



Toward a nonlinear Schrödinger equation for the description of geodesic-acoustic-modes in tokamaks

David Korger, Emanuele Poli, Fulvio Zonca,
Alberto Bottino, Matteo Valerio Falessi, Thomas Hayward-Schneider, Riccardo Stucchi

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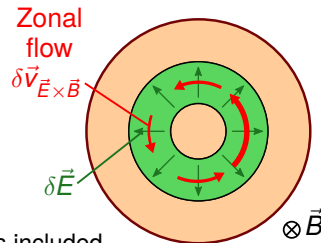
1 Motivation

2 Gyrokinetic derivation

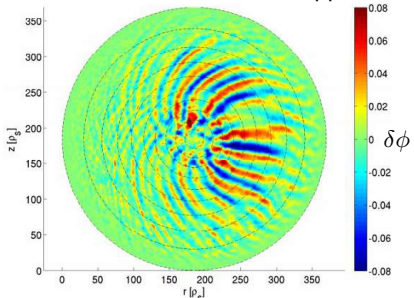
3 Outlook

Introduction: Turbulent Transport and Zonal Flows

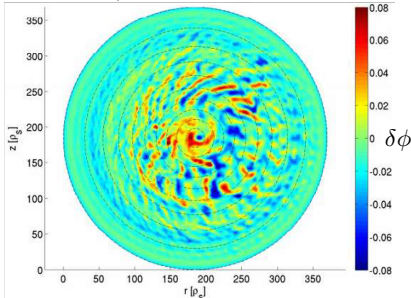
- Heat transport by turbulent structures to outer plasma layers
- Mitigation by zonal flows [e.g. Diamond 05]



GK simulation, zonal flows suppressed

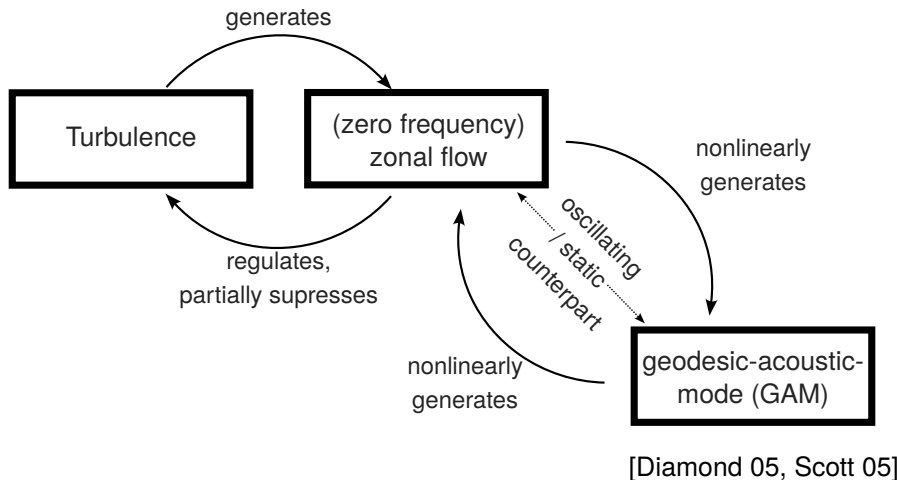


Simulation, zonal flows included

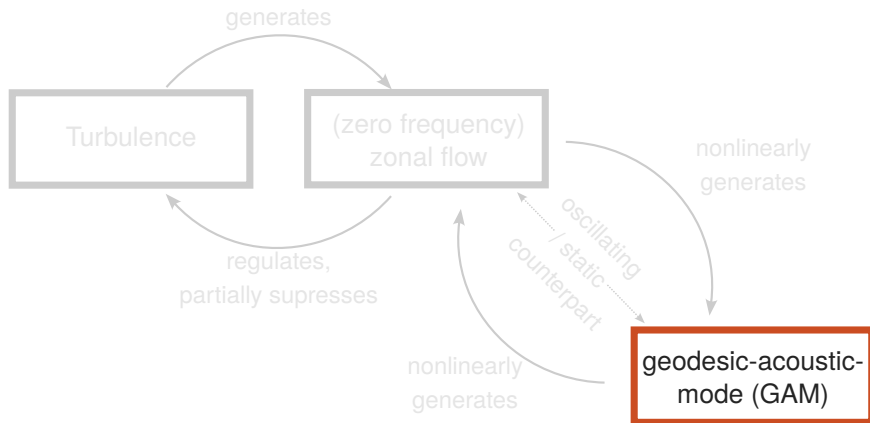


[courtesy
A. Bottino]

Introduction: Geodesic Acoustic Mode (GAM)



Introduction: Geodesic Acoustic Mode (GAM)



[Diamond 05, Scott 05]

Introduction: Geodesic Acoustic Mode (GAM)

- GAM = oscillatory counterpart to zonal flows
- Zonal $\vec{E} \times \vec{B}$ - flow velocity

$$\delta \vec{v}_{\vec{E} \times \vec{B}} = \frac{\delta \vec{E}_r \times \vec{B}}{B^2} \propto 1/|B|$$

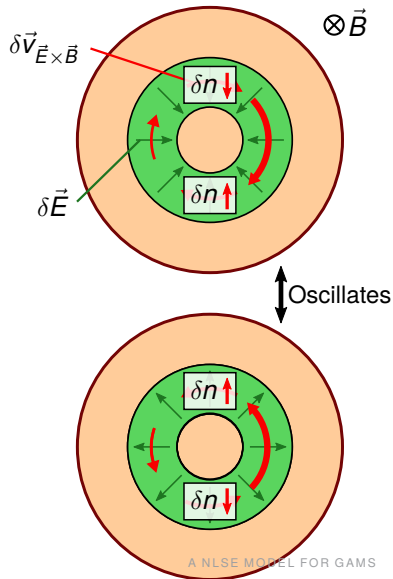
varies due to $B(\theta)$

⇒ GAM: **density perturbation** δn arises

$$\Rightarrow \delta n \propto \sin \theta$$

- $\delta \vec{E}_r$, $\delta \vec{v}_{\vec{E} \times \vec{B}}$, δn oscillate with

$$\omega_G \approx \mathcal{F} + \frac{G}{2} k_r^2$$



Nonlinear Schrödinger equation (NLSE) model

$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Potential term}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Momentum operator}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

$$\delta E_r(r,t) = \text{Re}[\psi](r,t)$$

Why can we use this model?

- Linear dispersion relation from GK theory

$$\delta E_r: \omega_G \approx \mathcal{F} + \frac{\mathcal{G}}{2}k_r^2$$

- Cubic nonlinear term $\propto |\psi|^2\psi$ is standard self-interaction extension [Spatschek 90]
- NLSE numerically verified [Poli 21, Korger 25]

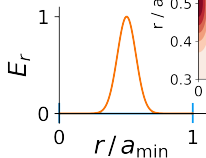
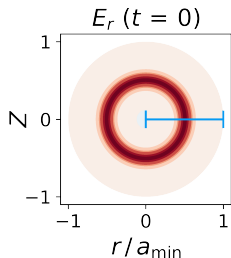
Why should we use this model?

- One of the most studied equations in physics
- Easier to understand
- Low computational cost

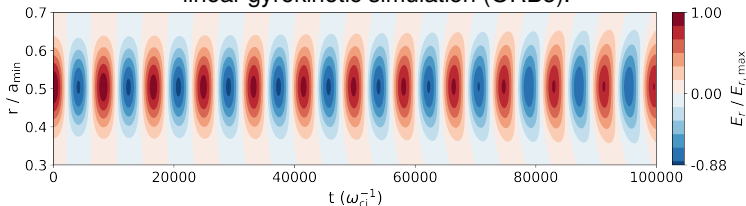
Performance of reduced (NLSE) model

Simple GK simulation:

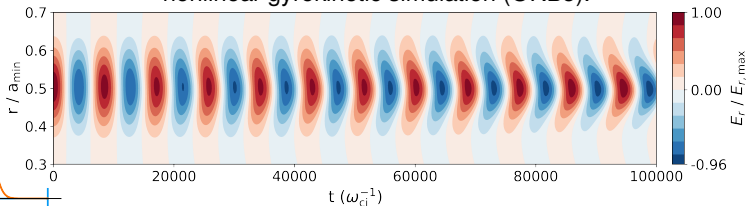
- Flat profiles (n_0 , T , q)
- circular concentric flux-surfaces
- Initialize $m = n = 0$ potential perturb. $\delta\phi$ (Rosenbl.-Hinton test)



linear gyrokinetic simulation (ORB5):



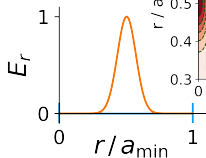
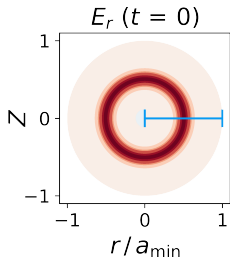
nonlinear gyrokinetic simulation (ORB5):



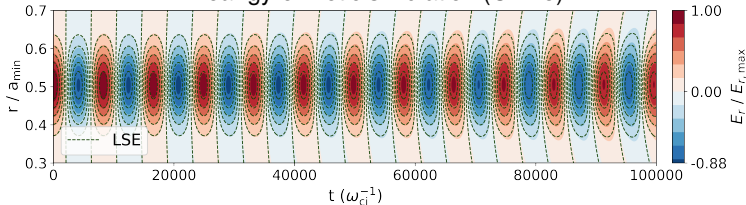
Performance of reduced (NLSE) model

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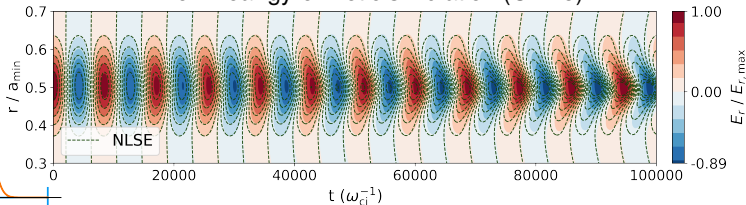
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- Initialize $m = n = 0$ potential perturb. $\delta\phi$ (Rosenbl.-Hinton test)



linear gyrokinetic simulation (ORB5):



nonlinear gyrokinetic simulation (ORB5):



NLSE: α_{NL} (so far) unknown $\rightarrow$ fit α_{NL}

Electrostatic ($\delta \vec{B} = 0$) gyrokinetic Vlasov-Maxwell system

Nonlinear GK equation

$$\begin{aligned} \left(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \right) \delta g_s & \stackrel{= \text{N.L.s}}{=} \\ & = \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi - \underbrace{\delta \vec{v}_{E \times B} \cdot \vec{\nabla}}_{\text{N.L.s}} \delta F_s. \end{aligned}$$

Quasineutrality equation

$$\begin{aligned} 0 &= \sum_s e_s \langle \delta F_s \rangle_v, \\ \hookrightarrow 0 &= \sum_s e_s \left[-\frac{e_s}{T_s} n_{0s} \delta \phi + \sum_s \langle J_0 \delta g_s \rangle_v \right]. \end{aligned}$$

1 δF -splitting

$$F_s = F_{0s} + \delta F_s,$$

$$\vec{B} = \vec{B}_0 + \delta \vec{B}, \quad (\text{electrostatic})$$

$$\vec{E} = -\vec{\nabla} \delta \phi = \delta \vec{E},$$

$$\dot{\vec{x}}_0 = \vec{v}_{\parallel} + \vec{v}_d,$$

$$\delta \dot{\vec{x}} \approx \delta \vec{v}_{E \times B},$$

$$\text{note: } \delta \dot{\mathcal{E}} \partial_{\mathcal{E}} \ll \delta \vec{v}_{E \times B} \cdot \vec{\nabla}.$$

2 Adiabatic splitting (Maxwellian F_{0s})

$$\delta F_s = -\frac{e_s}{T_s} F_{0s} \delta \phi + J_0 \delta g_s.$$

Three unknowns: δg_i , δg_e , $\delta \phi$.

Nonlin. GK equation (ions, electrons)

$$\underbrace{\left(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \right) \delta g_s}_{\mathcal{L}_{\text{GK}}\{\delta g_s\}} = \underbrace{N.L.s}_{= \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi - \delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s}.$$

Quasineutrality equation

$$0 = \sum_s e_s \left[-\frac{e_s}{T_s} n_{0s} \delta \phi + \langle J_0 \delta g_s \rangle_v \right].$$

Solving strategy:

1 Solve GK eq. for $\delta g_e\{\delta \phi\}$, $\delta g_i\{\delta \phi\}$

(a) Invert $\mathcal{L}_{\text{GK}}$

$$\delta g_s = \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi - \underbrace{N.L.s}_{\delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s} \right\}$$

2 Insert δg_e , δg_i in QN equation

Three unknowns: δg_i , δg_e , $\delta \phi$.

Nonlin. GK equation (ions, electrons)

$$\underbrace{\left(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \right) \delta g_s}_{\mathcal{L}_{\text{GK}}\{\delta g_s\}} = \underbrace{N.L.s}_{= N.L.s}$$

$$= \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi - \delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s.$$

Solving strategy:

1 Solve GK eq. for $\delta g_e\{\delta \phi\}$, $\delta g_i\{\delta \phi\}$

(a) Invert $\mathcal{L}_{\text{GK}}$

(b) Perturbation ansatz: $LIN_s \gg N.L.s$:

$$\delta g_s^L = \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi \right\}$$

$$\delta g_s^{NL} = \mathcal{L}_{\text{GK}}^{-1} \left\{ -\delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s^L \right\}$$

2 Insert δg_e , δg_i in QN equation

Quasineutrality equation

$$0 = \sum_s e_s \left[-\frac{e_s}{T_s} n_{0s} \delta \phi + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi \right\} \right\rangle_v - \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s \} \right\rangle_v \right]$$

Solving the GK Vlasov-Maxwell system [e.g. M. V. Falessi 2023, NJP]

Three unknowns: δg_i , δg_e , $\delta \phi$.

Solving strategy:

Nonlin. GK equation

Nonlinear Schrödinger equation

$$\underbrace{\left(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \nabla \right)}_{\mathcal{L}_{\text{GK}}\{\delta g_s\}} \delta E_r = \mathcal{F} \delta E_r - \frac{G}{2} \partial_r^2 \delta E_r - \alpha_{\text{NL}} |\delta E_r|^2 \delta E_r.$$

$$\begin{aligned} &= \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi - \underbrace{\delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s}_{\text{N.L.s.}} \\ &\quad \updownarrow \end{aligned}$$

1 Solve GK eq. for $\delta g_e\{\delta \phi\}, \delta g_i\{\delta \phi\}$

Satz: $LIN_s \gg N.L.s.$

$$\delta g_s = \mathcal{L}_{\text{GK}} \left\{ \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi \right\}$$

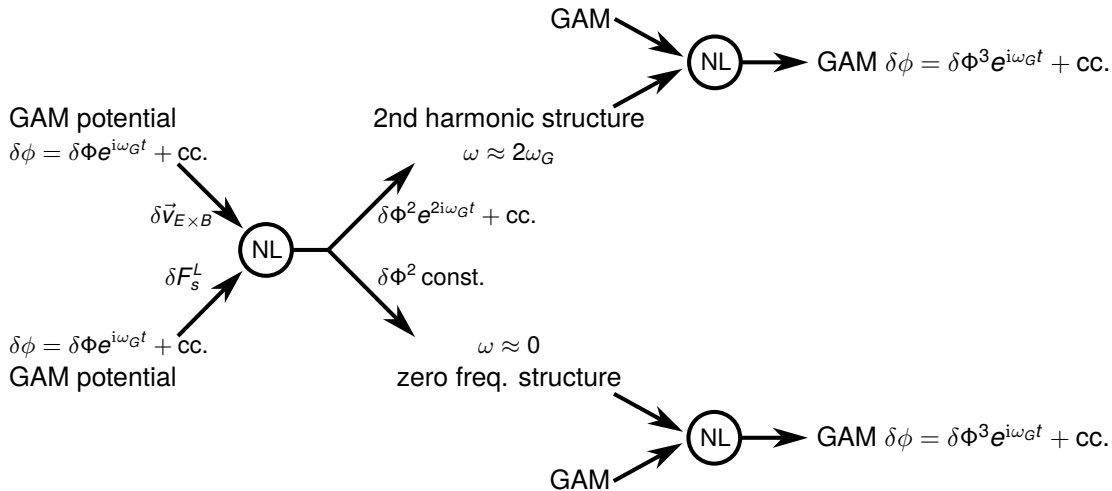
$$\delta g_s^{\text{NL}} = \mathcal{L}_{\text{GK}}^{-1} \left\{ -\delta \vec{v}_{E \times B} \cdot \vec{\nabla} \delta F_s^L \right\}$$

2 Insert $\delta g_e, \delta g_i$ in QN equation

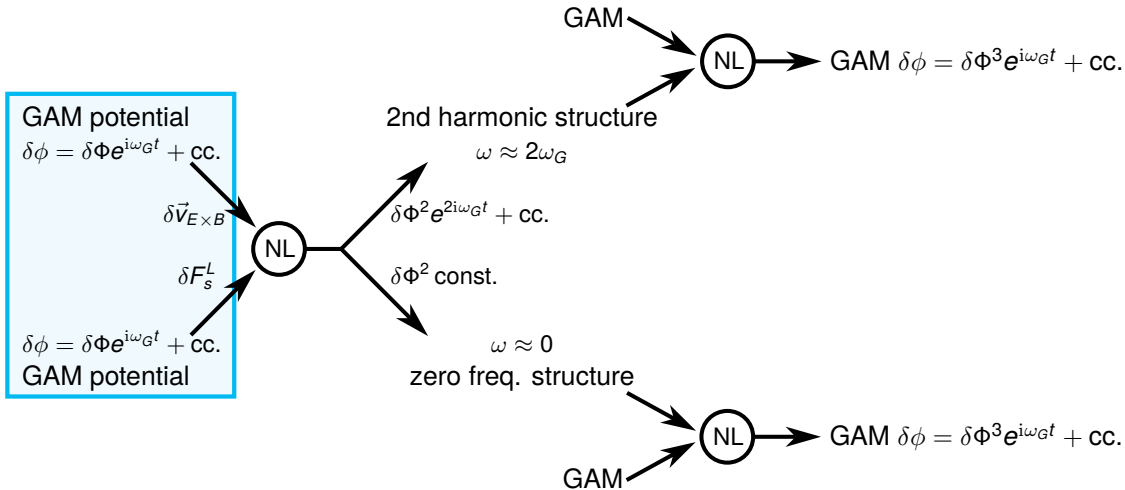
Quasineutrality equation

$$0 = \sum_s e_s \left[-\frac{e_s}{T_s} n_{0s} \delta \phi + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_{0s} J_0 \partial_t \delta \phi \right\} \right\rangle_v - \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \{ \text{N.L.s.} \} \right\rangle_v \right]$$

Cubic nonlinearity



Cubic nonlinearity



Linear GK equation (ions)

$$\begin{aligned} \overbrace{\left(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \right)}^{\mathcal{L}_{\text{GK}}} \delta g_i^L &= \frac{e}{T_i} F_0 J_0 \partial_t \delta \phi, \\ \hookrightarrow \left(1 - \frac{v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla}}{i\omega_G} \right) \delta g_i^{L, \omega_G} \\ &= \frac{e}{T_i} F_0 J_0 \delta \phi^{\omega_G} \end{aligned}$$

Fourier ansatz (in time):

$$\delta g_i^L \mapsto \delta g_i^{L, \omega_G} = \delta \hat{g}_i^{L, \omega_G} e^{-i\omega_G t}$$

Invert $\mathcal{L}_{\text{GK}}$:

$$\mathcal{L}_{\text{GK}}^{-1} = \frac{1}{-i\omega_G} \left(1 + \frac{v_{\parallel} \nabla_{\parallel} + v_d \cdot \nabla}{i\omega_G} + \dots \right)$$

Perturbation ansatz:

$$\omega_G \gg (\vec{v}_{\parallel} + \vec{v}_d) \cdot \vec{\nabla} \quad (q \gg 1, \rho_{Li} k_r \ll 1)$$

$$\delta g_i^{L, \omega_G} = \delta g_i^{L(0), \omega_G} + \delta g_s^{L(1), \omega_G} + \dots,$$

$$\delta \phi^{\omega_G} = \underbrace{[\delta \phi^{\omega_G}]_{\psi}}_{= \delta \phi^{(0)}} + \delta \phi^{(1), \omega_G} + \dots,$$

Solution:

$$\delta g_i^{L, \omega_G} = \overbrace{\frac{1}{-i\omega_G} \left(\cancel{1} + \frac{v_{\parallel} \nabla_{\parallel} + v_d \cdot \nabla}{i\omega_G} + \dots \right)}^{= \mathcal{L}_{\text{GK}}^{-1}} \frac{e}{T_i} F_0 J_0(-i\omega_G) \delta \phi^{\omega_G}$$

$$\boxed{\delta F_i^{L, \omega_G}} = -\frac{e}{T_i} F_0 \cancel{\delta \phi^{\omega_G}} + \delta g_i^{L, \omega_G} \approx \frac{e}{T_i} F_0 \frac{v_{\parallel} \nabla_{\parallel} + v_d \cdot \nabla}{i\omega_G} [\delta \phi^{\omega_G}]_{\psi} = \frac{e}{T_i} F_0 \overbrace{\frac{\hat{v}_{dr} \sin \theta \partial_r}{i\omega_G}}^{= v_{dr}} [\delta \phi^{\omega_G}]_{\psi}$$

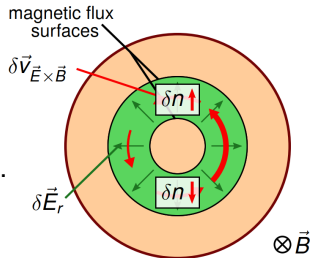
Approximations:

(a) Simple geometry: Circular concentric ψ , $\varepsilon \ll 1$

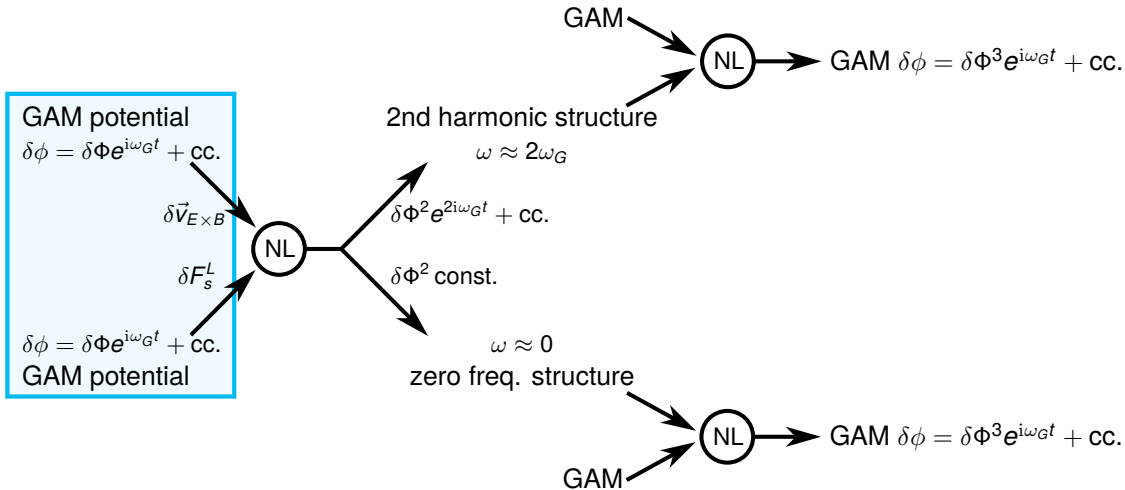
$$\Rightarrow [\dots]_{\psi} \approx \frac{1}{2\pi} \oint d\theta (\dots)$$

(b) No mirror force: $v_{\parallel}(\theta) = \sqrt{v^2 - \frac{\mu}{2B(\theta)}} \approx \sqrt{v^2 - \frac{\mu}{2\bar{B}_0}} = \bar{v}_{\parallel}$.

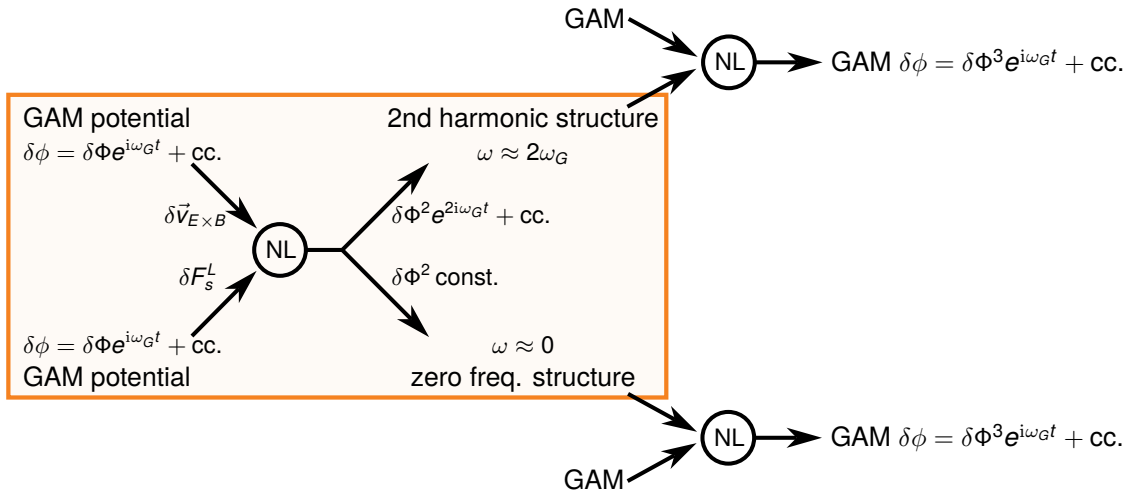
(see [H. Sugama 2006, Journ. Pl. Phys.])



Cubic nonlinearity



Cubic nonlinearity



Quadratic NL source (GAM $\times$ GAM - coupling)

$$\delta \vec{V}_{E \times B}^{\omega_G} \approx \frac{-\vec{\nabla}[\delta \phi^{\omega_G}]_\psi \times \vec{B}_0}{\bar{B}_0^2} \approx \frac{1}{\bar{B}_0} [E_r^{\omega_G}]_\psi \hat{e}_\theta \quad \delta F_i^{L, \omega_G} \approx -\frac{e}{T_i} F_0 \frac{\hat{v}_{dr} \sin \theta}{i \omega_G} [E_r^{\omega_G}]_\psi$$

$$\begin{aligned} N.L.i &\approx \delta \vec{V}_{E \times B} \cdot \vec{\nabla} \delta F_i^L = (\delta \vec{V}_{E \times B}^{\omega_G} + \text{cc.}) \cdot \vec{\nabla} (\delta F_i^{L, \omega_G} + \text{cc.}) \\ &= -\frac{1}{r \bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \cos \theta \left(\frac{1}{i \omega_G} [E_r^{\omega_G}]_\psi^2 + \frac{1}{(-i \omega_G^*)} [E_r^{\omega_G, *}]_\psi^2 \right. \\ &\quad \left. + \frac{1}{i \omega_G} [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi + \frac{1}{(-i \omega_G^*)} [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi \right) \end{aligned}$$

NL GK equation (ions)

$$\begin{aligned} & \overbrace{\left(-i2\omega_G + v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \right)}^{\mathcal{L}_{\text{GK}}} \delta g_i^{NL, 2\omega_G} = -N.L._i^{2\omega_G}, \\ \Rightarrow \delta g_i^{NL, 2\omega_G} &= \frac{1}{-2i\omega_G} \left(-N.L._i^{2\omega_G} + \dots \right) \\ &= \frac{1}{2i\omega_G} \left(-\frac{1}{r\bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \cos \theta \frac{1}{i\omega_G} [E_r^{\omega_G}]_{\psi}^2 \right). \end{aligned}$$

[H.S. Zhang 2009, Nucl. Fus.]

Invert $\mathcal{L}_{\text{GK}}$:

$$2\omega_G \gg v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla} \quad (! \text{ like GAM})$$

$$\hookrightarrow \mathcal{L}_{\text{GK}}^{-1} = \frac{1}{-2i\omega_G} \left(1 + \frac{v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla}}{2i\omega_G} + \dots \right)$$

We need $\delta F_i^{2\omega_G}$, $\delta \phi^{2\omega_G}$!

Insert into QN (at $\omega = 2\omega_G$)

$\hookrightarrow \delta \phi^{2\omega_G}$ arises such that QN is fulfilled

1 $[\delta \phi^{2\omega_G}]_{\psi}$: Not excited! (see [Zhang 09])

2 $m \neq 0$: $\delta \phi^{2\omega_G} \approx \frac{T_e}{en_0} \langle \delta g_s^{NL, 2\omega_G} \rangle_v \propto \cos \theta$

Comparison to simulations (compare [Zhang 2009])

Linear ($\omega = \omega_G$)

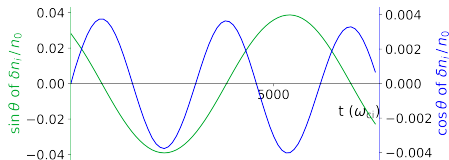
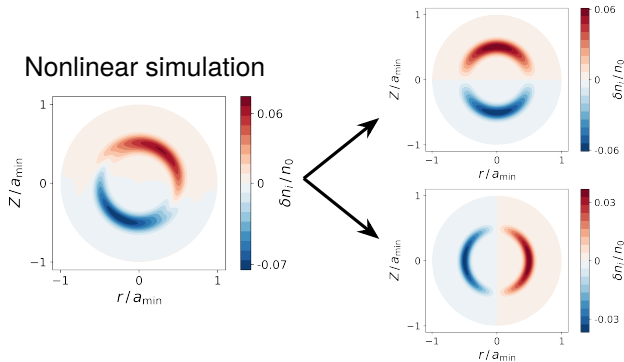
$$\langle \delta F_i^L \rangle_V \propto \sin \theta$$

Nonlinear ($\omega = 2\omega_G$)

$$\langle \delta F_i^{2\omega_G} \rangle_V \propto \cos \theta$$

Gyrokinetic simulation (ORB5)

Nonlinear simulation



Comparison to simulations (compare [Zhang 2009])

Linear ($\omega = \omega_G$)

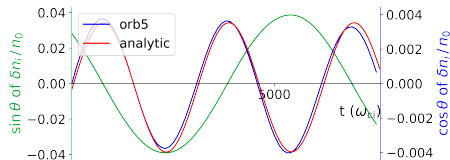
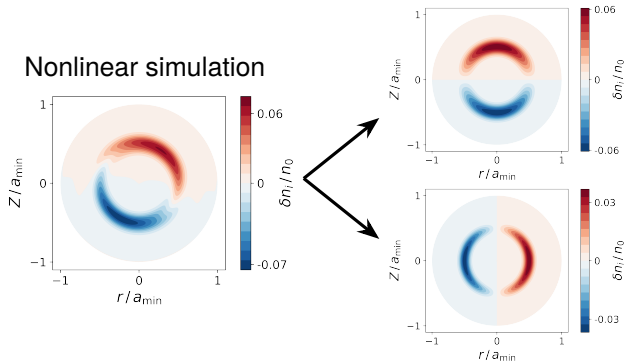
$$\langle \delta F_i^L \rangle_V \propto \sin \theta$$

Nonlinear ($\omega = 2\omega_G$)

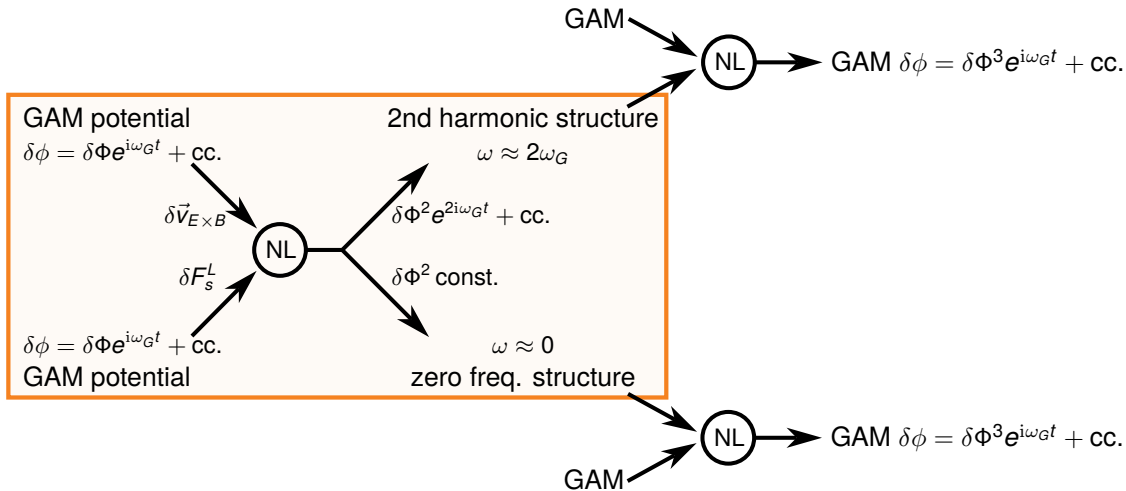
$$\langle \delta F_i^{2\omega_G} \rangle_V \propto \cos \theta$$

Gyrokinetic simulation (ORB5)

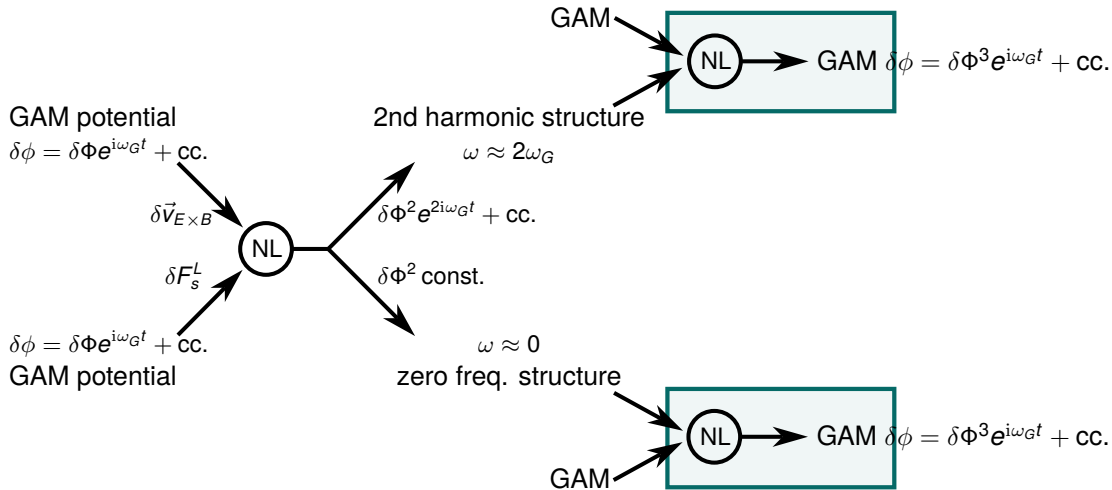
Nonlinear simulation



Cubic nonlinearity



Cubic nonlinearity



Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left[-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \langle J_0 \delta g_s^{\omega_G} \rangle_v \right]$$

$$\underbrace{\mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\}}_{\text{N.L.}_s^{\omega_G}} - \mathcal{L}_{\text{GK}}^{-1} \{ \text{N.L.}_i^{\omega_G} \}$$

To find $\text{N.L.}_i^{\omega_G} \rightarrow [\delta\phi]_\psi$ -coupling:

Look at flux-surf.-av. QN equation!



The relevant coupling is $\text{N.L.}_i^{\omega_G} \rightarrow [\delta\phi^{\omega_G}]_\psi$

Why $[\delta\phi^{\omega_G}]_\psi$?

Perturbation ansatz: $\delta\phi^{\omega_G} = [\delta\phi]_\psi + \delta\phi^{(1)} + \delta\phi^{(2)} + \dots$,

All quantities based on strongest contribution

E.g.: $\delta\phi^{(1)} = \frac{i\tau_e \hat{V}_{dr,th,i}}{\omega_G} \sin\theta \partial_r [\delta\phi]_\psi$

$$\delta\phi^{(2)} = \left(\frac{7}{4} + \tau_e \right) \frac{i\tau_e \hat{V}_{dr,th,i}^2}{2\omega_G^2} \cos 2\theta \partial_r^2 [\delta\phi]_\psi$$

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta \phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta \phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta \phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta \phi^{\omega_G} \right\} \right\rangle_v \right]_\psi - \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

Leading order: $J_0 \mathcal{L}_{\text{GK}}^{-1} \approx \frac{1}{-i\omega_G}$:

$$\begin{aligned} [\langle N.L.s^{\omega_G} \rangle_v]_\psi &= \lim_{\Delta\psi \rightarrow 0} \frac{1}{V_\psi} \int_{V_\psi} d^3x \int_W d^2v \mathcal{J}_x \mathcal{J}_v \delta \dot{\mathbf{Z}} \cdot \vec{\nabla}_z \delta F_s \\ &= \lim_{\Delta\psi \rightarrow 0} \frac{1}{V_\psi} \int_{V_\psi} d^3x \int_W d^2v \vec{\nabla}_z \cdot (\mathcal{J}_x \mathcal{J}_v \delta \dot{\mathbf{Z}} \delta F_s) \\ &= \lim_{\Delta\psi \rightarrow 0} \frac{1}{V_\psi} \int_{\substack{\partial(V_\psi, W) \\ \rightarrow \partial V_\psi}} d^4 \vec{\mathbf{S}} \cdot \mathcal{J}_x \mathcal{J}_v \delta \dot{\mathbf{Z}} \delta F_s \approx \left[\langle \delta \dot{\psi} \partial_\psi \delta F_s \rangle_v \right]_\psi, \quad [\text{Falessi N. J. Phys. 2023}]. \end{aligned}$$

Hamiltonian system
 $\vec{\nabla}_z \cdot (\delta \dot{\mathbf{Z}} \mathcal{J}_x \mathcal{J}_v) = 0$

Gauss theorem

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi - \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

Leading order: $J_0 \mathcal{L}_{\text{GK}}^{-1} \approx \frac{1}{-i\omega_G}$:

$$[\langle N.L.s^{\omega_G} \rangle_v]_\psi = \left[\left\langle \delta\psi \partial_\psi \delta F_s \right\rangle_v \right]_\psi \rightsquigarrow \left[\left\langle \delta \dot{r} \partial_r \delta F_s \right\rangle_v \right]_\psi = \left[\underbrace{\left\langle \frac{1}{r} \frac{\vec{B} \times \partial_\theta \delta\phi}{B^2} \cdot \hat{e}_r \partial_r \delta F_s \right\rangle_v}_{\substack{\rightarrow \text{varies} \\ \text{in } \theta}} \right]_\psi$$

must also vary in θ

$$\text{cubic NL: } \frac{1}{2\pi} \oint d\theta \left\langle \frac{1}{rB} \left(\partial_\theta \delta\phi^{2\omega_G} \partial_r \delta F_s^{-\omega_G} + \partial_\theta \delta\phi^{-\omega_G} \partial_r \delta F_s^{2\omega_G} \right) \right\rangle_v = 0. \quad \text{✗}$$

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi - \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

First order: $J_0 \mathcal{L}_{\text{GK}}^{-1} \approx \frac{1}{-i\omega_G} \frac{v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla}}{-i\omega_G}$:

$$\left[\left\langle \frac{v_{\parallel} \nabla_{\parallel} + \vec{v}_d \cdot \vec{\nabla}}{i\omega_G} N.L.i^{\omega_G} \right\rangle_v \right]_\psi \approx \left[\left\langle \underbrace{\frac{\hat{v}_{dr} \sin \theta \partial_r}{i\omega_G} \left(\frac{1}{r \bar{B}_0} \partial_r [\delta\phi^{-\omega_G}]_\psi \partial_\theta \underbrace{\delta F_i^{2\omega_G}}_{\propto \cos \theta} \right)}_{= \delta v_{E,B_\theta}} \right\rangle_v \right]_\psi$$


Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

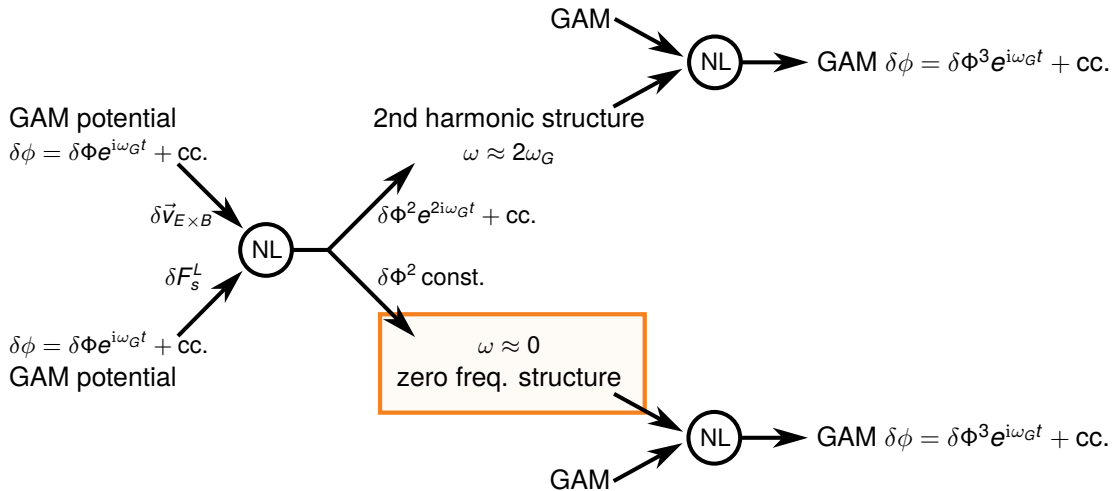
$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi - \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

First order (less approximations, see [Falessi NJP 2023]): $J_0 \mathcal{L}_{\text{GK}}^{-1}$

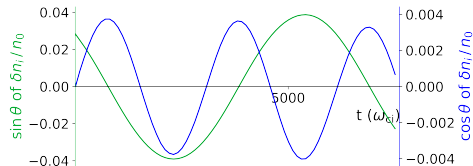
$$\begin{aligned} \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \{ N.L.i^{\omega_G} \} \right\rangle_v \right]_\psi &= \left[\left\langle \sum_{l=-\infty}^{\infty} \frac{i/\omega_{bi}}{\omega_G^2} \cos l\vartheta_c J_0 e^{-iQ_i} \overline{e^{-i/\vartheta_c + iQ_i} N.L.i^{\omega_G}}^{(\text{tr./b.})} \right\rangle_v \right]_\psi \\ &= \frac{4\pi^2}{V'_\psi} \sum_{\hat{\sigma}=\pm 1} \int d\mathcal{E} d\mu \sum_{l=-\infty}^{\infty} \frac{-l^2 \omega_{bi}}{\omega_G^2} \overline{\cos l\vartheta_c J_0 e^{-iQ_i}}^{(\text{tr./b.})} \int d\theta \frac{\mathcal{JRB}_t}{v_\parallel^2} \frac{\partial}{\partial \psi} (J_0 \delta\phi) e^{-i/\vartheta_c + iQ_i} \delta F_i. \end{aligned}$$

→ trapped particles may become important ($v_\parallel(\theta_b) = 0$)

Cubic nonlinearity

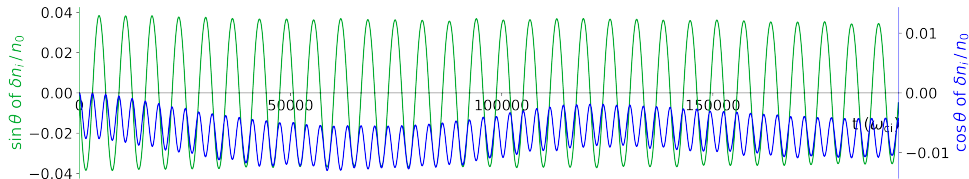


Zero frequency structure



⇒ According to simulations,

$$\delta n_i^{\omega \approx 0} \propto \cos \theta \text{ exists.}$$



Quadratic NL source (GAM $\times$ GAM - coupling)

$$\delta \vec{V}_{E \times B}^{\omega_G} \approx \frac{-\vec{\nabla}[\delta \phi^{\omega_G}]_\psi \times \vec{B}_0}{\bar{B}_0^2} \approx \frac{1}{r \bar{B}_0} [E_r^{\omega_G}]_\psi \hat{e}_\theta \quad \delta F_i^{L, \omega_G} \approx -\frac{e}{T_i} F_0 \frac{\hat{v}_{dr} \sin \theta}{i \omega_G} [E_r^{\omega_G}]_\psi$$

$$\begin{aligned} N.L.i &= \delta \vec{V}_{E \times B} \cdot \vec{\nabla} \delta F_i^L = (\delta \vec{V}_{E \times B}^{\omega_G} + \text{cc.}) \cdot \vec{\nabla} (\delta F_i^{L, \omega_G} + \text{cc.}) \\ &= -\frac{1}{r \bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \cos \theta \left(\frac{1}{i \omega_G} [E_r^{\omega_G}]_\psi^2 + \frac{1}{(-i \omega_G^*)} [E_r^{\omega_G, *}]_\psi^2 \right. \\ &\quad \left. + \frac{1}{i \omega_G} [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi + \frac{1}{(-i \omega_G^*)} [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi \right) \end{aligned}$$

$$\begin{aligned} N.L.i^{\omega \approx 0} &= -\frac{1}{r \bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \cos \theta \underbrace{\left(\frac{1}{i \omega_G} - \frac{1}{i \omega_G^*} \right)}_{= -\partial_t / |\omega_G|^2} [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi \\ &\text{see [Chen EPL 2014]} \end{aligned}$$

Quadratic NL source (GAM $\times$ GAM - coupling)

$$\begin{aligned}
 \delta \vec{V}_{E \times B}^{\omega_G} &\approx \frac{-\vec{\nabla}[\delta \phi^{\omega_G}]_\psi \times \vec{B}_0}{\bar{B}_0^2} \approx \frac{1}{r \bar{B}_0} [E_r^{\omega_G}]_\psi \hat{e}_\theta & \delta F_i^{L, \omega_G} &\approx -\frac{e}{T_i} F_0 \frac{\hat{v}_{dr} \sin \theta}{i \omega_G} [E_r^{\omega_G}]_\psi \\
 & & &+ \frac{e}{T_i} F_0 \frac{v_{\parallel} \nabla_{\parallel}}{i \omega_G} \frac{(\hat{v}_{dr} + \tau_e \hat{v}_{dti}) \sin \theta}{i \omega_G} [E_r^{\omega_G}]_\psi \\
 &\searrow & \swarrow & \\
 &(\delta \vec{V}_{E \times B}^{\omega_G} + \text{cc.}) \cdot \vec{\nabla} (\delta F_i^{L, \omega_G} + \text{cc.}) & & \\
 N.L._i^{\omega \approx 0} &= \frac{1}{r \bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \frac{\partial_t}{|\omega_G|^2} \cos \theta [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi \\
 &+ \frac{1}{r \bar{B}_0} \frac{e}{T_i} F_0 \hat{v}_{dr} \left(\frac{v_{\parallel} \nabla_{\parallel}}{(i \omega_G)^2} + \frac{v_{\parallel} \nabla_{\parallel}}{(-i \omega_G^*)^2} \right) \cos \theta [E_r^{\omega_G}]_\psi [E_r^{\omega_G, *}]_\psi
 \end{aligned}$$

Response at $\omega \approx 0$ (similar to [Chen EPL 2014])

NL GK equation (ions)

$$\overbrace{(\partial_t + v_{\parallel} \nabla_{\parallel} + \vec{v}_{dr} \partial_r)}^{\mathcal{L}_{GK}} \delta g_i^{NL, \omega \approx 0} = -N.L._i^{\omega \approx 0}$$

$$\hookrightarrow (\partial_t + v_{\parallel} \nabla_{\parallel}) \underbrace{e^{iQ} \delta g_i^{NL, \omega \approx 0}}_{\delta g_{Bi}^{NL, \omega \approx 0}} = -e^{iQ} N.L._i^{\omega \approx 0}$$

1 $\delta g_{Bi}^{NL, \omega \approx 0}$ is $m = 0$: $\Rightarrow \nabla_{\parallel} = 0$

$$\delta g_i^{NL, \omega \approx 0} = e^{-iQ} \partial_t^{-1} [-e^{iQ} N.L._i^{\omega \approx 0}]_{\psi}$$

2 $\delta g_{Bi}^{NL, \omega \approx 0}$ is $m \neq 0$: $\Rightarrow v_{\parallel} \nabla_{\parallel} \gg \partial_t$

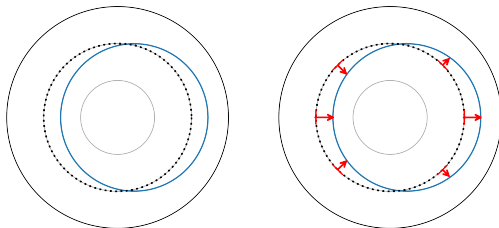
$$\delta g_i^{NL, \omega \approx 0} = e^{-iQ} (v_{\parallel} \nabla_{\parallel})^{-1} (-e^{iQ} \widetilde{N.L._i^{\omega \approx 0}})$$

Invert $\mathcal{L}_{GK}$:

$$\partial_t \ll v_{\parallel} \nabla_{\parallel} + \vec{v}_{dr} \partial_r \quad (! \text{ opposite to GAM})$$

$\hookrightarrow$ use tool from [Rosenbluth Hinton 1998]:

Shift operator e^{iQ}



$$iQ \approx \hat{\rho}_d \cos \theta \partial_r \ll 1$$

$\rightarrow$ accounts for radial drifts

Response at $\omega \approx 0$

1 $\delta g_{Bi}^{NL, \omega \approx 0}$ is $m = 0$: $\propto \partial_t / |\omega_G|^2$

$$\delta g_i^{NL, \omega \approx 0} = \underbrace{e^{-iQ} \partial_t^{-1} [-e^{iQ} N.L._i^{\omega \approx 0}]}_{= 1 + iQ + (iQ)^2/2 + \dots} \quad \times$$

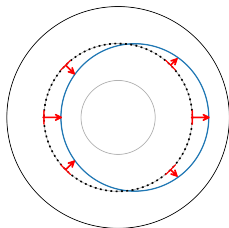
→ couples via 2nd order FOW effects to $\cos \theta$, $\propto (iQ)^2 |[\delta E_r]_\psi|^2$

2 $\delta g_{Bi}^{NL, \omega \approx 0}$ is $m \neq 0$: $\propto v_{\parallel} \nabla_{\parallel} / |\omega_G|^2$

$$\delta g_i^{NL, \omega \approx 0} = e^{-iQ} (v_{\parallel} \nabla_{\parallel})^{-1} (-\widetilde{e^{iQ} N.L._i^{\omega \approx 0}}) \quad \checkmark$$

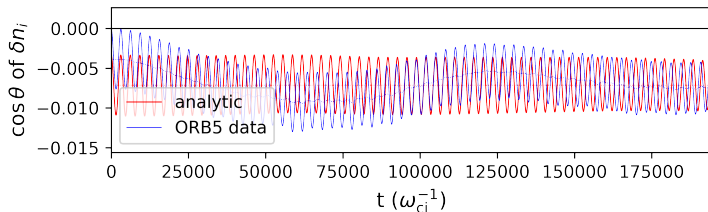
→ direct coupling to $\cos \theta$, $\propto |[\delta E_r]_\psi|^2$

Shift operator e^{iQ}



$$iQ \approx \hat{\rho}_d \cos \theta \partial_r \ll 1$$

Response at $\omega \approx 0$

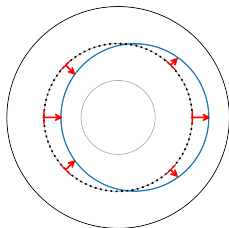


2 $\delta g_{Bi}^{NL, \omega \approx 0}$ is $m \neq 0$: $\propto \underbrace{v_{\parallel} \nabla_{\parallel} / |\omega_G|^2}$

$$\delta g_i^{NL, \omega \approx 0} = e^{-iQ} (v_{\parallel} \nabla_{\parallel})^{-1} (-\widetilde{e^{iQ} N.L.}_i^{\omega \approx 0}) \quad \checkmark$$

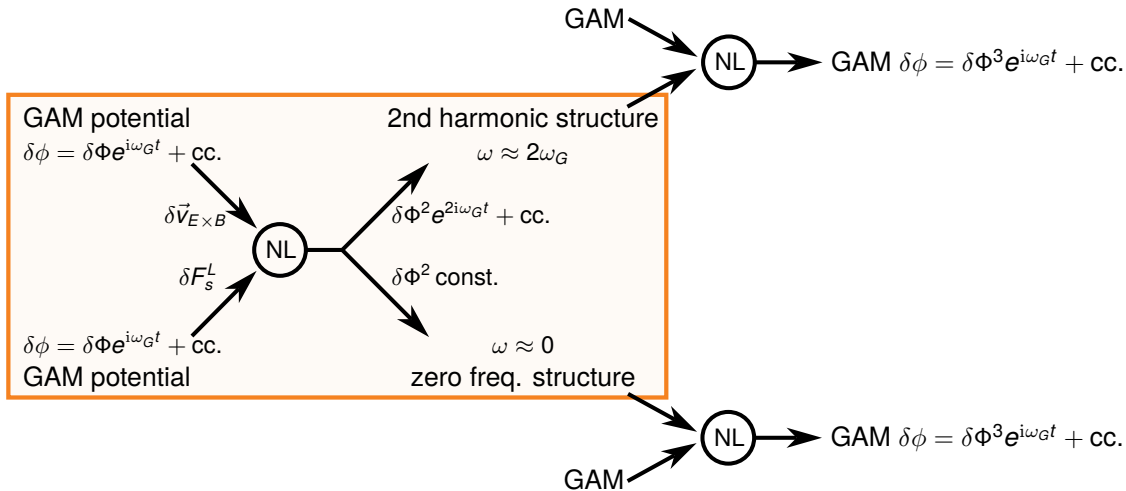
→ direct coupling to $\cos \theta$, $\propto |[\delta E_r]_{\psi}|^2$

Shift operator e^{iQ}

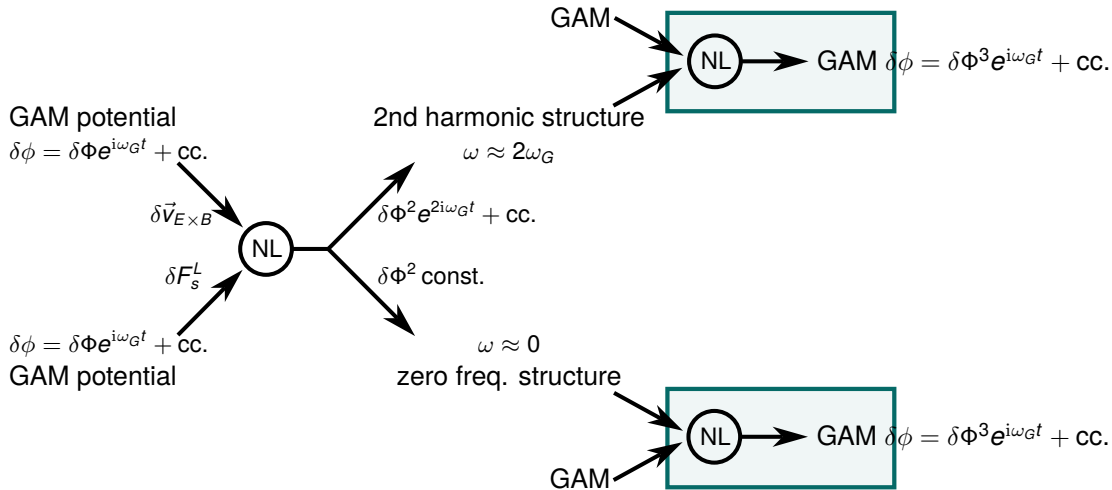


$$iQ \approx \hat{p}_d \cos \theta \partial_r \ll 1$$

Cubic nonlinearity



Cubic nonlinearity



Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi + \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

First order: $J_0 \mathcal{L}_{\text{GK}}^{-1} \approx \frac{1}{-i\omega_G} \frac{\vec{v}_d \cdot \vec{\nabla}}{i\omega_G}:$

$$\left[\left\langle \frac{\vec{v}_d \cdot \vec{\nabla}}{i\omega_G} N.L.i^{\omega_G} \right\rangle_v \right]_\psi = \left[\left\langle \frac{\hat{v}_{dr} \sin \theta \partial_r}{i\omega_G} \left(\underbrace{\frac{1}{r\bar{B}_0} \partial_r [\delta\phi^{-\omega_G}]_\psi}_{= \delta v_{E \times B, \theta}} \partial_\theta \underbrace{\delta F_i^{2\omega_G}}_{\propto \cos \theta} + \frac{1}{r\bar{B}_0} \partial_r [\delta\phi^{\omega_G}]_\psi \partial_\theta \underbrace{\delta F_i^{\omega \approx 0}}_{\propto \cos \theta} \right) \right\rangle_v \right]_\psi$$

$\begin{array}{cccc} -[\delta E_r^{-\omega_G}]_\psi & [\delta E_r^{\omega_G}]_\psi^2 & -[\delta E_r^{\omega_G}]_\psi & |[\delta E_r^{\omega_G}]_\psi|^2 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}$

$$\propto \partial_r \left(|[\delta E_r^{\omega_G}]_\psi|^2 [\delta E_r^{\omega_G}]_\psi \right) = -|[\delta E_r^{\omega_G}]_\psi|^2 \partial_r^2 [\delta\phi^{\omega_G}]_\psi$$

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \left\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \right\rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi + \left[\left\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \right\rangle_v \right]_\psi \right)$$

Linear GAM dispersion relation

$$\left[\left\langle \frac{\vec{v}_d}{i\omega} \right. \right. \underbrace{\left(1 - \frac{\omega_{G0}^2}{\omega_G^2} - \frac{1}{q^2} D_{1,q} + \frac{1}{2} D_{1,k} \rho_{Ti}^2 \partial_r^2 \right)}_{= \delta v_{E \times B, \theta}} \underbrace{\frac{1}{2} \rho_{Ti}^2 \partial_r^2}_{\propto \cos \theta} \left. \left. \delta\phi^{\omega_G} \right]_\psi \right]_\psi \underbrace{\left(\right)}_{= \delta v_{E \times B, \theta}} \underbrace{\left(\right)}_{\propto \cos \theta} \left. \right]_\psi$$

$$\propto \partial_r \left(\left| [\delta E_r^{\omega_G}]_\psi \right|^2 [\delta E_r^{\omega_G}]_\psi \right) = - \left| [\delta E_r^{\omega_G}]_\psi \right|^2 \partial_r^2 [\delta\phi^{\omega_G}]_\psi$$

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \left\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \right\rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi + \left[\left\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \right\rangle_v \right]_\psi \right)$$

Nonlinear GAM dispersion relation

$$0 = \frac{e^2 n_0}{T_i} \left(1 - \frac{\omega_{G0}^2}{\omega_G^2} - \frac{1}{q^2} D_{1,q} + \underbrace{\frac{1}{2} D_{1,k} \rho_{Ti}^2 \partial_r^2}_{= \delta V_{E \times B, \theta}} + \underbrace{\tilde{\alpha}_{\text{NL}} | [E_r^{\omega_G}]_\psi |^2}_{\propto \cos \theta} \right) \underbrace{\frac{1}{2} \rho_{Ti}^2 \partial_r^2}_{= \delta V_{E \times B, \theta}} [\delta\phi^{\omega_G}]_\psi \Bigg] \Bigg\rangle_v \Bigg]_\psi$$

$$\propto \partial_r \left(| [\delta E_r^{\omega_G}]_\psi |^2 [\delta E_r^{\omega_G}]_\psi \right) = - | [\delta E_r^{\omega_G}]_\psi |^2 \partial_r^2 [\delta\phi^{\omega_G}]_\psi$$

Quasineutrality equation ($\omega = \omega_G$)

$$0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_{0s} \delta\phi^{\omega_G} + \left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v - \langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right)$$

$$\xrightarrow{\text{fsa}} 0 = \sum_s e_s \left(-\frac{e_s}{T_s} n_0 [\delta\phi^{\omega_G}]_\psi + \left[\left\langle J_0 \mathcal{L}_{\text{GK}}^{-1} \left\{ \frac{e_s}{T_s} F_0 J_0 \partial_t \delta\phi^{\omega_G} \right\} \right\rangle_v \right]_\psi + \left[\langle \mathcal{L}_{\text{GK}}^{-1} \{ N.L.s^{\omega_G} \} \rangle_v \right]_\psi \right)$$

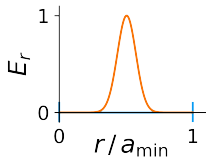
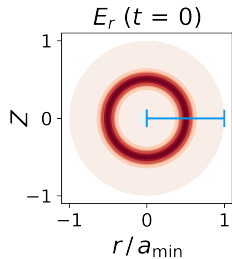
Nonlinear GAM dispersion relation

$$0 = \frac{e^2 n_0}{T_i} \left(1 - \frac{\omega_{G0}^2}{\omega_G^2} - \frac{1}{q^2} D_{1,q} + \frac{1}{2} D_{1,k} \rho_{Ti}^2 \partial_r^2 + \tilde{\alpha}_{\text{NL}} |[E_r^{\omega_G}]_\psi|^2 \right) \frac{1}{2} \rho_{Ti}^2 \partial_r^2 [\delta\phi^{\omega_G}]_\psi$$

$$\hookrightarrow \omega_G \approx \omega_{G0} - \frac{\omega_{G0}}{2q^2} D_{1,q} + \frac{\omega_{G0}}{4} D_{1,k} \rho_{Ti}^2 \partial_r^2 + \frac{1}{2} \omega_{G0} \tilde{\alpha}_{\text{NL}} |[E_r^{\omega_G}]_\psi|^2$$

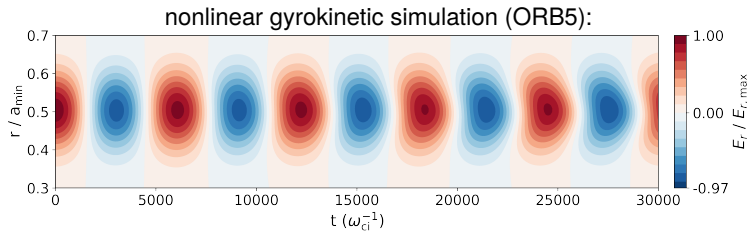
$$\text{with } \alpha_{\text{NL}} = \frac{1}{2} \omega_{G0} \tilde{\alpha}_{\text{NL}} = \frac{63}{64} \frac{1}{r^2 \bar{B}_0^2} \frac{1}{\omega_{G0} (7/4 + \tau_e)}.$$

Does $\alpha_{\text{NL}} = \frac{63}{64} \frac{1}{r^2 \bar{B}_0^2} \frac{1}{\omega_{G0}(7/4 + \tau_e)}$ match simulations?

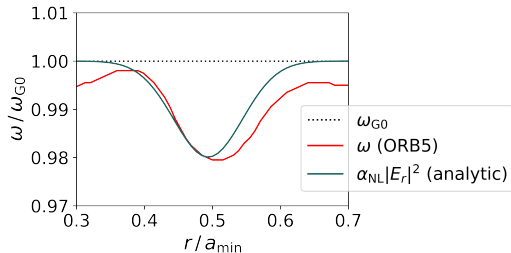


Note: $\omega_G = \omega_{G0} - \frac{1}{2} \mathcal{G} k_r^2 - \alpha_{\text{NL}} |[E_r]_\psi|^2$,

where: $\mathcal{G} = 0$ at $\frac{T_e}{T_i} = 5.5$.

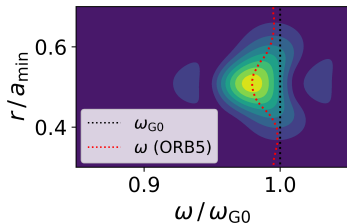


Does $\alpha_{\text{NL}} = \frac{63}{64} \frac{1}{r^2 \bar{B}_0^2} \frac{1}{\omega_{\text{G0}}(7/4 + \tau_e)}$ match simulations?

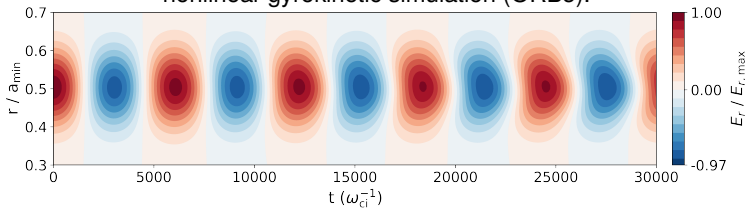


Note: $\omega_{\text{G}} = \omega_{\text{G0}} - \frac{1}{2} \mathcal{G} k_r^2 - \alpha_{\text{NL}} |[E_r]_{\psi}|^2$,

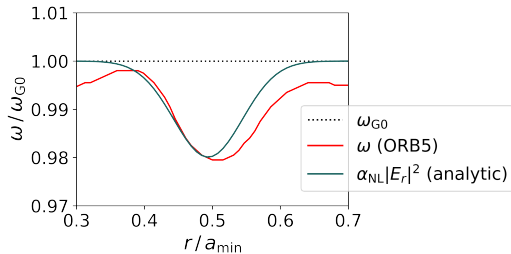
where: $\mathcal{G} = 0$ at $\frac{T_e}{T_i} = 5.5$.



nonlinear gyrokinetic simulation (ORB5):

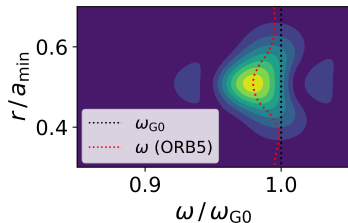


Does $\alpha_{\text{NL}} = \frac{63}{64} \frac{1}{r^2 \bar{B}_0^2} \frac{1}{\omega_{\text{G0}}(7/4 + \tau_e)}$ match simulations?

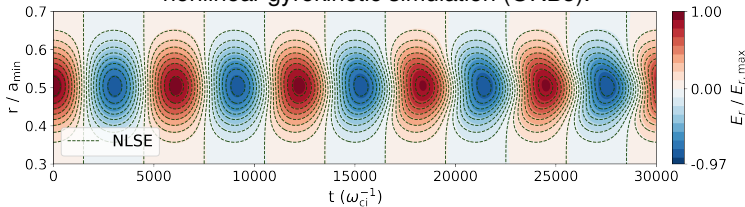


Note: $\omega_{\text{G}} = \omega_{\text{G0}} - \frac{1}{2} \mathcal{G} k_r^2 - \alpha_{\text{NL}} |[E_r]_{\psi}|^2$,

where: $\mathcal{G} = 0$ at $\frac{T_e}{T_i} = 5.5$.



nonlinear gyrokinetic simulation (ORB5):



Derivation of the NLSE model

- Nonlinear Schrödinger equation for GAMs has been derived using many approximations
- Nonlinear frequency shift is formally at same order as linear dispersive effects

Simulations

- Intermediate analytical results (quadratic structures) match simulation data
- Final nonlinear term $\alpha_{\text{NL}}|\psi|^2\psi$ matches
 - well for zero ($\tau_e = 5.5$) or anomalous ($\tau_e > 5.5$) dispersion
 - qualitatively in “self-focusing” regime ($\tau_e < 5.5$), inclusion of damping necessary

Outlook

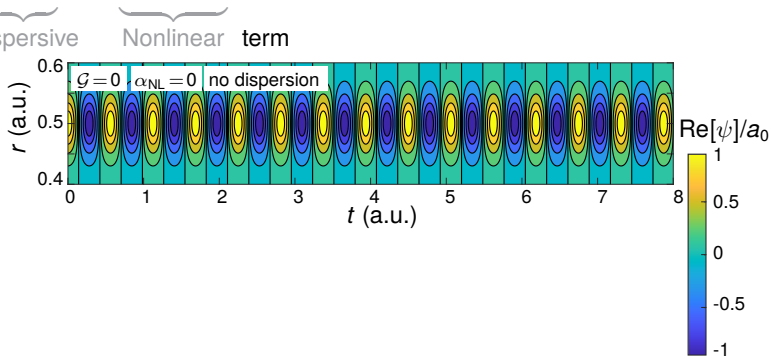
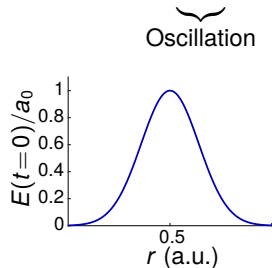
- Weaken approximations (reconsider geometric effects, mirror force, ...)
- Simulations with physical GAM source

Appendix

Nonlinear Schrödinger equation (NLSE) model

$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Oscillation}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Dispersive}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

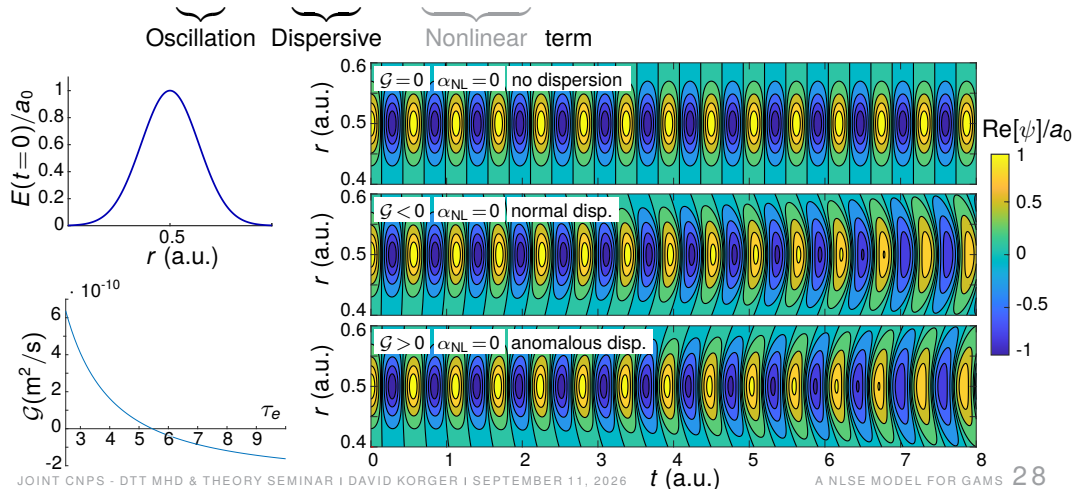
$$\delta E_r(r,t) = \text{Re}[\psi](r,t)$$



Nonlinear Schrödinger equation (NLSE) model

$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Oscillation}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Dispersive}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

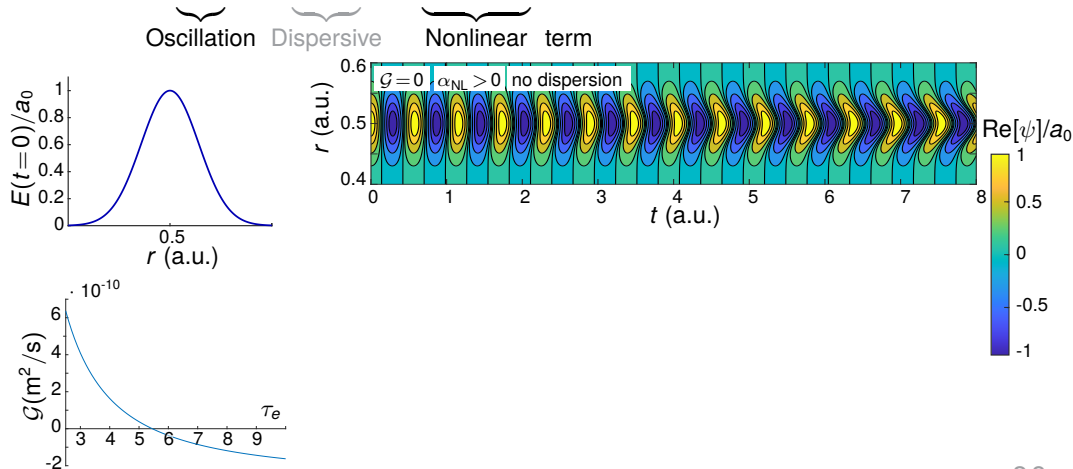
$$\delta E_r(r,t) = \text{Re}[\psi](r,t)$$



Nonlinear Schrödinger equation (NLSE) model

$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Oscillation}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Dispersive}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

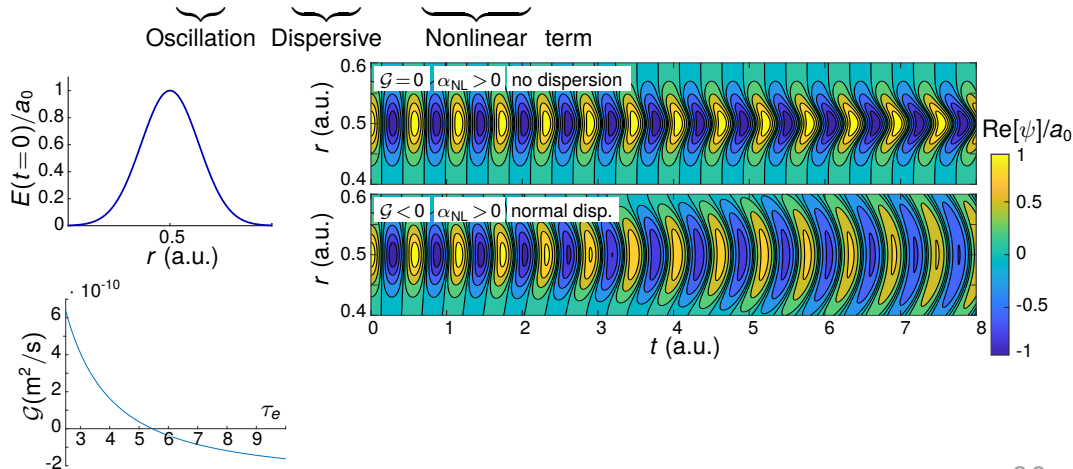
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Nonlinear Schrödinger equation (NLSE) model

$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Oscillation}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Dispersive}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

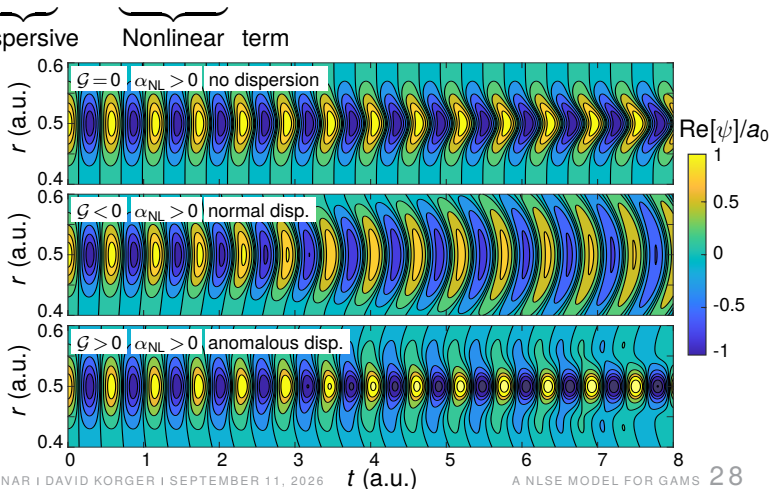
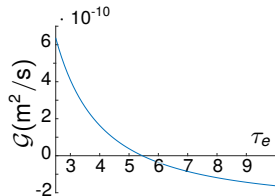
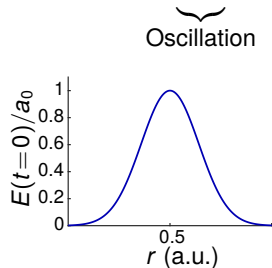
$$\delta E_r(r,t) = \text{Re}[\psi](r,t)$$



Nonlinear Schrödinger equation (NLSE) model

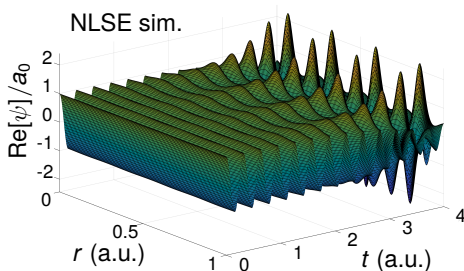
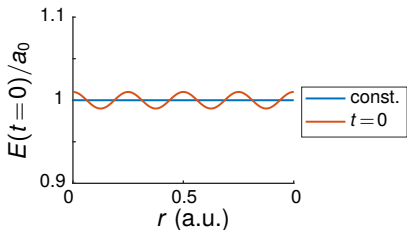
$$i\partial_t\psi(r,t) = \underbrace{\mathcal{F}\psi}_{\text{Oscillation}} - \underbrace{\frac{\mathcal{G}}{2}\partial_r^2\psi}_{\text{Dispersive}} - \underbrace{\alpha_{\text{NL}}|\psi|^2\psi}_{\text{Nonlinear term}}. \quad [\text{Poli 21}]$$

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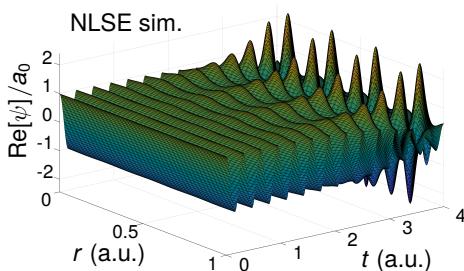
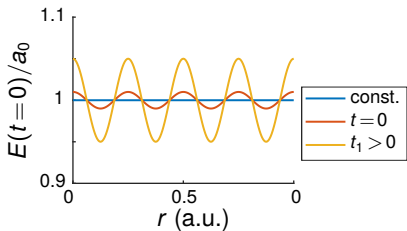
- Wave envelope (of GAM el. field) unstable to specific range of wavenumbers k
- “Pure form”: constant background ($k = 0$) perturbed by single wavenumber k_{pert}

$$\delta E_r(r, t = 0) = \psi(r, t = 0) = a_0 + a_1 \cos(k_{\text{pert}} r), \quad a_1 \ll a_0$$



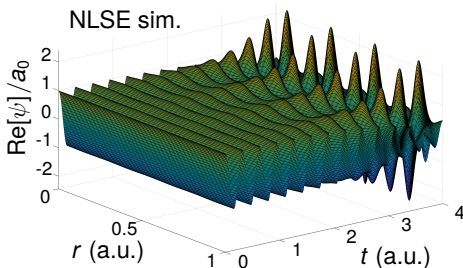
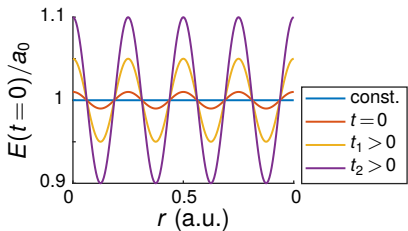
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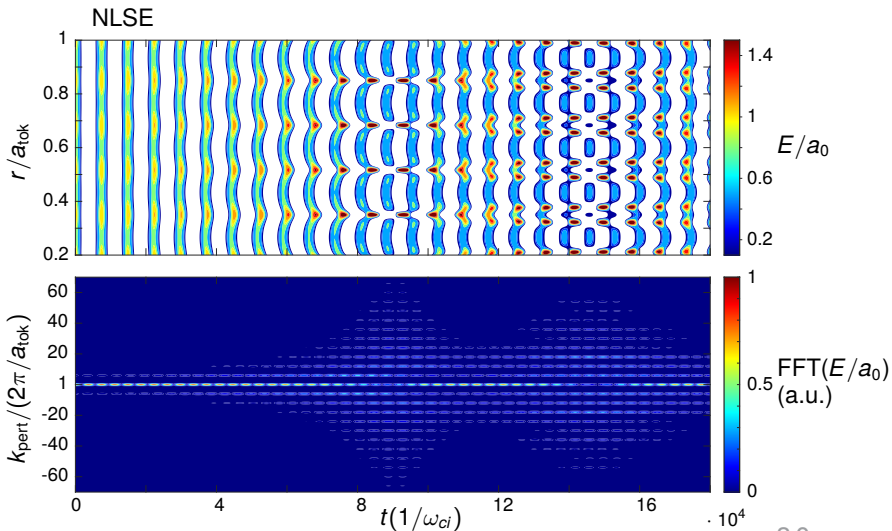


Modulational Instability

NLSE sim. with MI
init. condition

Parameters:

- $\tau_e = 3$,
- $L_x = 400$,
- $q = 15$,
 $B = 1.9\text{ T}$
- $a_{\text{tok}} = 0.13\text{ m}$,
- $R_0 = 1.3\text{ m}$,
- $A_0 = 2.5 \cdot 10^{-4}$,
- $w_0 = 0.35$,
- $r_0 = 0.6$,
- $p = 4$,
- $k_{\text{pert}} = 6 \frac{2\pi}{a_{\text{tok}}}$.



Modulational Instability: Results

Observe MI in ORB5 simulation:

- Growth in ORB5 sim. is slower than NLSE
- Nonlinearity α_{NL} is not constant along r
- Steep gradients in saturation phase not reproduced

Parameters: $k_{pert} = 6 \frac{2\pi}{a_{tok}}$,
 $\tau_e = 3$, $L_x = 400$, $q = 15$,
 $B = 1.9 T$, $A_0 = 2.5 \cdot 10^{-4}$,
 $w_0 = 0.35$, $p = 4$, $r_0 = 0.6$

