

On the kinetic continuous spectrum in toroidal plasmas

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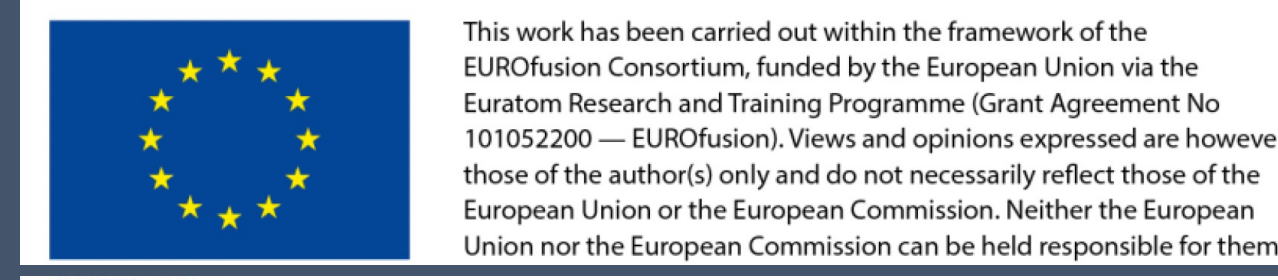
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ABSTRACT

- At low frequencies comparable to characteristic thermal-particle frequencies, kinetic effects modify the compressional response in high-temperature toroidal plasmas, with important consequences for the stability of drift Alfvén waves (DAWs) driven by both thermal free energy and resonant energetic particles (EP) [1].
- Using linear gyrokinetic theory, we analyze the shear Alfvén wave (SAW) and slow sound wave (SSW) continuous spectra and show how mode polarization and dispersion are altered by Landau damping and kinetic compressibility, including finite parallel electric field effects.
- Floquet solutions [2,3] can be constructed for general tokamak geometries, and the renormalized plasma response is expressed analytically in quadrature form. At short wavelengths the formulation recovers the kinetic Alfvén wave (KAW) dispersion and damping, providing a robust test case for gyrokinetic codes.
- Within the general fishbone-like dispersion relation (GFLDR) framework [4,5], the theory yields a compact description of resonant low-frequency DAW excitation by EP and clarifies why Alfvénic polarization is preferentially excited.

THEORETICAL MODEL

- Use straight magnetic field line flux coordinates (Boozer) (r, θ, ζ) for the description of a generic axisymmetric tokamak equilibrium with magnetic field $B_0 = \nabla(\zeta - q\theta) \times \nabla\psi$.
- Low-frequency kinetic continuous-spectrum problem is formulated by means of **quasineutrality condition** and **gyrokinetic vorticity equation** [1,2,3]. Field variables are $\delta\phi$ and $\delta A_{\parallel}$ but for convenience we work with

$$\delta\psi = -\frac{1}{c}\nabla_{\parallel}^{-1}\partial_t\delta A_{\parallel}, \quad \delta\phi_{\parallel} \equiv \delta\phi - \delta\psi \rightarrow \delta E_{\parallel} = -\nabla_{\parallel}\delta\phi_{\parallel}.$$

- We formulate the problem in ballooning space because it is particularly convenient to capture the local structure of the fluctuations and their intrinsic multiscale character, including parallel mode structure and wave-particle resonances.
- With the ϑ – ballooning space representation of $-\nabla_{\perp}^2/k_{\vartheta}^2$ and $k_{\vartheta} = -nq/r$

$$\delta\phi_{\parallel} \mapsto |\nabla r|\delta\bar{\Phi}_{\parallel 0}/\hat{\kappa}_{\perp}, \quad \delta\psi \mapsto |\nabla r|\delta\bar{\Psi}_0/\hat{\kappa}_{\perp}.$$

- In the low-frequency limit $|\omega|/\omega_A \ll 1$ fluctuation structures are characterized by two length scales, $\vartheta_0 \sim O(1)$ and $\vartheta_1 \sim O(\omega_A/|\omega|) \gg 1$. Asymptotic expansions in $|\omega|/\omega_A$

$$\delta\bar{\Phi}_{\parallel 0} = \delta\bar{\Phi}_{\parallel 0}^{(0)} + \delta\bar{\Phi}_{\parallel 0}^{(1)} + \dots, \quad \delta\bar{\Psi}_0 = \delta\bar{\Psi}_0^{(0)} + \delta\bar{\Psi}_0^{(1)} + \dots,$$

with $\delta\bar{\Psi}_0$ being a function of ϑ_1 only up to first order included.

- Use action angle coordinates: θ_c and ζ_c such that $\omega_b = \dot{\theta}_c$ and $\bar{\omega}_d = \dot{\zeta}_c$ are, respectively, the bounce/transit and the magnetic drift precession frequency; $\tilde{\epsilon}_c$ parameterizing the equilibrium particle motion as $\zeta = \zeta_c + \tilde{\epsilon}_c$ at constant actions $(\mathcal{E}, \mu, P_{\phi})$, $\mathcal{E} = v^2/2$, $\lambda = \mu\bar{B}_0/\mathcal{E}$ and $\bar{B}_0$ the magnetic field on axis.
- Solve for the magnetic-drift/banana-center gyrokinetic particle compression response $\delta\hat{K}_B = e^{i\hat{Q}_B}\delta\hat{K}$, with $\hat{Q}_B \cong RB_{\phi}/(d\psi/dr)(k_{\vartheta}v_{\parallel}/\Omega)s\vartheta_1$, $\omega_d \cong \dot{\vartheta}_0\partial_{\vartheta_0}\hat{Q}_B$

$$\delta\hat{K}_B = \left\{ \sum_{p \in \mathbb{Z}} \frac{e^{e^{i\hat{Q}_B}Q\bar{F}_0}(J_0|\nabla r|\delta\bar{\Phi}_{\parallel 0}/\hat{\kappa}_{\perp})e^{-i\Lambda \text{sgn}(\vartheta_1)\bar{\theta}_c - ip\bar{\theta}_c}}{n\bar{\omega}_d + p\omega_b + \Lambda\dot{\bar{\theta}}_c \text{sgn}(\vartheta_1) - \omega} + \sum_{p \in \mathbb{Z}} \frac{e^{e^{i\hat{Q}_B}Q\bar{F}_0}(\omega_d/\omega)(J_0|\nabla r|\delta\bar{\Psi}_0^{(0)}/\hat{\kappa}_{\perp})e^{-i\Lambda \text{sgn}(\vartheta_1)\bar{\theta}_c - ip\bar{\theta}_c}}{n\bar{\omega}_d + p\omega_b + \Lambda\dot{\bar{\theta}}_c \text{sgn}(\vartheta_1) - \omega} \right\} e^{i\Lambda \text{sgn}(\vartheta_1)\bar{\theta}_c + ip\bar{\theta}_c},$$

and, for a (canonical) Maxwellian, with $\omega_{*p} = \omega_{*n} + \omega_{*T}$

$$Q\bar{F}_0 = -\frac{m}{T_0}\omega \left[1 - \frac{\omega_{*n}}{\omega} - \frac{\omega_{*T}}{\omega} \left(\frac{m\mathcal{E}}{T_0} - \frac{3}{2} \right) \right] \bar{F}_0$$

- Quasineutrality condition** ($\mathfrak{J}$ is the Jacobian, $\tau_b = 2\pi/\omega_b$)

$$\sum \frac{n_0 e^2}{T_0} \frac{|\nabla r|}{\hat{\kappa}_{\perp}} \delta\bar{\Phi}_{\parallel 0} = \sum_{\ell \in \mathbb{Z}} e^{i\ell\vartheta_0} \sum \int \mathcal{E} d\mathcal{E} d\lambda \tau_b \sum_{\sigma} \left[e^{-i\ell\vartheta_0} \frac{eJ_0(\lambda_L)\delta\hat{K}}{\mathcal{J}\bar{B}_0} \right]. \quad (\overline{\dots}) = \tau_b^{-1} \oint \frac{d\theta}{\bar{\theta}} (\dots)$$

- Gyrokinetic vorticity equation** in the limit of vanishing FLR $\lambda_L^2 = k_{\vartheta}^2 \hat{\kappa}_{\perp}^2 \rho_L^2 \rightarrow 0$

$$\partial_{\vartheta_1}^2 \delta\bar{\Psi}_0^{(0)} + \Lambda^2 \delta\bar{\Psi}_0^{(0)} = 0, \quad \Lambda^2 = \lim_{\lambda_L \rightarrow 0} \left[\frac{\omega^2 \mathcal{J}^2 B_0^2}{v_A^2} \left(1 - \frac{\omega_{*pi}}{\omega} \right) \left(1 + \delta\bar{\Phi}_{\parallel 0}/\delta\bar{\Psi}_0^{(0)} \right) \right]_{\vartheta_0} + \sum \int \mathcal{E} d\mathcal{E} d\lambda \overline{(\dots)}_{\vartheta_0} = (2\pi)^{-1} \oint (\dots) d\vartheta_0$$

$$\times \sum_{\sigma} \tau_b \left[\frac{4\pi\omega \mathcal{J} B_0/\bar{B}_0}{k_{\vartheta} c} \kappa_g \text{sgn}(s\vartheta_1) \frac{m(\mu B_0 + v_{\parallel}^2)J_0(\lambda_L)\delta\hat{K}}{|\nabla r|\delta\bar{\Psi}_0^{(0)}} \right].$$

- In the limit $\lambda_L^2, \hat{Q}_B^2 \rightarrow 0$, $\hat{\kappa}_{\perp}^2 \rightarrow \infty$, the coefficients and kinetic-response kernels become 2π -periodic in the ballooning coordinate within each asymptotic region of large positive or negative ϑ_1 , with globl consistency given by the GFLDR, $i\Lambda = \delta\bar{W}_f + \delta\bar{W}_k$ [4,5].
- The corresponding kinetic-response kernels inherit the same periodic-lattice structure, in the sense of invariance under simultaneous translation of their ballooning-space arguments by 2π .

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KINETIC CONTINUOUS SPECTRUM AND FLOQUET SOLUTIONS

- Solutions can be decomposed into Bloch sectors satisfying the quasi-periodicity

$$\begin{pmatrix} \delta\Psi_0(\vartheta_0 + 2\pi) \\ \delta\Phi_{\parallel 0}(\vartheta_0 + 2\pi) \end{pmatrix} = e^{2\pi i\Lambda} \begin{pmatrix} \delta\Psi_0(\vartheta_0) \\ \delta\Phi_{\parallel 0}(\vartheta_0) \end{pmatrix}$$

- Here, $e^{2\pi i\Lambda}$ is the Floquet multiplier and the generalized inertia is the corresponding characteristic exponent [2,6,7].
- The Bloch–Floquet reduction remains applicable because the limiting problem retains its periodic-lattice structure, despite the nonlocality of the kinetic response represented by nonlocal integro-differential kernels.
- The solution is not generally determined by a finite-dimensional set of local initial data. Consequently, the kinetic Floquet problem cannot, in general, be reduced to a finite-dimensional monodromy matrix [8].

PREFERENTIAL EXCITATION OF ALFVÉNIC POLARIZATION

- The obtained expressions are generally valid and are applicable to long wavelength drift Alfvén waves (DAW) and show that one can solve for the low frequency fluctuation spectrum by means of quadratures involving equilibrium quantities only.
- These results generalize earlier ones [9,10,11,12] to arbitrary tokamak equilibria and particle distribution functions.
- They show that Alfvénic polarization is characterized by weaker damping and stronger excitation by EPs [13,14], since $\delta\bar{W}_f + \delta\bar{W}_k$ in the GFLDR provides an effective $k_{\parallel}R$.

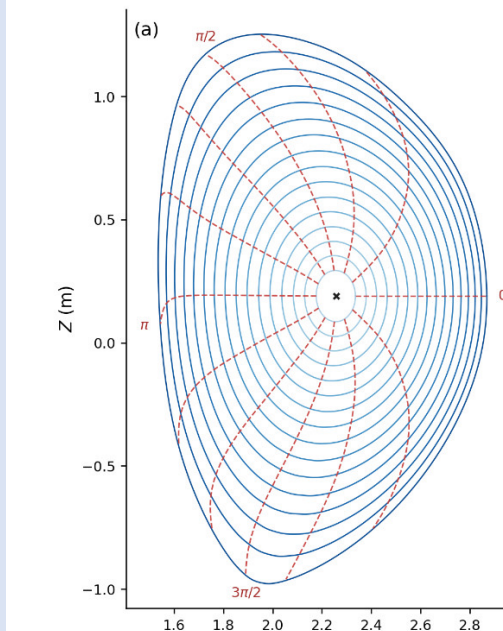


Fig. 1: DTT Reference scenario

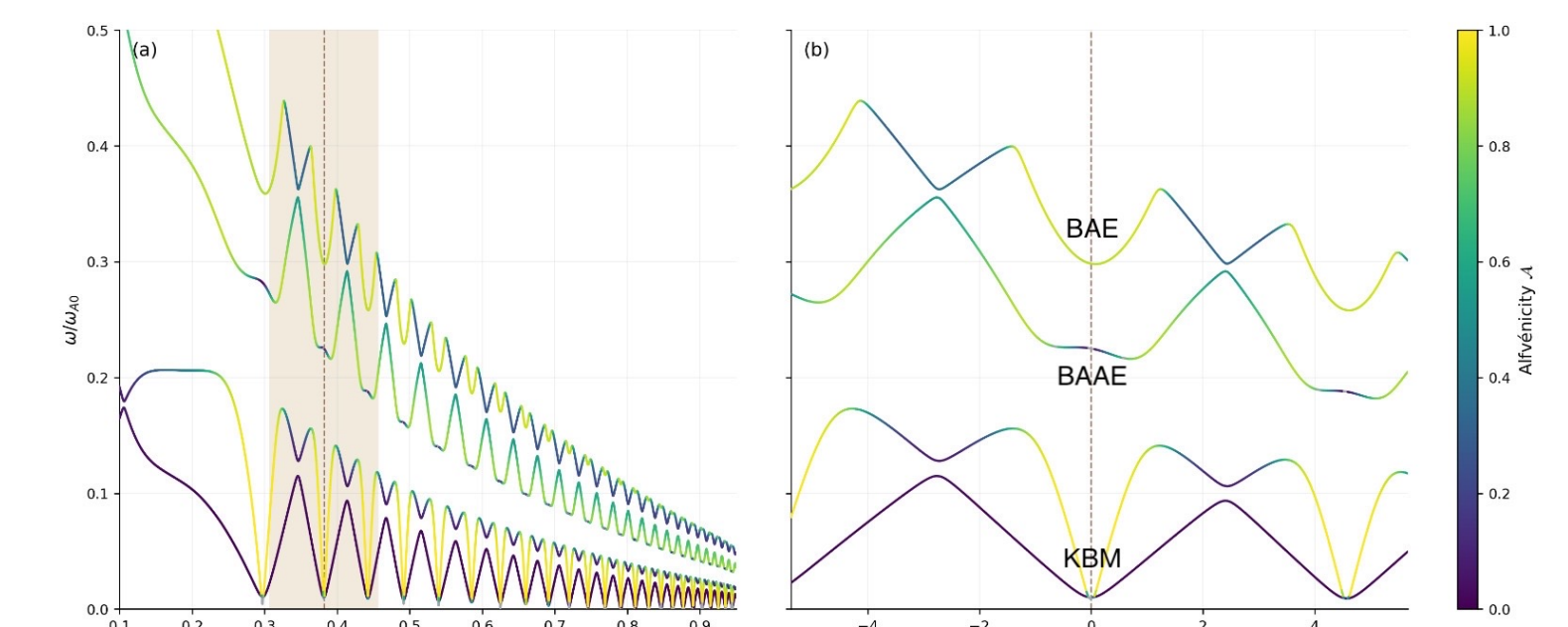


Fig. 2: SAW/SSW continuous spectra in DTT

KINETIC ALFVÉN WAVES IN TOKAMAK PLASMAS

- In order to explore the properties of KAW in tokamak plasmas, we need to take into account FLR and FOW effects in the vorticity equation that reads [4,5,15]

$$\partial_{\vartheta_1}^2 \delta\bar{\Psi}_0^{(0)} + \left\{ \frac{\omega^2 \mathcal{J}^2 B_0^2}{v_A^2} \left[\left(1 - \frac{\omega_{*ni}}{\omega} \right) \frac{1 - \Gamma_0(b)}{b} - \frac{\omega_{*Ti}}{\omega} (\Gamma_0(b) - \Gamma_1(b)) \right] (\delta\bar{\Psi}_0^{(0)} + \delta\bar{\Phi}_{\parallel 0}) \right\}_{\vartheta_0} - \sum \int \mathcal{E} d\mathcal{E} d\lambda \sum_{\sigma} \tau_b \left[\frac{4\pi\omega \mathcal{J} B_0^2/\bar{B}_0}{k_{\vartheta}^2 c^2} e^{-\frac{\omega_d}{\hat{\kappa}_{\perp}|\nabla r|} J_0(\lambda_L)\delta\hat{K}} \right] = 0.$$

- Here, $\Gamma_{0,1}(b) = e^{-b} I_{0,1}(b)$, $I_{0,1}(b)$ are modified Bessel functions, $b = k_{\vartheta}^2 \hat{\kappa}_{\perp}^2 \rho_L^2$.
- Letting $\tau = T_e/T_i$, the quasineutrality condition becomes

$$\left(1 + \frac{1}{\tau} \right) \frac{|\nabla r|}{\hat{\kappa}_{\perp}} \delta\bar{\Phi}_{\parallel 0} + \left[\left(1 - \frac{\omega_{*ni}}{\omega} \right) (1 - \Gamma_0(b)) - \frac{\omega_{*Ti}}{\omega} b (\Gamma_0(b) - \Gamma_1(b)) \right] \frac{|\nabla r|}{\hat{\kappa}_{\perp}} \delta\bar{\Psi}_0^{(0)} = \frac{T_{0i}}{n_0 e^2} \sum_{\ell \in \mathbb{Z}} e^{i\ell\vartheta_0} \sum \int \mathcal{E} d\mathcal{E} d\lambda \tau_b \sum_{\sigma} \left[e^{-i\ell\vartheta_0} \frac{eJ_0(\lambda_L)\delta\hat{K}}{\mathcal{J}\bar{B}_0} \right].$$

- These equation extend to toroidal geometry the well-known KAW equations in a slab [16,17,18,19], and provide a benchmark case for gyrokinetic codes in general tokamak geometry.

CONCLUSIONS AND DISCUSSIONS

- We developed a **linear gyrokinetic formulation** of the shear-Alfvén and slow-sound continuous spectra in general toroidal geometry.
- The formulation consistently includes **Landau damping, kinetic compressibility, finite parallel-electric-field effects, finite-Larmor-radius (FLR) physics, and finite-orbit-width (FOW) effects**.
- An extended **Bloch–Floquet framework** describes the non-self-adjoint, nonlocal kinetic problem and yields the renormalized plasma response in quadrature form.
- The kinetic continuous spectrum is governed by a **generalized inertia**, which can be evaluated from equilibrium quantities and particle-orbit integrals.
- Mode polarization controls collisionless damping**: the framework explains the preferential resonant excitation of **Alfvénic modes by energetic particles**, while acoustic and the mixed-polarization response is comparatively weaker.
- The short-wavelength limit recovers **kinetic-Alfvén-wave (KAW) propagation and damping**, providing a benchmark for gyrokinetic simulations.

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