^{-1}\mathbf{b}\times\nabla\bar{F}_0\cdot\nabla$
- ∂_t accounts for the temporal meso-scale variation
- toroidal geometry effects through the radial profile of $\bar{\omega}_{d0}(r)$, precession frequency dependence on the constants of motion, and bounce averaging

Bounce averaged expressions left implicit: to be made explicit as shown above

Dyson Schrödinger Model III

- Physical interpretation [Dupree 66]:
  - Δ: nonlinear complex frequency shift
  - F: nonlinear anti-symmetric resonance distortion (in radius)
  - D: nonlinear resonance broadening (in radius)

$$\Delta = -\sum_{k\neq -k_0}\left(1+\frac{n}{n_0}\right)\left(\frac{n_0 c}{d\psi/dr}\right)^2 \overline{e^{iQ_{k_0}}\partial_r \left\langle \delta L_g\right\rangle_{-k} e^{-iQ_{k+k_0}}}\frac{i}{(\bar{\omega}_d-\omega)_{k+k_0}}\overline{e^{iQ_{k+k_0}}\partial_r \left\langle \delta L_g\right\rangle_{k} e^{-iQ_{k_0}}}$$

$$F = \sum_{k\neq -k_0} n n_0 \left(\frac{c}{d\psi/dr}\right)^2 \overline{e^{iQ_{k_0}}\partial_r \left\langle \delta L_g\right\rangle_{-k} e^{-iQ_{k+k_0}}}\frac{i}{(\bar{\omega}_d-\omega)_{k+k_0}}\overline{e^{iQ_{k+k_0}} \left\langle \delta L_g\right\rangle_{k} e^{-iQ_{k_0}}}$$
$$-\sum_{k\neq -k_0} n n_0 \left(\frac{c}{d\psi/dr}\right)^2 \overline{e^{iQ_{k_0}} \left\langle \delta L_g\right\rangle_{-k} e^{-iQ_{k+k_0}}}\frac{i}{(\bar{\omega}_d-\omega)_{k+k_0}}\overline{e^{iQ_{k+k_0}}\partial_r \left\langle \delta L_g\right\rangle_{k} e^{-iQ_{k_0}}}$$

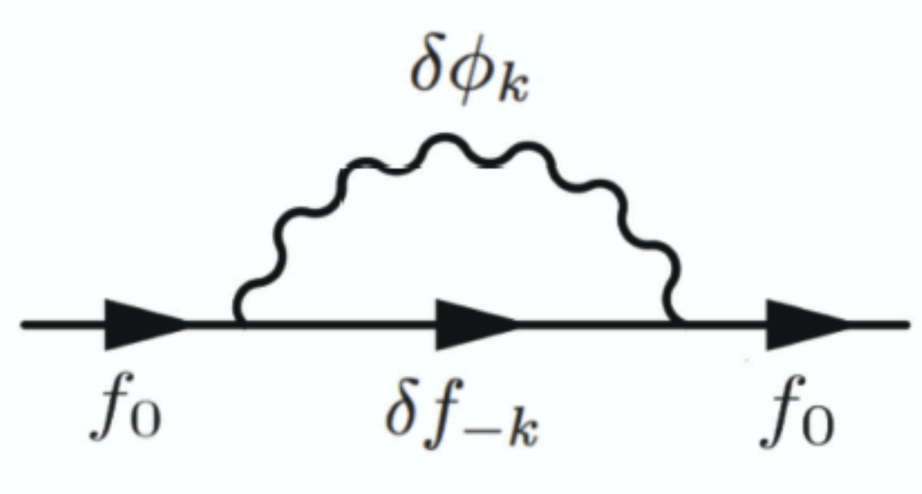
$$D = \sum_{k\neq -k_0} \left(\frac{nc}{d\psi/dr}\right)^2 \overline{e^{iQ_{k_0}} \left\langle \delta L_g\right\rangle_{-k} e^{-iQ_{k+k_0}}}\frac{i}{(\bar{\omega}_d-\omega)_{k+k_0}}\overline{e^{iQ_{k+k_0}} \left\langle \delta L_g\right\rangle_{k} e^{-iQ_{k_0}}}$$

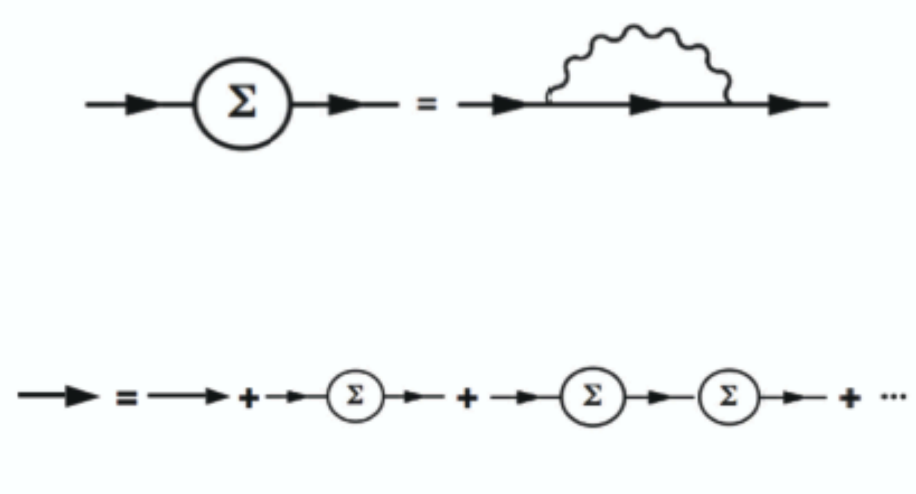
Evidence of avalanches

- Dropping for simplicity radial modulations by ZS, and writing formal solution for the particle response ($\delta \bar{G}_k$) keeping relevant NL terms

$$\partial_t \delta \bar{G}_z \sim -i \sum_{\boldsymbol{k}} \frac{nc}{d\psi/dr} \frac{\partial}{\partial r} \left[\overline{e^{iQ_z} \langle \delta L_g \rangle_{-k} e^{-iQ_k}} \frac{(nc)/(d\psi/dr)}{(\bar{\omega}_{dk} - \omega_k - i\partial_t + i\Delta \ldots)} \overline{e^{iQ_k} \langle \delta L_g \rangle_k e^{-iQ_z}} \frac{\partial}{\partial r} \delta \bar{G}_z \right]$$

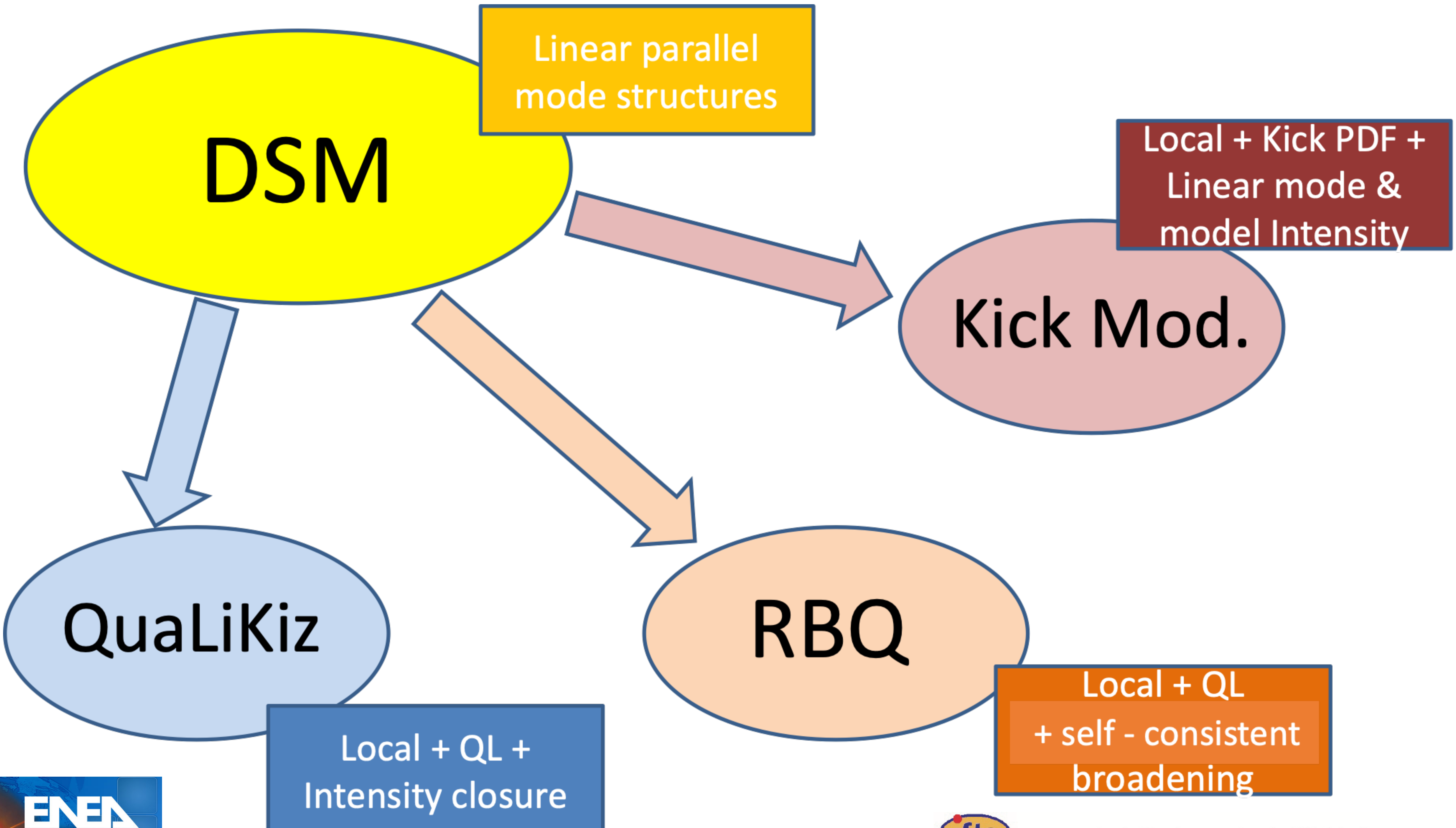
- Importance of fluctuation spectrum in determining NL PSZS evolution:
  - broad spectrum $\Rightarrow$ QL diffusion [Al'tshul' & Karpman 65]
  - narrow (quasi-coherent) spectrum: ballistic transport of phase-locked particles and convective amplification of radially propagating wave-packets (NLSE) [C&Z RMP16]





Dyson Schrödinger Model IV

Recovering QL limit: … for a broad spectrum



Motivation II

- Theoretical framework based on two pillars [C&Z RMP16]
    - The so-called General Fishbone-Like Dispersion Relation [GFLDR; Z&C POP 21 072120 and 072121] (Ruirui Ma's talk)
    - The derivation of the renormalized energetic particle response [Z&C NJP 17 013052)].
    - It is fully deployed and explained in the 2016 Rev. Mod. Phys. article
- Outline of this talk
    - Schematic derivation of the theoretical framework
    - Simple applications of two routes to NL physics:
        - nonlinear wave-wave interactions
        - nonlinear SAW-EP interactions
    - Extensions to broader scope of plasma physics:
        - Earth's chorus chirping rate

Nonlinear wave-wave interactions - I

- Parametric decay of SAW [Sagdeev&Galeev 69] (ideal MHD) and/or KAW [Hasegawa&Chen 76] into sideband SAW/KAW and ion sound wave.
- Branching ratio of scattering cross sections: [KIN/MHD]$\sim (\Omega_i/\omega_0)^2(k_\perp\rho_i)^4$.
- Mode converted KAW $\Rightarrow \boldsymbol{k}_{0\perp} \simeq k_{0r}\hat{\boldsymbol{r}}$:
  - Ideal MHD macro-scale theories [Sagdeev&Galeev 69]: $\boldsymbol{k}_{0\perp} \parallel \boldsymbol{k}_{-\perp}$ $\Rightarrow \boldsymbol{k}_{-\perp} \simeq k_{-r}\hat{\boldsymbol{r}} \Rightarrow$ no P_θ breaking $\Rightarrow$ little transport!
  - Gyrokinetic micro/meso-scale theory [C&Z, EPL 2011]: $\boldsymbol{k}_{0\perp} \perp \boldsymbol{k}_{-\perp}$ $\Rightarrow \boldsymbol{k}_{-\perp} \simeq k_{-\theta}\hat{\boldsymbol{\theta}} \Rightarrow$ large P_θ breaking $\Rightarrow$ significant transport!
- With SAW/KAW excitation by EPs $\Rightarrow$ impact on cross-scale couplings.
- For $|k_\perp\rho_i|^2 > |\omega_0|/|\Omega_i| \sim \mathcal{O}(10^{-2})$, kinetic effects are crucial for SAW turbulence dynamics and associated transports. [Simulations by Y. Lin *et al.* PRL 12]

Recent ISEP talk: … Liu Chen

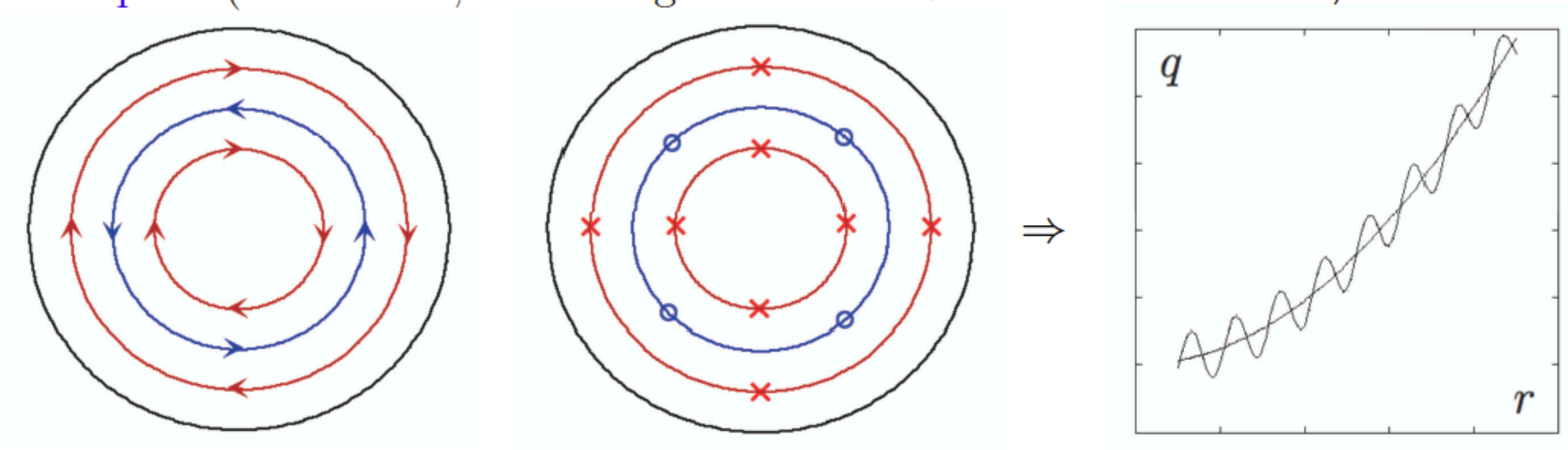
Nonlinear wave-wave interactions - II

- Zonal Structures by Toroidal Alfvén Eigenmodes

[C&Z 12; Qiu *et al.* 13]

  - Zonal structures ⇒ coherent micro/meso-scale radial corrugations of equilibrium in toroidal device plasmas.

Examples: ($n = m = 0$; finite magnetic shear ⇒ no convective cells)



Zonal Flow   Zonal Current

[More generally: phase-space zonal structures]   [Z&C NJP15]

Nonlinear wave-wave interactions - III

⇒ Zonal structures spontaneously excited by micro/meso-scale turbulence due to plasma instabilities. (includes both drift waves and SAWs)

⇒ Zonal structures scatter instability turbulence to shorter-radial wavelength stable domain ⇒ nonlinearly damp the instability.

⇒ Self-regulation of plasma instabilities!

- SAWs in toroidal plasmas ⇒ continuous and discrete spectra
  - Continuous spectrum and incompressible plasma displacement
    $\Rightarrow \omega^2 = k_\parallel^2(r)V_A^2(r) \Rightarrow \mathbf{Re} + \mathbf{Mx} \simeq 0$ Alfvénic State
    ⇒ negligible nonlinear contributions [Alfvén 42,50; Walén 44]
  - Discrete spectrum and/or finite plasma/EP compressibility
    ⇒ AEs ⇒ finite nonlinear contribution
    ⇒ Spontaneous excitation of zonal structures via modulational instability of the radial envelope of a finite-amplitude AE pump wave.

[Qiu et al PRL18, PST18, NF19a,b, PPCF20]

Nonlinear SAW-EP interactions - I

- EPM: convectively amplified solitons in active media

- Nonlinear dynamics of phase-space structures that evolve non-adiabatically ($\dot{\omega} \sim \omega_B^2$) on the same time scale of the corresponding fluctuations
  $\Rightarrow$ **Phase-space zonal structures** [Z&C NJP15; C&Z RMP16]

- NLSE for EPM envelope: look for solutions that can be expressed as the convectively amplified propagating (self-similar) wave-packet

$$A(\xi, t) = U(\xi) e^{\int^t \gamma(t')dt'} \equiv W(\xi) e^{i\varphi(\xi) + \int_0^t \gamma(t')dt'} \quad ,$$

$$\xi \equiv k_{n0}\,(r - r_0) \equiv k_{n0} \left(r - r_0(0) - \int_0^t v_g(t')dt' \right) \quad ,$$

with k_{n0} denoting the NL wave-vector and v_g the NL group velocity.

- Solution is obtained by balancing the nonlinear term with the linear dispersiveness.

Nonlinear SAW-EP interactions - II

- This optimal balance fixes $k_{n0} v_g$ but not the nonlinear group velocity $\Rightarrow$ One-parameter family ($\lambda_g \lesssim 1$) of EPM wave packets

$$\dot{r}_0^2 = v_g^2 = \lambda_g^2 v_{E\times B}^2 \quad ; \quad v_{E\times B} = (\boldsymbol{E}\times\boldsymbol{B})_r \ \text{ velocity}$$

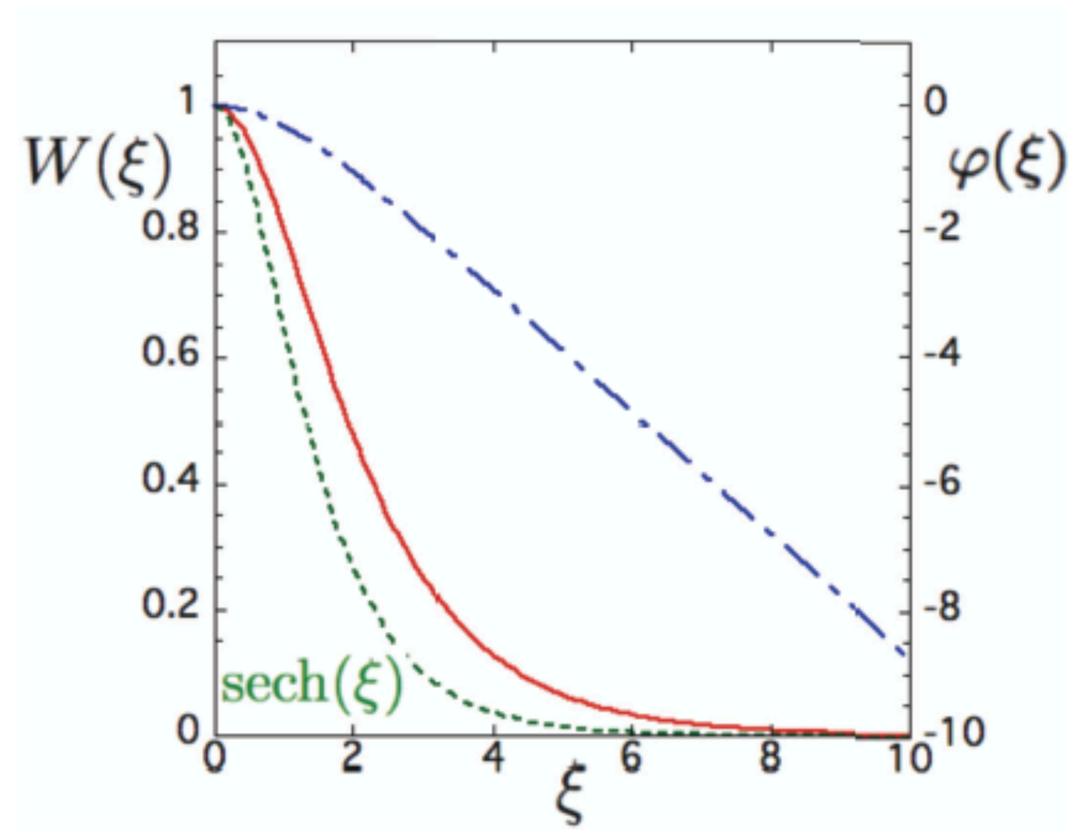
$$k_{n0}^2 = \kappa^2 k_\vartheta^2 / \lambda_g^2 \quad ; \quad \kappa = \mathcal{O}(1) \ \text{ const.}$$

$$\omega_{EPM} = \Omega(r_0, \lambda_g; \ldots) \ \Rightarrow \ \lambda_g \ : \ \partial \mathbb{I}\mathrm{m}\Omega / \partial \lambda_g = 0 \quad (\text{Max wave EP power exchange})$$

while $U(\xi) = W(\xi) e^{i\varphi(\xi)}$ satisfies the nonlinear Schrödinger equation

$$\boxed{\partial_\xi^2 U = \lambda_0 U - 2iU|U|^2} \ .$$

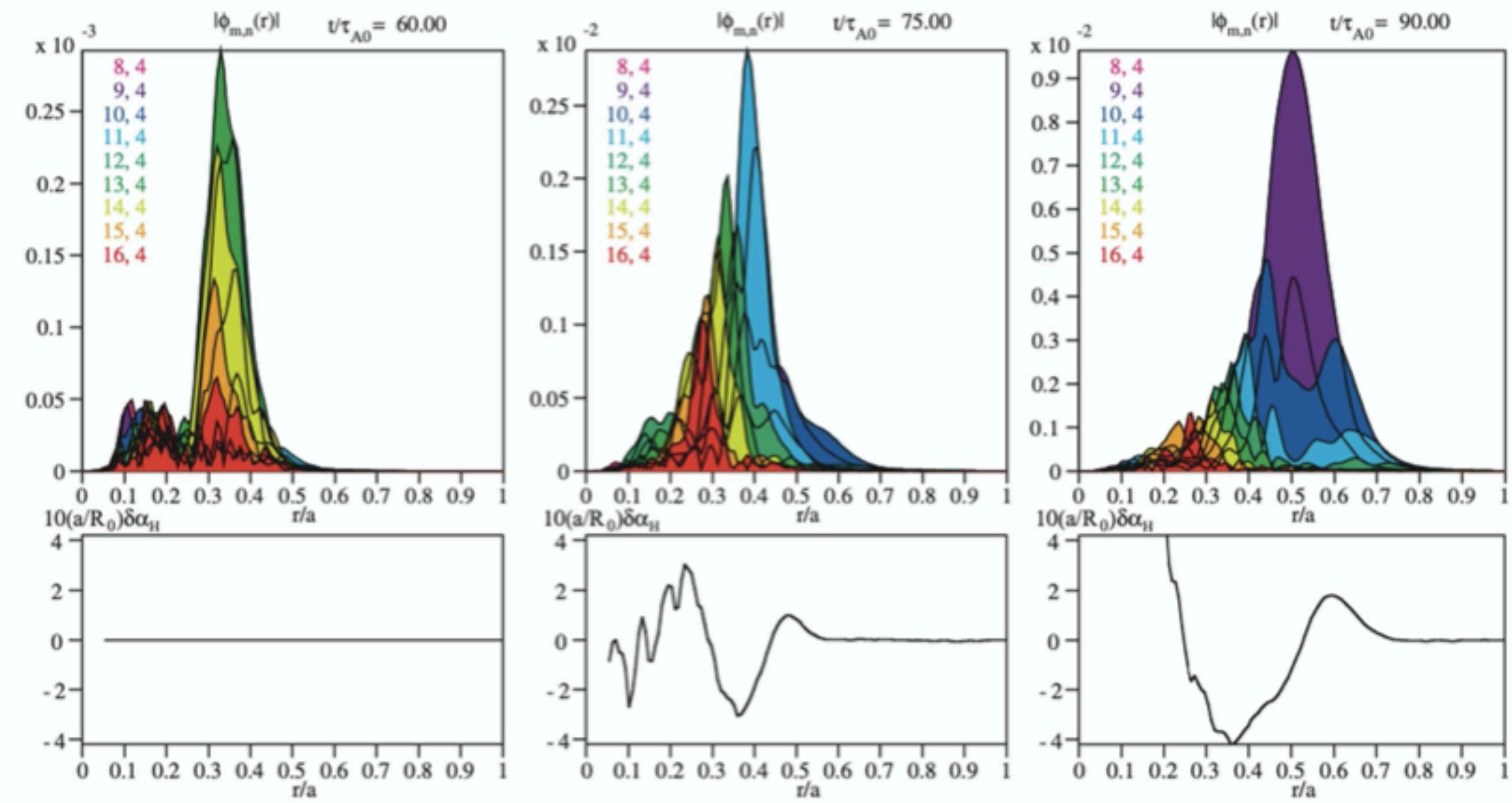
- The value $\lambda_0 \simeq -0.47 + i1.33$ corresponds to the ground state of the corresponding complex nonlinear oscillator.



Nonlinear SAW-EP interactions - III

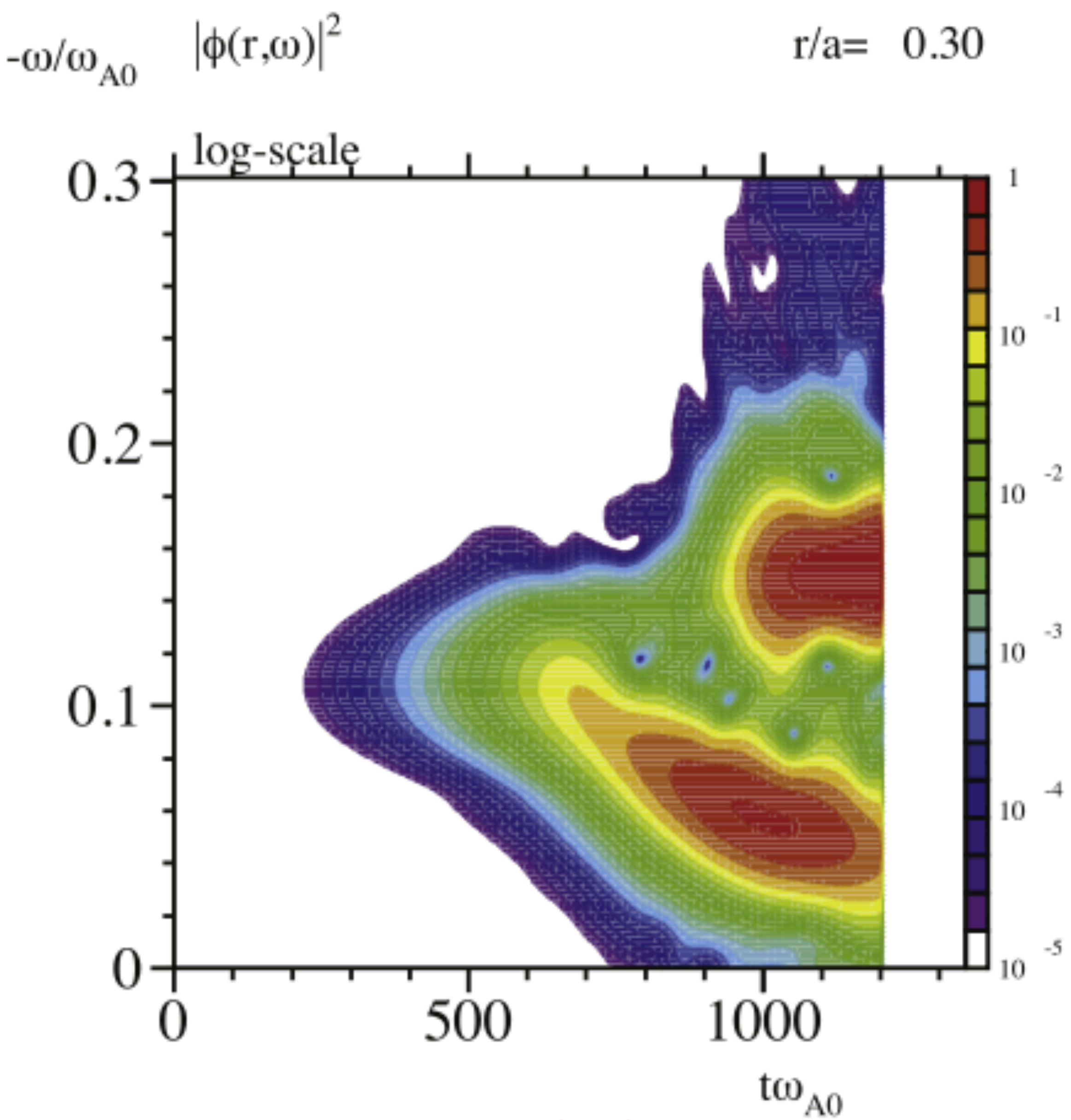
Theoretical framework systematically verified by NL Hybrid MHD GK simulations (HMGC): (Xin Wang AAPPS Invited; EPM chirping explained with hybrid simulations)

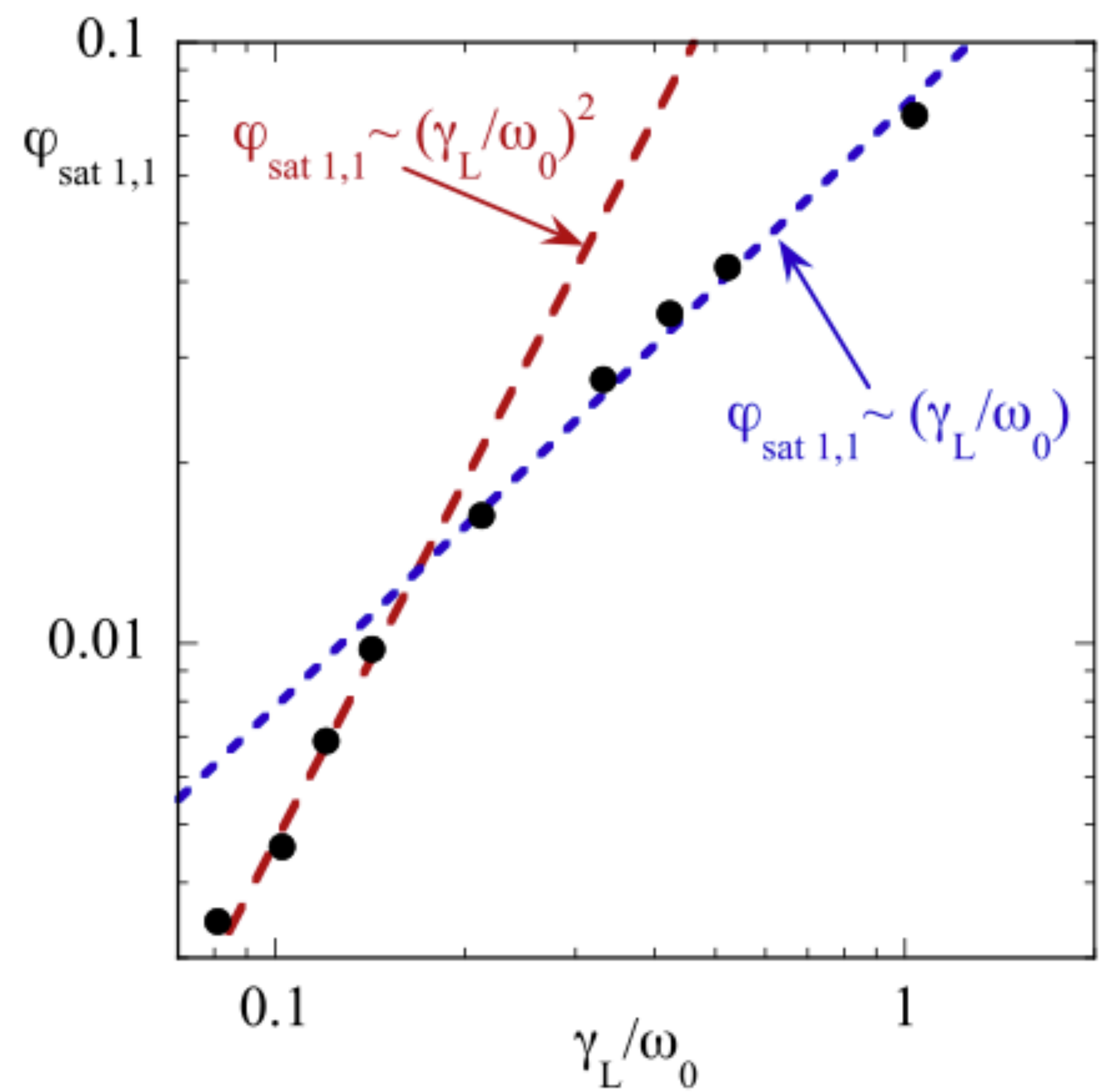
Evidence of EP avalanche [Z. et al Nucl. Fusion 45 (2005) 477 – 484]



Nonlinear SAW-EP interactions - IV

Theoretical framework systematically verified by NL Hybrid MHD GK simulations (HMGC&HYMAGYC): fishbones (G. Vlad et al. New J. Phys. 18 (2016) 105004)

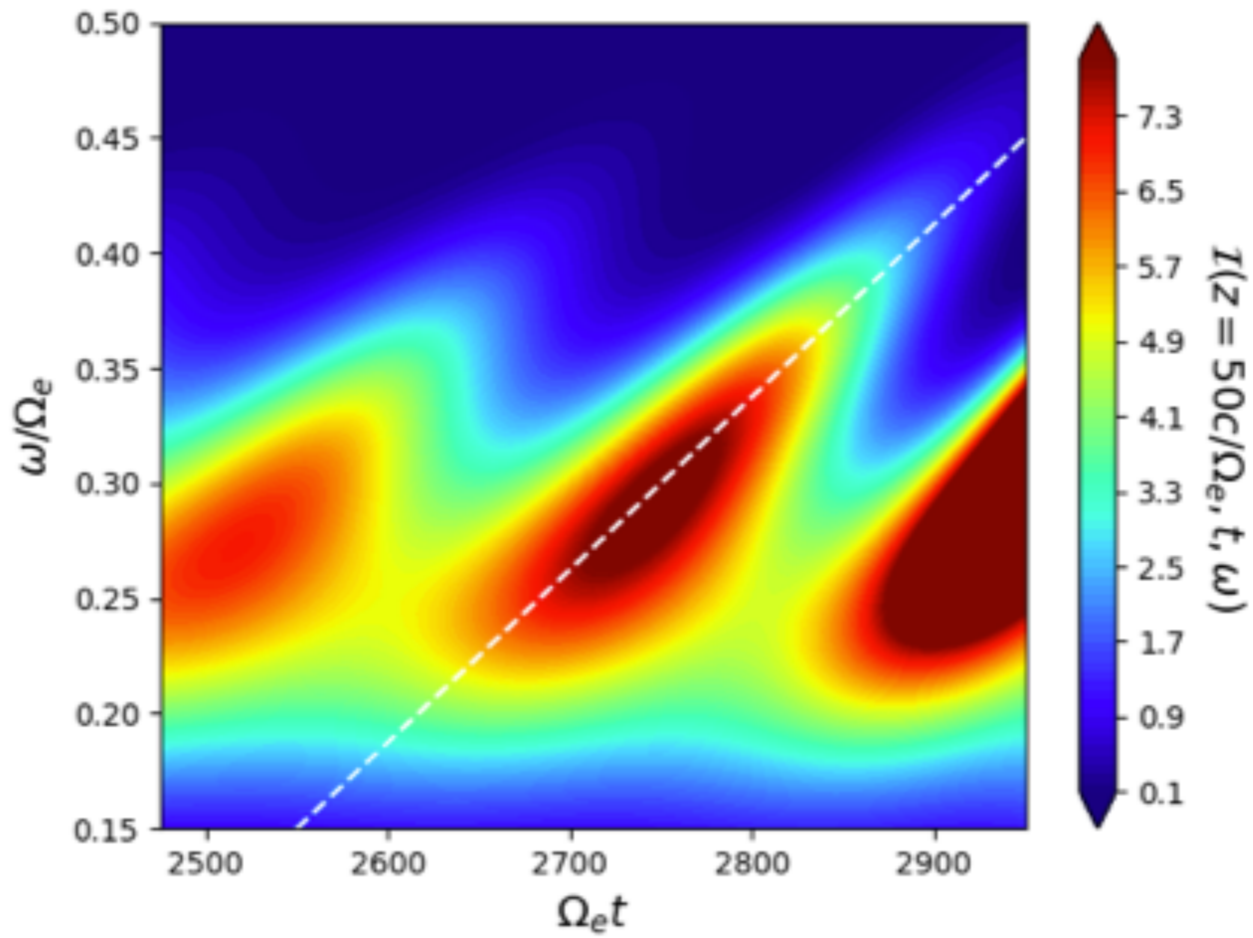




XHMGC e-fishbones; weakly hollow q profile

Extensions to broader scope of plasma physics

- Example from Space Physics: chorus emission & chirping
- Adopt the Dyson approach to construct the nonlinear growth rate and frequency shift & Solve WKE [FZ et al, sub.to RMPP/JGR]



(b)

$$\frac{\partial \omega}{\partial t} = \pm \frac{1}{2} \frac{\left\langle\left\langle \omega_{\mathrm{tr}k}^4 \right\rangle\right\rangle^{1/2}}{\left(1 - v_{r\omega}/v_{g\omega}\right)^2}$$

See also X. Tao, F. Zonca, L. Chen and Y. Wu, Theoretical and numerical studies of chorus waves: A review, Science China Earth Sciences 63, 78-92, (2020).

X. Tao, F. Zonca, L. Chen, A “Trap-Release-Amplify” model of Chorus Waves, JGR: Space Physics, 126, e2021JA029585

FZ, XT, LC, submitted to RMPP/JGR

Quasilinear limit

Dyson – like equation (DSE)

$$(\partial_t + v_{\parallel}\partial_z) f_0 = \frac{e}{m}\frac{v_\perp}{4}\mathbb{R}\mathrm{e}\sum_k i\left\{\delta\bar{E}_k\left[\bar{\partial}_{\mathcal{E}k} + \frac{1}{v_\perp^2}\right.\right.$$
$$\left.\left. \times\left(1-\frac{kv_{\parallel}}{\omega_k}\right)\right]\delta\bar{f}_k^* - \delta\bar{E}_k^*\left[\bar{\partial}_{\mathcal{E}k} + \frac{1}{v_\perp^2}\left(1-\frac{kv_{\parallel}}{\omega_k}\right)\right]\delta\bar{f}_k\right\}$$

Closed by …

$$\delta\bar{f}_k = \frac{e}{m}\mathcal{P}_k v_\perp \delta\bar{E}_k \bar{\partial}_{\mathcal{E}k} f_0 \; ,$$
$$\mathcal{P}_k \equiv \left[\Omega + kv_{\parallel} - \omega_k - i(\partial_t \cancel{+ v_{\parallel}\partial_z}) \cancel{-\Delta\omega_k}\right]^{-1}$$

resonance broadening

slow spatiotemporal dep.

1. Propagator becomes the linear one
2. The spectrum is considered broad (w.r.t. ω_{trap}) small Kubo #
3. Resonances overlap and $\omega_{trap} >> 1/\tau_{diff} \sim 1/\tau_{NL}$
4. Near marginal stability: γ even smaller than $1/\tau_{diff} \sim 1/\tau_{NL}$

Conclusions

- Nonlinear GK theory of Alfvénic fluctuations and EP transport in burning plasmas is mature
  - Based on wave-wave and SAW-EP interactions treated on the same footing: 2 routes of NL physics
- Overarching and unified theoretical framework with different levels of approximation for self-consistent description of EP transport in tokamaks has been successfully derived and demonstrated
  - complete NL GK/Hybrid
  - Dyson Schrödinger Model (DSM)
  - Quasilinear transport
- Numerical applications have been made for selected reference cases and verification of the approach is complete

Conclusions

- Applications to realistic cases of practical interest (CFETR, BEST, DTT, etc.) are in progress along with the extension to 3D stellarator equilibria [EUROfusion projects ATEP/TSVV#10 (Lauber/Mishchenko)]

Thank you for your attention!

謝謝

Your questions are welcome