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ABSTRACT
Chirping Alfvén modes are considered as potentially harmful for the confinement of energetic particles in burning tokamak plasmas. In fact, by changing their frequency, they are able to extract as much power as possible from these particles, possibly increasing their transport. In this paper, the nonlinear evolution of a single-toroidal-number chirping mode is analyzed by numerical particle simulation. The relevant resonant structures are identified by numerical techniques based on the use of a coordinate system including two constants of motion: the magnetic moment and a suitable function of the initial particle coordinates. The analysis is focused on the dynamics of two different resonant structures in the particle phase space: those yielding the largest drive during the linear and the nonlinear phase, respectively. It is shown that, for each resonant structure, a density-flattening region is formed around the respective resonance radius, with a radial width that increases as the mode amplitude grows. It is delimited by two steepened negative density gradients, drifting inwards and outward. If the mode frequency were constant, phase-space density flattening would quench the resonant-structure drive as the steepened gradients leave the original resonance region. The frequency chirping, however, causes the resonance radius and the resonance region to drift inwards. This drift, along with a relevant increase in the resonance width, delays the moment in which the inner density gradient reaches the inner boundary of the resonance region, leaving it. On the other hand, the island evolves consistently with the resonance radius; as a consequence, the steepened density gradient further moves inward. This process continues as long as it allows to keep the steepened gradient within the resonance region. When this is no longer possible, the resonant structure ceases to be effective in driving the mode. To further extract energy from the particles, the mode has to tap a different resonant structure, possibly making use of additional frequency variations.

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I. INTRODUCTION
Alfvén modes can be driven unstable, in tokamak plasmas, by the resonant interaction with alpha particles produced by fusion reactions and/or energetic ions produced by auxiliary heating methods.1–9 The interaction with the mode can in turn deteriorate the confinement of these particles, preventing their thermalization in the core plasma and eventually damaging the first wall. Some of the Alfvén modes, like the toroidal Alfvén eigenmodes10 (TAEs), have a MHD counterpart; that is, they exist as marginally stable modes even in the absence of energetic-particle drive. Other modes, like the energetic-particle modes11 (EPMs), have no such counterpart: they are oscillations of the Alfvén continuum driven unstable when the energetic-particle drive exceeds the continuum damping; in the absence of this drive, they are strongly damped. Both types of modes can exhibit frequency chirping.12–16 For weakly driven modes near marginal stability, this phenomenon has been explained on the basis of a 1D bump-on-tail model with sources and sinks,17,18 which predicts an adiabatic chirping; that is, a chirping characterized by a frequency variation, in the bounce time of wave-trapped particles, much smaller than their bounce frequency squared. More generally, it has been shown9,19,20 that non-perturbative energetic-particle effects generally yield nonlinear dynamics without time scale separation between nonlinear mode evolution and phase space transport. The theoretical framework of Refs. 9, 19, and 20 also reproduces the Berk–Breizman paradigm as the limiting case in the weak drive regime, where the frequency sweeping is adiabatically slow.17,18 It is nonetheless worthwhile noting the crucial role...
of non-perturbative energetic-particle response that yields phase locking and non-adiabatic frequency chirping in both laboratory and space plasmas. Note also that experimental results concerning chirping frequency fishbones (analogous, in some respects, to EPMs) are explained, in Ref. 24 in terms of the reactive torque exerted on the plasma during the instability burst.

In this paper, we want to investigate the nonlinear dynamics of chirping EPMs by numerical particle simulation. Recently, chirping dynamics was analyzed by means of the orbit-following code ORBIT, with particular attention to the phenomenon of field pulsation, in the frame of a semiperturbative approach: the effects of energetic-particle dynamics on the field structure were neglected. Here, we adopt a selfconsistent approach, using the extended Hybrid Magnetohydrodynamics Gyrokinetic Code (XHMGC), to investigate the succession of resonant phase-space structures in driving the chirping mode. For the sake of simplicity, we consider the case of an Alfvén spectrum characterized by a single toroidal mode number \( n \). We had already studied the case of a single-\( n \) mode, with constant frequency \( \omega \) in Refs. 31 and 32. Our approach consisted of adopting a coordinate system that includes two invariants: namely, the magnetic moment \( M \) and the quantity \( C \equiv \omega \rho_{\phi} - \Delta E \), with \( \rho_{\phi} \) and \( \Delta E \) being, respectively, the toroidal angular momentum and the energy of the particle. The phase space can then be seen as a set of slices characterized by given values of \( M \) and \( C \). Because of the invariance of these two coordinates, particles cannot cross their birth slice and the gradients of the distribution function orthogonal to the slices do not play any direct role in mode–particle power exchange: the slices can then be treated as isolated ones. Once the most relevant resonances (that is, the slices yielding the largest contribution to the mode drive or damping) are identified, saturation dynamics can be analyzed by focusing on the evolution of each of them. To this aim, in Refs. 31 and 32 we sampled the selected slices by a large number of test particles moving in the fields computed in a full-population, self-consistent simulation. Their interaction with the mode was then analyzed in detail. It was shown that an island-like structure, enclosing the bounded orbits of particles instantaneously trapped in the potential well of the wave, forms around the resonance radius in the plane \( (\Theta, r_{eq}) \); here, \( \Theta \) corresponds to the wave-phase seen by the particle and \( r_{eq} \) is the “equatorial radius” (see Secs. V and II, respectively, for their definition). Mixing of trapped particles originating from the opposite sides of the resonance radius occurs. This gives rise to a flattened-density region delimited by steepened negative density gradients moving inward and outward as the island width grows with the increasing mode amplitude. Saturation is reached when the density flattening completely covers the resonant-interaction region. The width of this region is typically limited by the smaller of the resonance width and mode width: in the former case, the saturation mechanism has been named resonance detuning; in the latter, radial decoupling.

We will see that, in the case of chirping modes, the ability of the mode to change its frequency (a downward chirping, in the considered case) affects the above mechanism. Indeed, it causes the resonance radius and the resonance region to move inward. This drift allows to delay the moment in which the inner density gradient reaches the inner boundary of the resonance region and leaves it. Conversely, the island reconstitutes around the new resonance radius, drifting inward too. As a consequence, the steepened density gradient further moves inwards. This process goes on as long as the frequency can decrease and the resonance region can move inward in such a way to keep the gradient within itself. When this is no longer possible, either because a further change in frequency is disadvantageous in terms of drive/damping balance, or because such a change does not result in a significant inward shift of the resonance radius and the resonance region, the gradient ceases to be effective in driving the mode. To further extract energy from the energetic particles, the mode has to tap a different resonant structure, possibly making use of additional frequency variations. We will analyze the analogies and the differences between the behaviors of two resonant structures, identified as those dominating the linear stage and the nonlinear one, respectively.

The paper is structured as follows. In Sec. II, we show how to identify the most relevant phase-space structures for the mode–particle dynamics. For this purpose, it is useful to adopt coordinate systems that include some constants of particle motion. In the view of a numerical approach, the only requirement is that the adopted coordinates of each particle are known at each time; in such a view, a suitable set of constants of motion can be immediately recognized in the initial coordinates of the particle. The numerical experiment performed to analyze the dynamics of a chirping mode is described in Sec. III. Section IV presents the search for the most relevant phase-space resonant structures. In Sec. V, the Hamiltonian-mapping test-particle approach is adopted to investigate some aspects of the nonlinear dynamics of a resonant structure. The relationship between mode–particle power transfer and fulfillment of the resonance condition is examined in Sec. VI. Section VII analyzes the phenomenon of particle trapping and de-trapping in detail. Section VIII shows how the density-flattening region and the resonance region evolve during the mode chirping; it also describes the succession of different resonant structures in driving the mode during the nonlinear stage. A summary of the paper and a discussion of the critical points of our approach are presented in Sec. IX. The details of the power-transfer calculation in a coordinate system different from the one adopted for particle pushing are presented, for interested readers, in the Appendix.

II. IDENTIFYING RELEVANT RESONANCES: CONSTANT-OF-MOTION COORDINATES

In analyzing the interaction between Alfvén modes and energetic particles, it is important to identify the phase-space regions where the mode–particle power exchange takes place. If \( \hat{E}_H \) is the total energy of the energetic (“hot”) particle population, the overall power transfer is given by

\[
\frac{d\hat{E}_H}{dt} = -\frac{1}{m_H} \left[ d\phi Z D_{\phi} - e_H \frac{dE_H}{dt} \right].
\]

Here, the gyrocenter coordinate system \( Z \equiv (r, \Theta, \phi, M, U, \vartheta) \) has been used to describe the phase space, with \( r \) being the radial coordinate, \( \Theta \) and \( \vartheta \) the poloidal and toroidal angle, respectively (cf. Fig. 2 below), \( M \) the conserved magnetic moment, \( U \) the parallel (to the equilibrium magnetic field) velocity, and \( \vartheta \) the gyrophase. Moreover

\[
E \simeq m_H U^2/2 + M \Omega_H + e_H \langle \delta\phi \rangle - e_H U/c \langle \delta A_i \rangle
\]

is the single-particle energy, \( m_H \) and \( e_H \) are the energetic-particle mass and electric charge, respectively; \( c \) is the speed of light; \( \langle \delta\phi \rangle \) and \( \langle \delta A_i \rangle \) are the gyro-averages of the fluctuating scalar potential and the parallel (to the equilibrium magnetic field) component of the fluctuating
vector potential, respectively; and \( F_\theta \) and \( D_{\varphi \to Z} \) are the \( \varphi \)-independent energetic-particle distribution function and the Jacobian of the transformation from canonical to gyrocenter coordinates, respectively.

It is worth reducing the dimensionality of the phase space by averaging the contribution of each marker over the poloidal and toroidal angles. In other words, we can define a power-transfer density in the 3D space \((r, M, U)\) in the following way:

\[
- \frac{dE}{dt} \equiv \frac{1}{m_H} \int dP dM dU (r, M, U, t),
\]

that is,

\[
P(r, M, U, t) \equiv -2\pi \int d\phi d\varphi D_{\varphi \to Z} F_\theta \frac{dE}{dt}.
\]

In the Appendix, it is shown how to compute this quantity as a sum of the contributions of the individual markers, each representing the physical particles contained in a microscopic phase space volume element. It is also shown how a coordinate transformation \( Z \to \tilde{Z} = (\tilde{Z}^1, \theta, \phi, \tilde{Z}^2, \tilde{Z}^3) \) does not complicate the calculation of the power-transfer density \( P(\tilde{Z}^1, \tilde{Z}^2, \tilde{Z}^3, t) \), provided the new coordinates are known, at each time, for all the markers. In particular, there is no need to deal with the explicit form of the Jacobian \( D_{\varphi \to Z} \).

A useful choice of the coordinate system is that resorting to exact invariants of the (perturbed) motion, if it is possible to identify them. There is an immediate advantage in adopting coordinate systems that include such invariants: namely, the unambiguous identification of the most active particles in driving or damping the mode at any time. This allows, in particular, to discern whether the mode is driven by the same group of particles, possibly with nonlinearly modified orbits, in any stage of its evolution or by a succession of different groups. In the former case, we could describe the mode–particle power exchange as a process in which the mode is locked to that group of resonant particles, adapting itself to any change in their orbits to further extract energy from them. In the latter case, the interaction between mode and resonant particles should be described as a continuous search by the mode for the particles that can transfer power more efficiently.

Let us assume that the coordinates \((\tilde{Z}^1, \tilde{Z}^4, \tilde{Z}^5)\), or, more generally, some of them, are exact invariants of the perturbed motion. If we define \( \tilde{Z}_{\text{max}}(t), \tilde{Z}^4_{\text{max}}(t), \tilde{Z}^5_{\text{max}}(t) \) the coordinates of the maximum power transfer at time \( t \), any variation of the value corresponding to an exact invariant would indicate that new particles have replaced the previous ones in destabilizing the mode.

In practice, \( M \) can be considered as a conserved quantity up to the required asymptotic-expansion order in the collisionless gyrokinetic limit. So, a succession of different values of \( M \) would readily be recognized as a succession of different relevant particles taking part in the considered physical process. If, instead, the coordinates of the maximum power transfer correspond to \( r_{\text{max}}(t), M_{\text{max}}, U_{\text{max}}(t) \), with constant \( M_{\text{max}} \) (typical for modes driven by transit resonances), the \( Z \) coordinates do not allow us to distinguish between the same driving group and a succession of different driving groups. In order to succeed in this task, we have to identify more invariants, if they exist. In particular, if two more invariants can be selected, we get a complete identification, at any time, of the most destabilizing particles.

Another advantage in adopting a phase-space coordinate system including exact invariants is that we can reduce the dimensionality of the free-energy source. This can be seen in the following way. The Vlasov equation for the energetic-particle distribution function can be written in the form

\[
\frac{\partial \tilde{F}_\theta}{\partial t} + \frac{d\tilde{Z}^i}{dt} \frac{\partial \tilde{F}_\theta}{\partial \tilde{Z}^i} = 0
\]

with \( \tilde{F}_\theta(t, \tilde{Z}) = F_\theta(t, Z(\tilde{Z})) \). If \( \tilde{Z}^i \) is an invariant of the motion, \( d\tilde{Z}^i/dt = 0 \). Then, the gradient \( \partial \tilde{F}_\theta/\partial \tilde{Z}^i \) does not take part in the mode–particle dynamics. We can express the same concept in a different way: if we cut the phase space into slices orthogonal to the invariant coordinate, there will be no particle flux from one slice to the other; that is, the nonlinear particle evolution will not yield any change of the distribution function along that coordinate. The dynamics of a given slice will then depend on the other slices only through the fields, not because of mixing of the respective populations. This feature allows us to investigate separately each slice. Once the slices providing the most relevant resonances are identified, the details of mode–particle interaction can be analyzed, e.g., by the so-called Hamiltonian-mapping technique.31 It greatly improves the resolution with which the resonance is examined by sampling the corresponding slice with a large number of test particles; these particles are evolved in the self-consistent fields computed by the considered simulation. The advantage of this approach is twofold: on the one hand, the dynamics of resonant particles is not obscured by the behavior of nonresonant ones; in particular, the local modifications of the distribution function can be detected even in the case of weak resonances, despite a low sensitivity of the overall population to the presence of the mode. On the other hand, examining the evolution of a single-slice dynamics can make the description of the nonlinear dynamics easier.

In previous papers,13,15,32,33 we have investigated the saturation mechanisms of Alfvén modes characterized by a single toroidal number \( n \) and a constant frequency \( \omega \). In such cases, the quantity \( C = \omega P_\phi - n E \) is an exact invariant of the perturbed motion. Indeed, the equations of motion in the Hamiltonian form read

\[
\frac{dP_\phi}{dt} = -\frac{\partial H}{\partial \phi} - \frac{\partial \omega P_\phi}{\partial t} \quad \text{(5)}
\]

and

\[
\frac{dE}{dt} = \frac{\partial H}{\partial t}, \quad \text{(6)}
\]

where \( H \) is the single particle Hamiltonian. The conservation of \( C \) immediately follows33 from the time and toroidal-angle dependence of the Hamiltonian in the form \( H = H(\omega t - n \phi) \). In those cases, we adopted a coordinate system \( Z = (r_{eq}, \theta, \phi, M, C) \), performing the Hamiltonian-mapping analysis of phase-space slices characterized by fixed values of \( M \) and \( C \). Here and in the following, the subscript “eq” indicates the equatorial coordinates; that is, the values that particle coordinates would assume at the next crossing of the equatorial plane at \( \theta = 0 \) if its motion were unperturbed (for banana orbits, the outmost crossing is considered). As explained in Ref. 31, once a relevant slice has been fixed, the analysis proceeds storing the information concerning each test-particle orbit at its equatorial-plane crossing; in particular, its coordinates \( r_{eq} \) and \( \phi \) (\( \theta = 0 \) by definition) and the power
exchanged with the mode during the poloidal orbit just completed. In this way, the distribution function for particles belonging to the considered slice can be computed; its integral over \( \phi \), which depends only on \( r_{\text{eq}} \), represents the free-energy source with which the slice can contribute to the destabilization of the mode. Time evolution of such integral can then be related to that of the power transfer, getting insight into the saturation dynamics.

If multiple toroidal numbers are present and/or the frequency is not constant, \( C \) is no longer an exact invariant of the perturbed motion. This is the general case, of course, both because a large \( n \) spectrum of Alfvén modes can be destabilized in tokamak equilibria and chirping modes are often observed in experiments.\(^{12–16} \) In these cases, slices orthogonal to the \( C \) direction would not be isolated. The dynamics would then also be influenced by the gradient of the distribution function along \( C \) and the corresponding fluxes crossing each slice. In other words, the evolution of the free-energy source associated with a certain slice could not be adequately analyzed by simply looking at the gradient of the distribution function along \( r_{\text{eq}} \).

However, it is possible to resort to alternative coordinate transformations, which give the same advantages as the “exact-invariant” one. In this paper, we propose to look at simple constants of particle motion. Here, we do not need to know whether global invariants,\(^{30,36} \) suitable for replacing \( C \), exist, nor what form they take. The reason is that any constant of motion could be used instead of the additional global invariant (isolating integral), as it will be shortly discussed at the end of this section. Identifying constants of motion is an elementary task, as the initial coordinates of a particle are, by definition, conserved along its motion. The transformation that links a certain coordinate system, \( \tilde{Z} \), to the system represented by its initial values \( \tilde{Z}_0 \) can be formally obtained by the equations of motion

\[
\frac{d\tilde{Z}'}{dt} = \tilde{V}'(\tilde{Z}, t)
\]

in the following form:

\[
\tilde{Z}'_0 = \tilde{Z}' - \int_0^t dt' \tilde{V}'(\tilde{Z}, t').
\]

More generally, we can consider a coordinate transformation in which only some of the \( \tilde{Z} \) are replaced by the corresponding \( \tilde{Z}'_0 \), or by any function of them; in the following, any of these coordinate systems will be referred to as an initial-value coordinate system. Note that the choice of an initial-value coordinate system is, in general, impractical in the context of an analytical treatment, as it requires complicated backward transformations from the current particle coordinates to the initial ones. In the frame of a numerical approach, however, it is sufficient to memorize the initial coordinates of each marker, in addition to its current coordinates.

In concluding this section, we would like to emphasize the following points. For the purpose of a complete and unambiguous identification of instantaneous power transfer maxima, the use of any triad of invariant coordinates is equally effective, regardless of whether they are global invariants or not. As far as the goal of simplifying the description of nonlinear dynamics is concerned, the conclusion is similar, but requires some clarification. First of all, it is apparent how the choice of any invariant coordinate (besides \( M \)) allows a subdivision of the phase space into slices that do not exchange particles with each other. The fact, however, that these slices are treated as isolated is a mere artifice, consisting of treating the fluctuating field as an exogenous field: this is what is done in sampling each of the considered slices by a large number of test particles that passively evolve according to the field previously calculated in a self-consistent simulation. In the evolution of the real system, however, each slice communicates with the others, though not exchanging particles, precisely through the field. Moreover, one slice can take over from another in destabilizing the mode, due to the modification of the frequency and/or the structure of the mode. One cannot therefore expect \textit{a priori} that the relevant dynamics will remain confined to a particular slice. In principle, it is also possible that such a confinement occurs in correspondence with the choice of a given invariant, but, so far, no recipe is known to guide this choice in the general case: indeed, having such a recipe would be equivalent to a full ability to predict the nonlinear evolution of the system, even in very complex situations. All this is independent of the particular invariant (global or not) included in the coordinate system.

From this point of view, there is, therefore, no qualitative difference between the adoption of a coordinate system including a global invariant and that of a system containing any other quantity conserved during the particle’s motion; in both cases, only \textit{a posteriori} can it be established whether the choice made is effective for the purpose of a simplified description: namely, whether it allows to restrict the analysis to a small number of relevant slices.

III. SETTING OF THE NUMERICAL EXPERIMENT

In this section, the numerical experiment designed for analyzing the dynamics of a chirping Alfvén mode is presented.

We have simulated, by the hybrid MHD-gyrokinetic code XHMGC,\(^{28–30} \) the evolution of an \( n=3 \) energetic-particle mode,\(^{11} \) with poloidal harmonics \( m=4–9 \), in a large aspect ratio \( (R_0/a=10, a \) being the minor radius of the torus) Tokamak equilibrium, characterized by shifted circular magnetic-flux surfaces. The safety factor \( q \), monotonically increasing from 1.6 to 2.8, is shown vs \( s \equiv \sqrt{1 - \psi_{\text{equil}}/\psi_{\text{equil0}}} \) in Fig. 1. Here, \( \psi_{\text{equil}}(r, \theta) \) is the poloidal

![Fig. 1. Radial profile of the safety factor q. Here s = sqrt{1 - psi_equil/psi_equil0}.](https://example.com/fig1.png)
flux of the equilibrium magnetic field \( B_{\text{eq}} \equiv R_0 B_{\phi 0} \nabla \phi + R_0 \nabla \psi_{\text{eq}} \times \nabla \phi \) and the subscript 0 indicates the quantities computed at the equilibrium magnetic axis. The spatial coordinate system is shown in Fig. 2, with \( R \) being the major-radius coordinate; note that the minor-radius coordinate \( r \) is not a flux-surface label. A deuterium plasma is considered, along with an energetic deuteron population. Both bulk-ion and energetic-ion populations are treated kinetically. The relevant dimensionless parameters are the following: \( n_{i0}/n_0 = 0.01 \), \( v_{i0}/v_A = 0.3 \), \( \rho_{i0}/a = 0.01 \), \( \psi_0/v_A = 0.06 \), \( \rho_H/a = 0.002 \). Here, \( n_H \) and \( n_i \) are the densities of the two species; \( v_H \) and \( v_i \) and \( \rho_H \) and \( \rho_i \) are their thermal velocities and Larmor radii, respectively; \( v_A \) is the Alfvén speed and speed. The density profile of energetic particles is shown in Fig. 3, while their temperature profile, as well as the bulk-ion density and temperature profiles, is assumed flat. The initial distribution function is Maxwellian for both species. For the energetic particles, it is multiplied by an anisotropy factor \( \Xi(x; z_0, \kappa) \), with

\[
\Xi(x; z_0, \kappa) = \frac{4}{\kappa} \frac{1}{\pi} \frac{1}{\exp \left( \frac{1 + \cos z_0}{\kappa} \right) + \exp \left( \frac{1 - \cos z_0}{\kappa} \right)}.
\]

Figure 4 shows the time evolution of mode energy and frequency. The first saturation of the mode is followed by a stage in which the mode gets further drive and saturates again, several times. A significant initial relaxation due to the fact that such functions are not equilibrium ones is inhibited.

Our purpose is yielding a detailed description of the nonlinear evolution of the mode. To this aim, we consider the power-transfer density \( P(Z', Z^+, Z^-, t) \), computed, after discretizing the energetic-particle distribution function, in the form given by Eq. (A12). As discussed in Sec. II, different coordinate systems \( Z \) can be adopted to describe the phase space.

The usual choice made in investigations based on gyrokinetic simulation consists of referring the power transfer to the actual gyrocenter coordinates \( (Z \equiv Z) \). However, it can be more interesting referring the power transfer to an equatorial-coordinate system, \( (Z_{eq}, 0, \phi, M, U_{eq}) \). Indeed, the triad \( (r_{eq}, M, U_{eq}) \) fully identifies

\( \Xi(x; z_0, \kappa) \equiv \frac{4}{\kappa} \frac{1}{\pi} \frac{1}{\exp \left( \frac{1 + \cos z_0}{\kappa} \right) + \exp \left( \frac{1 - \cos z_0}{\kappa} \right)} \)

IV. SEARCH FOR THE MOST RELEVANT PHASE-SPACE STRUCTURES

FIG. 2. Toroidal coordinate system \((r, \theta, \phi)\) for a tokamak plasma equilibrium with major radius \( R_0 \) and minor radius \( a \).

FIG. 3. Energetic-particle density profile.

FIG. 4. Time evolution of the mode energy (red) and frequency (black). Here, \( T_{eq} \equiv m_H v_{eq}^2 \).
the (unperturbed) poloidal orbit of the particle; this choice is then justified by the interest in considering the contribution of each particle along its whole orbit, rather than the instantaneous one. This can be stated in a slightly different way: until a certain particle maintains its unperturbed orbit, it will contribute to the power transfer to the same point in the space \((r_{eq}, M, U_{eq})\), while it would spread its contribution on a curve \(U(h) = U(r(h), M)\) in the space \((r, M, U)\).

Figure 7 shows the time evolution of the coordinates of the maximum of \(\tilde{P}(r_{eq}, M, U_{eq}, t)\) for the considered case. We see that the drive is always peaked at \(M = 0, U_{eq} > 0\), consistent with the deeply co-passing character of the distribution function. During the nonlinear stage, the maximum power transfer coordinates \(r_{eq\max}(t)\) and \(U_{eq\max}(t)\) change, while \(M_{eq\max}(t)\) remains constant. This time dependence is consistent with both the following pictures: a nonlinear drive yielded by the same particle group responsible for the linear destabilization, with coordinates progressively modified by the interaction with the mode; or a drive yielded by a succession of different groups. As discussed in Sec. I, the fact that there is no time dependence of \(M_{eq\max}\) (with \(M\) being the only invariant coordinate in the considered system) does not allow to determine which of these two interpretations is correct. To do so unambiguously, we have to include additional constants of motion in the coordinate system. As stated before, the frequency variation causes \(C\) to lose its exact invariance, although, in the present case, the rate of variation of \(C\) remains quite small. Instead of
controlling the amount of nonconservation and evaluating its relevance, we prefer to include the initial values of the equatorial minor radius and/or the equatorial parallel velocity, $r_{eq0}$ and $U_{eq0}$. Then, while pushing particles in the usual gyrocenter coordinates $Z$, we adopt two alternative systems: a three-constants system, $\tilde{Z}_3 = (r_{eq0}, \theta, \psi, M, U_{eq0})$ for the identification of instantaneous maxima of the power transfer, and a two-constant one, $\tilde{Z}_2 = (r_{eq}, \theta, \psi, M, U_{eq0})$, for investigating the nonlinear saturation dynamics. Note that in the latter system the first coordinate is the current value of the equatorial radial coordinate: it is not a constant of motion. The corresponding gradient of the distribution function plays the role of a free-energy source for the mode destabilization, and the associated fluxes contribute to mode saturation.

The coordinates of the instantaneous maxima of the power-transfer density in the space $(r_{eq0}, M, U_{eq0})$ are shown in Fig. 8. In this case, the time variation of $r_{eq0\max}(t)$ and $U_{eq0\max}(t)$ clearly indicates that the maximum drive is generally due, at different times, to different groups (characterized by different initial coordinates).

Figure 9 shows the contour plots of the power transfer, for $M = 0$, in the space $(r_{eq0}, U_{eq0})$, at three different times: $t/\tau_{A0} = 252.0, 480.0, 558.0$. We see that the region dominating the drive during the linear stage progressively loses importance, being replaced by regions less contributing to the linear destabilization.

Computing the power transfer in the coordinate system $\tilde{Z}_3$ is worth to label particles providing the maximum drive at different times, but it is not suitable for investigating the saturation dynamics. To address this issue, we resort to the $\tilde{Z}_2 = (r_{eq}, \theta, \psi, M, U_{eq0})$ coordinate system, containing only two constants of motion, $M$ and $U_{eq0}$. In this way, the gradient of the distribution function along the equatorial radial coordinate $r_{eq}$ represents the free-energy source, and the corresponding flow contributes to mode saturation.

In principle, we can expect that in the present case the mode is driven by resonances other than the linear-stage one during the nonlinear stage, unlike the cases characterized by a single toroidal number and a constant frequency. Then, we have first to identify the most relevant structures in the 2D reduced phase space $(M, U_{eq0})$. Once these structures are identified, we can analyze each of them independently of the others. Indeed, the phase space can be cut into distinct slices, orthogonally to the subspace $(M, U_{eq0})$, with flows entering or leaving the slices being forbidden by the conservation of $M$ and $U_{eq0}$.

Figure 10 shows the contour plots of the power transfer in the plane $(M, U_{eq0})$ (integrated over all the other coordinates) at
The succession of different slices in driving the mode is apparent. This can be seen more clearly from Fig. 11, where the time behavior of the power yielded by the most relevant slices is represented. With reference to the grid adopted to show the contour plots of Fig. 10, only those grid points hosting the maximum power transfer at some time in the considered interval have been considered. Consistently with the previous observation, we find that all the relevant slices are characterized by $M = 0$. Figure 12 shows the time evolution of the 2D maximum coordinates obtained by approximating, as done for Fig. 8, the average power transfer near the grid points by a second-order polynomial and looking for the closest maximum. In the following, we will focus our analysis on two slices only: the slice dominating the power exchange during the linear stage ($M = 0, U_{eq0} = 1.92$) and the slice dominating around $t = 564.0 \tau_{A0}$ ($M = 0, U_{eq0} = 1.63$), after the contribution of the former one has strongly decreased. We will conventionally refer to these two slices as the linear slice and the nonlinear one.

V. HAMILTONIAN MAPPING ANALYSIS

In order to analyze the behavior of each of the two phase-space structures identified in Sec. IV, we apply the Hamiltonian-mapping approach, described in previous papers. The slice is sampled by a large number of test particles, initialized with the following coordinates: $r = r_{min} - r_{max}$ with $[r_{min}, r_{max}]$ being a radial interval covering the whole radial domain of interest (here, $r_{min} = 0.4a, r_{max} = 0.65a$), $\theta = 0, \phi = 0 - 2\pi, M = 0, U = U_{eq0}$, with $U_{eq0}$ being the value characterizing the slice. As all particles are initialized at $\theta = 0$, the initial values of $r$ and $U$ assume the meaning of the equatorial coordinates $r_{eq}$ and $U_{eq}$. Test particles are pushed in the fields computed self-consistently, at each time step, in the considered simulation.
coordinates are detected at each crossing of the equatorial plane, and the wave-particle phase \( \Theta \equiv \int_{t_j}^{t_{j+1}} d\theta(t') + m\theta - n\phi \) is calculated. Note that we have defined the phase taking into account that the mode frequency is not a constant. The wave phase at the \( j \)th crossing will then be given by \( \Theta_j = \int_{t_j}^{t_{j+1}} d\theta(t') + 2\pi jm\sigma - n\phi_j \), where \( \sigma = \text{sign}(U) \), \( t_j \) is the time at which the crossing occurs, and \( \phi_j \) is the corresponding toroidal angle. The resonance condition can be written as

\[
\Delta \Theta_j \equiv \Theta_j - \Theta_{j-1} = \int_{t_{j-1}}^{t_j} dt' \omega(t') + 2\pi m\sigma - n\Delta \phi_j = 2\pi k
\]

with \( k = 0, \pm 1, \pm 2, \ldots \) being the transit harmonic. Here, \( \Delta \phi_j \equiv \phi_j - \phi_{j-1} \). The average power exchanged with the mode along the poloidal orbit is also computed.

Figure 13 shows, for each test particle, a colored marker in the plane \((\Theta, r_{eq})\), where \( \Theta \equiv \Theta \text{ mod } 2\pi \). From Eq. (10), we see that the resonance conditions reads in terms of \( \Theta \)

\[
\Delta \Theta_j = 0.
\]

To get a clearer view of the evolution of particles, a companion marker is also drawn at \( \Theta + 2\pi \). Marker color is chosen according to the birth \( r_{eq} \) value of the particle. Three times are considered, relative to linear \((t = 100.8\tau_{A0})\), early-nonlinear \((t = 351.0\tau_{A0})\), and fully nonlinear stage of mode evolution \((t = 480.6\tau_{A0})\), respectively. In the unperturbed motion, \( r_{eq} \) is constant. Then, during the linear phase of the mode evolution [Fig. 13(a)], particle trajectories in the \((\Theta, r_{eq})\) plane essentially reduce to fixed points for \( r_{eq} = r_{eq0} \) (in this case, \( r_{eq0} \approx 0.544 \)), while they correspond to drift along the \( \Theta \) axis in the negative (positive) direction, for \( r_{eq} \) greater (less) than \( r_{eq0} \). As the mode amplitude grows [Fig. 13(b)], \( r_{eq} \) varies because of the mode-particle interaction (e.g., radial \( E \times B \) drift). Even particles that were initially resonant are brought out of resonance, drifting in phase and eventually undergoing an inversion of the drift in \( r_{eq} \). Particles that cross the \( r_{eq} = r_{eq0} \) line revert the phase drift as well. Thus, their trajectories are bounded, giving rise to the formation of an island-like structure in the \((\Theta, r_{eq})\) plane [Fig. 13(c)]. Its radial extension grows with mode amplitude, consistent with equations of motions accounting for radial \( E \times B \) drift and particle motion due to radial magnetic field perturbation. Particles outside the island undergo a secular drift in phase, as the \( E \times B \) drift is not able to cause them crossing the resonance radius.
The formation of the island mixes particles born on opposite sides of the resonance radius, causing a density flattening around this radius. This can be seen from Fig. 14, where the initial density profile and the flattened ones at different times are compared. At the same time, steepened negative density gradients emerge at the boundaries of this flattening region. This flattening process is the basis for mode saturation, or, more precisely, for the exhausting of the destabilizing contribution of the considered phase-space slice. If the mode were constrained to keep its frequency and radial structure constant, saturation would be reached as soon as the flattened-density region extends over the whole resonant interaction region. Such region, in turn, would be limited by the intersection between the resonance region and the radial extension of the mode. The former can be defined as the region, around the resonance, in which the mismatching between the resonance frequency and the mode one is smaller than a quantity of the order of the linear growth rate 

\[ |\omega - \omega_{\text{res}}| \lesssim \gamma_L. \]  

(12)

Its width \( \Delta r_{\text{res}} \) can then be estimated as

\[ \Delta r_{\text{res}} \propto \frac{\gamma_L}{|\partial \omega_{\text{res}} / \partial r|}. \]  

(13)

When the most stringent limit comes from the width of the resonance region, the saturation mechanism has been dubbed resonance detuning; when it comes from the mode width, radial decoupling. If the equilibrium allows the mode to modify its frequency and/or structure, several elements can make it able to further extract energy from particles, prolonging its growth. First, modification of the radial structure can prevent or delay radial decoupling, provided it is the relevant mechanism. Second, frequency modifications cause the resonance radius and the resonance region to move radially, thus managing to reach an area with a density profile not yet flattened. Third, different sensitivities of different resonances to the former two processes can lead to a succession of dominating resonances in driving the mode in the nonlinear stage, such that the exhaustion of the contribution of one of these resonances does not imply a definitive saturation of the mode. We will see in Sec. VIII that, in the present case, the latter two elements are essential in determining the nonlinear evolution of the mode.

VI. POWER TRANSFER RATE AND RESONANCE CONDITION

Before addressing saturation mechanisms in more detail, we want to investigate the behavior of the most destabilizing particles, within the same slice, and its connection with their ability to fulfill the resonance condition.

Figure 15 shows plots analogous to those presented in Fig. 13. In this case, however, markers are colored according their "instantaneous" power-transfer rate (averaged over the last poloidal orbit). Two times are considered, in the linear phase and in the nonlinear one, respectively. In the linear phase, the power transfer maxima are located at the resonance radius; in the nonlinear phase, consistent with island formation, density flattening and the appearance of steep
gradients, they separate from each other, drifting inwards and outward, respectively. We wonder whether the most destabilizing particles at $t = 248.4\tau_{A0}$ (represented by the red markers) are the same that drive the mode at $t = 419.4\tau_{A0}$. To answer this question, we plot in Fig. 16 the same markers shown in Fig. 15, fixing their color according to the power-transfer rate they have at $t = 248.4\tau_{A0}$. Figure 16(a) is, by definition, coincident with Fig. 15(a). Figure 16(b) shows the distribution of the $t = 248.4\tau_{A0}$ most destabilizing particles. Comparing it with Fig. 15(b), it is evident that the mode is driven by different particle groups at the two considered times.

Since a particle does not permanently belong to the group of the most destabilizing particles, we now want to check whether the power exchange between the mode and a certain particle is limited to only a short time interval or can recur later. To this aim, we follow (Fig. 17) a small group of test particles in the plane $(\Theta, r_{eq})$. Particles are chosen by selecting the most destabilizing ones at a given time. Markers are colored according to the same prescription as in Fig. 15. All positions recorded at different times are plotted, in order to get information about particle trajectories in that plane. Several elements are noteworthy. First, the same particles can assume, at different times, the role of driving or damping particles. In the former case, particles move radially inward; in the latter case, outward. In both situations, particles satisfy the resonance condition, Eq. (11), quite well and, indeed, they are wave-trapped particles [Fig. 17(b)]. Second, the time interval in which a particle is in resonance with the mode is of the order of few poloidal orbits. Third, particle radial excursion is of the order of the island

**FIG. 15.** Power transfer structures in the plane $(\Theta, r_{eq})$. Each marker is colored according to its “instantaneous” power-transfer rate (averaged over the last poloidal orbit). Two times are considered: $t = 248.4\tau_{A0}$ (a) and $t = 419.4\tau_{A0}$ (b).

**FIG. 16.** Same as Fig. 15, but with markers colored according to their power-transfer rate at $t = 248.4\tau_{A0}$. Two times are considered: $t = 248.4\tau_{A0}$ (a) and $t = 419.4\tau_{A0}$ (b). The frame (a) is, by definition, coincident with Fig. 15(a). Comparing the frame (b) with Fig. 15(b) shows that the mode is not driven, at different times, by the same particle clusters.
width. Fourth, trajectories show an overall inward radial drift, corresponding to an analogous drift of the resonance radius [Fig. 17(c)]; this will be discussed in Secs. VII and VIII. We observe, here, that the radial displacement of the orbit in a bounce time is of the same order of magnitude as its width; this shows that chirping in this case has a non-adiabatic character. These facts are better shown in Fig. 18: the radial position and the power transfer are reported, vs time, along with the wave-phase variation, for the most destabilizing of the particles considered in Fig. 17. It is easy to see that maximum power transfer (alternatively, positive and negative) and radial excursion (respectively, inward and outward) occur when the particle is in phase with the mode \( \Delta \Theta = \Delta \Theta_{k=-1} \) close to zero). The overall inward drift of the radial motion is also evident; it can also be appreciated looking at the inward drift of the island, shown in Fig. 19.

VII. TRAPPING/DE-TRAPPING PROCESS

The evolution of the island structure shown in Fig. 19 can be analyzed as follows: after dividing test particles into bands according to their birth \( r_{eq} \) values, as done in Fig. 19 (but with a finer subdivision), we conventionally consider a band as “wave trapped” if at least one of its particles has its last phase variation smaller than a certain value (0.015, in this case), corresponding to the resonance condition fairly well satisfied. Figure 20 reports, at each time, the indices of wave-trapped bands. We observe a progressive, asymmetric (mainly inward) trapping of bands. From this analysis, a de-trapping process is not apparent. The reason is that a band is considered wave trapped if at least one of the particles it includes is fairly resonant. A different picture can be obtained by looking at the individual behavior of test particles. In Fig. 21, only markers corresponding to particles that have

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**FIG. 17.** Trajectories of a small group of test particles in the plane \((\Theta, r_{eq})\). Only particles yielding, at a certain time \((t = 415.8 \tau_{a0})\), more than 95% of the maximum power transfer at the same time are considered. Markers are colored according to their instantaneous power transfer level. Different frames refer to three different times: \(t = 469.8 \tau_{a0}\) (a), \(t = 563.4 \tau_{a0}\) (b) and \(t = 779.4 \tau_{a0}\) (c). All the previous-time positions are plotted, in order to get information about the whole trajectories.

**FIG. 18.** Time evolution of equatorial radial position (a) and power transfer (b), compared with that of \( \Delta \Theta_{k=-1} \), for the most destabilizing, at \( t = 415.8 \tau_{a0} \), of the particles, considered in Fig. 17. Here, \( \Delta \Theta_{k=-1} \) is the wave-phase variation given by Eq. 10 for \( k = -1 \).
already satisfied the above conventional resonance condition, getting then trapped by the wave, are plotted ("already-trapped" particles). Color is red for particles that appear still trapped in the wave ("actually-trapped" particles); it switches to blue (de-trapped particle) as the particle cumulates a phase drift greater than a certain conventional threshold ($2.5\pi$, in this case). The same two times considered in Fig. 19 are examined. We see that, at $t = 726.0\tau_{A0}$, a significant amount of particle de-trapping takes place on the outer side of the island. Here, the de-trapping phenomenon becomes apparent due to the fact that each particle is considered according to its specific dynamics, which can be different from that of closely born particles.

The continuous trapping/de-trapping phenomenon can also be observed by resorting to the Lagrangian coherent structure (LCS) technique.42–46 It is well known that these structures generalize dynamical patterns observed in autonomous and periodic systems to temporally aperiodic flows like the one that we are analyzing. In particular, as already shown in Refs.47 and 48, we can describe the shape of the structures enclosing instantaneously trapped particles and, thus, partition the phase space into different sub-domains where particle motion is qualitatively different, i.e., trapped/unbounded trajectories. The application of this technique yields the results shown in Fig. 22. For each marker, the forward Lyapunov exponent, $k_f$, and the backward one, $k_b$, are defined, respectively, as $k_f \equiv \log \left( \frac{\Delta r_f}{\Delta t} \right)$ and $k_b \equiv \log \left( \frac{d_f}{\Delta t} \right)$, where $d_f$ is the distance in the plane $(\Theta, r_{eq})$ at time $t + \tau$ ($\tau = 0, \pm \Delta t$), between a marker and its nearest neighbor at time $t$. In this figure, $t = 504.0\tau_{A0}$ and $\Delta t = 200.0\tau_{A0}$. Markers are colored according to the values of $k_f$ [Fig. 22(a)], $k_b$ [Fig. 22(b)] or the largest of the two [Fig. 22(c)]. Large values of the forward exponent identify the so-called repulsive lines [shown in Fig. 22(a)]; particles on opposite sides of such lines tend to diverge with increasing time. Large values of the backward exponent correspond to the attractive lines [Fig. 22(b)]; particles on opposite sides of these lines have converged to close positions from distant ones. The superposition of the two kind of lines [Fig. 22(c)] would correspond, in the case of constant mode frequency and amplitude, to the separatrix delimiting the island. In the present case (both amplitude and frequency depending on time), we observe that the attractive and repulsive lines do not match at the top and bottom of the island. The channels left open between the attractive and repulsive lines allow for an incoming particle flow from the lower
edge and an outgoing flow from the upper edge. The former yields particle trapping; the latter, particle de-trapping. This fact can be better appreciated from Fig. 23, showing the LCSs at \( t = 558.0 T_{A0} \) with the same choice of \( \Delta t \) as in Fig. 22. The trajectories of some particles are also shown. Particles have been chosen in such a way that they are located, at \( t = 558.0 T_{A0} \), just below the lower channel (the blue big dot), within the channel (violet and light blue big dots) and just above the channel (the black big dot). We observe that, at the reference time, the first and second particles lay on the opposite sides of a repulsive line; the second and third particles are not separated neither by repulsive nor by attractive lines; the third and fourth particles lay on the opposite sides of an attractive line. Coherent with such observation, the trajectories of the different particles show that the first and second particles come from relatively close positions in the unbounded-orbit region, and tend to diverge from each other, as the first particle maintains an unbounded orbit, while the second gets trapped. The second and third particles tend to move together (both come from unbounded-orbit region; both get trapped). The third and fourth particles come from well separated positions (the fourth particle comes from the bounded-orbit region) and converge toward close positions (both in the bounded-orbit region). This results, that is, substantially independent of the specific particles examined, show that particles that enter the lower channel become trapped. Figure 24 shows the same LCSs, along with the trajectories of four different particles, located, at the reference time, just below (the first particle), within (second and third particles) and just above (the fourth particle) the upper channel.
It can be seen that the second and third particles become de-trapped, while the first particle remains trapped and the fourth one maintains an unbounded orbit. These findings can be easily justified repeating, *mutatis mutandis*, the above considerations concerning the relative positions of each couple with respect to the repulsive and attractive lines. We then arrive at the specular conclusion: particles that enter the lower channel become trapped.

The possibility of distinguishing actually trapped particles from already-trapped ones (Fig. 21) allows us to evidence a subtle distinction between the island structure and the density-flattening region. Indeed, the island only includes, by definition, actually trapped particles. However, particles that become de-trapped continue to take part to the density flattening, as they contribute to the mixing phenomenon around the resonance radius. Then, the density flattening region extends over the whole region including the set of already-trapped particles (both actually trapped and de-trapped particles). This can be seen from Fig. 25, which shows the time evolution of the inner and outer radial boundaries of the regions including the set of actually trapped particles and that of the already trapped particles, respectively. The radial position of the maximum negative density gradients, delimiting the density-flattening region, are also shown. Despite the numerical noise affecting the computation of density gradients, it can be seen that identifying the boundaries of the already-trapped set with the radial position of the maximum gradients is a fair approximation. Then, in the following, we will assume that such boundaries adequately represent the steep-gradient positions and, then, the boundaries of the flattened-density region. Conversely, we have observed that, due to the inward island drift (consequent to the downward frequency chirping), trapping of new particles takes place mainly on the inner side of the island structure; particle de-trapping, on the outer side only. Therefore, the inner boundary is the same for the region including the actually trapped particles (the island) and that including the already trapped ones (the flattening region); outer boundaries differ instead from each other. In this respect, as far as the inner boundary of the density-flattening region is concerned, we can identify its radial position (and that of the corresponding density gradient) with the inner boundary of the island as well.
As last remark, we want to emphasize that this trapping and de-trapping process that accompanies chirping fluctuations has been recently pointed out by hybrid simulations and corresponding theoretical framework of "chorus emission" in the Earth’s magnetosphere, suggesting the universal nature of the underlying nonlinear dynamics.

VIII. FREQUENCY CHIRPING AND RESONANCE EVOLUTION

We want to investigate how the collective power transfer is related to the fulfillment of the resonance condition and the evolution of the free-energy source (that is, the radial density gradient). Let us start from considering the linear test-particle set.

We have already observed that, during the linear stage [Fig. 15(a)], the maximum power transfer is yielded by resonant particles, around \( r_{\text{eq}} \approx 0.54a \). In the early nonlinear stage, the island formation and growth and the corresponding flattening of the density profile around the resonance radius is accompanied by the splitting of the power transfer maximum into two separate maxima, each of them following one of the two steep negative density gradients delimiting the flattening region and moving inwards and outward, respectively [Fig. 15(b)].

Figure 26 compares the radial boundaries of the island and the density-flattening regions (shown in Fig. 25) with those of the resonance region, conventionally defined as the region containing particles that satisfy the condition \( |\Delta \Theta| \leq 2\gamma_{ij}(t_{j}\tau_{p,j})_{\text{ref}} \sim 4\pi\gamma_{jh}R_{0}/U_{\text{eq}} \approx 0.153 \pi \); this choice corresponds to include in the resonance width up to one fifth of the resonance peak. Some increase in the width of the resonance region is observed. Such widening keeps the drifting gradients in the resonance region for a slightly longer time than the "linear" resonance width would be able to do, providing further drive to the mode. This would however go quite soon to an end if the mode were not able to vary its frequency. As soon as the density-flattening region completely covered the broadened resonance region, the drive would be exhausted. In the present case, however, the frequency can chirp down. This moves the resonance radius and the entire resonance region inward. The symmetry of the two large density gradients is, thus, broken: the inner gradient can contribute to resonant drive much more efficiently than the outer one. Conversely, wave-trapped trajectories [Figs. 17(c) and 19] and the wave-trapping process (Fig. 20) will follow the new resonance radius, along with the island structure (cf. the black lines in Fig. 25), and the inner gradient will further drift inward, requesting further frequency variation.

We can expect this process to continue until the frequency change becomes either detrimental to the growth of the mode from the point of view of the drive/damping balance, or unable to cause a significant inward shift of the resonance radius and/or the resonance region. The latter fact effectively occurs, as seen from Fig. 27, in which all markers are reported, at four different times, in the plane \((r_{\text{eq}}, \Delta \Theta)\), with \( j \) referring, for each marker, at the last poloidal orbit completed. We observe that the resonance widening (corresponding to the widening of the red region around \( \Delta \Theta = 0 \)) is favored by the spindle-shaping of the resonant structure; contrasted by its migration toward the left-bottom corner of the plot (due to the downward frequency chirping). At a certain time, this migration, which moves the resonance condition toward the narrow spindle end, prevails, and the inward drift of the resonance region suddenly slows down.

This is clearly shown in Fig. 28, which compares the trajectories, in the space \((\nu, r_{\text{eq}})\) of the inner boundary of the density-flattening region and that of the resonance region. The time evolution of the mode frequency is also reported to make it clear how these trajectories are traveled over time. It is possible to see that, as the frequency falls below \( \nu_{\text{res}} \approx 0.18 \) (that is, at \( t \approx 500\tau_{R,0} \)), the inner gradient remains irretrievably outside the resonance region. Correspondingly, the inner maximum of the power transfer loses its prevalence, as shown in Fig. 29: the same boundaries as in Fig. 26 are plotted, along with the radial coordinates of the time dependent maximum of the power transfer and the time evolution of the slice-integrated power. No further frequency decrease is able to prolong the relevance of the destabilizing contribution yielded by the linear slice, and the mode has to tap to a different slice in order to extract more power from the energetic particles. In the following, we examine the behavior of the nonlinear slice; that is, the slice that takes the role of most destabilizing slice after the linear one has exhausted its drive capability.

The first fact we observe is that, for the nonlinear slice, the island formation is delayed when compared to the linear one. This can be seen from Fig. 30, where the plot of Fig. 13(c), relative to the linear slice, is compared with the plots obtained, for the nonlinear slice, at the same time \( t = 480\tau_{R,0} \) and at a later time \( t = 554.4\tau_{R,0} \). The delay in the island formation can also be observed from Fig. 31, which compares the time evolution of the island radial width for the two slices. The second observation (Fig. 32) is that during the linear stage the maximum power transfer of the nonlinear slice neither occurs around the resonance radius, nor is directly related to the density gradient: it is influenced by the fluctuating field localization determined by the interaction with the linear slice. Only later, because of resonance-region widening and frequency chirping, the maximum power transfer matches a full resonance condition and appears dominated by the inner density gradient. The third relevant element shown by Fig. 32 is that the de-trapping is negligible, in the considered time interval, for such slice, so that even the outer actually trapped particle boundary...
and the already trapped particle one essentially coincide. Finally, and more important, the decoupling between density-flattening and resonance-region boundaries is not only delayed, but also less pronounced than in the linear-slice case, as shown in Fig. 33. This is consistent with the fact that the inner power-transfer maximum maintains its prevalence, as shown in Fig. 32, for a longer time and that the power transfer yielded by the slice, after reaching its maximum, falls down only at \( t = 600 \, \sigma_{\text{AO}} \).

**IX. SUMMARY AND DISCUSSION**

The dynamics of a single-\( n \) chirping Alfvén mode has been investigated. We have shown how adopting a coordinate system including two constants of particle motion allows one to analyze the nonlinear evolution of the system in terms of isolated resonances.

The evolution of the particle-mode power exchange shows that a succession of different resonances takes place in driving the mode. We have identified two phase-space slices, which yield the dominant contribution to mode destabilization during the linear and nonlinear phases, respectively. The analysis of the dynamics of these slices has been performed by Hamiltonian-mapping test-particle techniques. Wave-particle trapping yields the formation of a relatively closed island-like structure around the resonance radius, characterized by mixing of particles originating from the opposite sides of that radius. A density-flattening region is then produced, delimited by steepened (negative) density gradients. If frequency were constant, saturation
would occur as the density gradients reach the limit of the resonance region. With decreasing frequency, however, this region drifts inward and keeps the inner gradient within the region where the power transfer is effective, helped by a significant resonance widening. The island then grows around the new resonance radius, by trapping new particles on its inner boundary, while other particles get de-trapped from the outer one. De-trapped particles still continue to contribute to the density flattening, as they have previously taken part in the mixing phenomenon. While the inner density gradient is still associated with the inner boundary of the island, the outer gradient, located at the outer boundary of a wider region, is no longer linked to the island and falls definitively out of the resonance region. The inward drift of the island and its further growth cause the inner gradient to drift inward as well; to further maintain the drive capability of the slice, further frequency decreasing is needed. The process continues as long as the frequency change is effective in causing an inward shift of the resonance region able to recapture the density gradient. At a certain time, frequency chirping causes the widening of the resonance region to saturate; its inner boundary is no longer able to precede the inner density gradient, and the process goes to an end. After that time, the drive yielded by the considered slice decreases. To further extract energy from the energetic particles, the mode has to resort to the interaction with a different resonant structure (the “nonlinear slice”), using, if needed, additional frequency variations. Such slice is poorly resonant during the linear phase. It becomes fully resonant because of frequency chirping and resonance widening. The corresponding island formation is delayed with respect to the linear slice. Moreover, its resonance widening is more effective than that of the linear set, and it is able to prolong the permanence of the large density gradient in the resonance region. This gives the mode further drive and causes it to reach larger amplitudes than that driven by the linear slice.

**FIG. 28.** Trajectories, in the space \((\omega t, r_{eq})\) of the inner boundary of the density-flattening region (red) and that of the resonance region (blue). The time evolution of the mode frequency is also reported (green dashed line).

**FIG. 29.** Same boundaries as in Fig. 28, along with the radial coordinates of the time dependent maximum of the power transfer (orange dots) and the time evolution of the slice-integrated power (green line; cf. Fig. 11).

**FIG. 30.** Test-particle markers in the \((\Theta, r_{eq})\) at \(t = 480.6\tau_{A0}\) for the linear slice (a) and for the nonlinear one (b). The plot relative to the nonlinear slice at a later time \((t = 554.4\tau_{A0})\) is also shown (c). Marker colors follow the recipe adopted in Fig. 13. The delay in the island formation for the nonlinear slice is evident.
In this paper, the analysis has been focused on the drive evolution during the "early" nonlinear stage; that is, the stage that brings to the first fall of the destabilizing contribution of the linearly dominant resonant structure (the linear slice) and to the growth and successive fall of the nonlinear-slice contribution. We have not addressed the subsequent rich and complicate evolution. Moreover, we have looked at the drive evolution, without investigating the nonlinear damping mechanisms. In this respect, our comprehension of the nonlinear dynamics of a chirping mode is still far from being complete, although we hope to have successfully enlightened some relevant aspects.

We have also to stress that including a constant of motion (in our case, the initial value of the equatorial parallel velocity) in the coordinate system, though useful for treating a case characterized by varying frequency (and, then, by nonconservation of the global invariant $C$), does not represent a unique recipe. In particular, there is no a priori certainty that the nonlinear dynamics can be described in terms of the contribution of a single slice, characterized by unique values of $M$ and the chosen constant. In principle, it is possible that the existence of a particular constant of motion (global invariant or not) is able to yield such a result, but there is no known way to identify such a constant a priori. Thus, only a posteriori we can discern whether the particular choice made is suitable enough for our scopes: that is, whether it allows the relevant dynamics to be described in terms of a single slice or, more modestly, a few separate slices.

Finally, we observe that, as stated above, the rate of variation of $C$ is quite small in the case examined here. In this respect, neglecting such rate and treating $C$ as an invariant produces, in fact, very similar results: this means that the fluxes along $C$ do not play a relevant role in the nonlinear dynamics of the mode. A completely different conclusion can be obtained in the case of Alfvén spectra characterized by multiple toroidal mode numbers. This case will be the subject of future investigations.

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AUTHOR DECLARATIONS
Conflict of Interest
The authors have no conflicts to disclose.

DATA AVAILABILITY
The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: MODE–PARTICLE POWER EXCHANGE AND PHASE-SPACE COORDINATES

In this appendix, we want to show how the calculation of the mode–particle power exchange can be performed in the framework of a gyrokinetic simulation, using different sets of phase space coordinates.

A gyrokinetic simulation is performed by (a) discretizing the whole phase space into microscopic volumes, (b) representing each volume by a marker (macro-particle) that carries the total electric charge contained in the volume, and, at each time step, (c) computing the electromagnetic fields in terms of the momenta of the marker distribution function, and (d) updating the phase-space coordinates of each marker according to the fields it experiences. Here, we consider for simplicity the energetic-particle population, though all the following treatment can be immediately generalized to any species.

Let us adopt, for the phase space, the gyrocenter coordinate system \( Z \equiv (r, \theta, \phi, M, U, \vartheta) \). Discretizing the phase space allows us to represent the particle distribution function in terms of markers in the following way:

\[
D_{z \rightarrow \tilde{z}} F_{H}(Z, t) = \int d^{3}Z' D_{z \rightarrow \tilde{z}} F_{H}(Z', t) \delta^{(5)}(Z - Z')
\]

\[
= \sum_{l} \Delta_{l} F_{H}(Z_{l}(t)) \delta^{(5)}(Z - Z_{l}). \tag{A1}
\]

Here, \( Z_{l} = Z_{l}(t) \) represents the coordinates of the \( l \)th macroparticle, which evolve according to the equations of motion. Moreover

\[
\delta^{(5)}(Z - Z_{l}) \equiv \delta(r - r_{l})\delta(\theta - \theta_{l})\delta(\phi - \phi_{l})\delta(M - M_{l})\delta(U - U_{l})
\]

\[
\Delta_{l} \equiv \Delta_{l}(t) D_{z \rightarrow \tilde{z}}(Z_{l}(t)) \tag{A2}
\]

and

\[
\Delta_{l}(t) = [\Delta \Delta \Delta \partial \delta \Delta \partial \Delta M \Delta U]_{l} \tag{A3}
\]

Note that the integration in Eq. (A1) extends to a 5D space, as \( D_{z \rightarrow \tilde{z}} F_{H} \) does not depend on the gyrophase \( \vartheta \). Note also that \( \Delta_{l} \), the phase-space volume element corresponding to the \( l \)th macroparticle, is a constant of motion of the \( l \)th macro-particle (Liouville’s theorem).

The 3D power transfer from energetic particles to the wave is given by \(-d\mathcal{E}_{H}/dt\), where

\[
\mathcal{E}_{H} = \frac{1}{m_{H}} \int d^{3}Z D_{z \rightarrow \tilde{z}} F_{H} E
\]

is the total energy of fast particles. We can then write

\[
-\frac{d\mathcal{E}_{H}}{dt} = -\frac{1}{m_{H}} \int d^{3}Z \frac{\partial}{\partial t} \left( D_{z \rightarrow \tilde{z}} F_{H} E + D_{z \rightarrow \tilde{z}} F_{H} \frac{\partial E}{\partial t} \right)
\]

\[
= -\frac{1}{m_{H}} \int d^{3}Z \left[ \frac{\partial}{\partial Z} \left( D_{z \rightarrow \tilde{z}} F_{H} \frac{dZ'}{dt} \right) E + D_{z \rightarrow \tilde{z}} F_{H} \frac{\partial E}{\partial t} \right]
\]

\[
= -\frac{1}{m_{H}} \int d^{3}Z dZ dE \frac{dZ}{dt} \frac{\partial E}{\partial t} + \frac{dZ}{dt} \frac{\partial E}{\partial Z} \right]
\]

\[
= -\frac{1}{m_{H}} \int d^{3}Z dZ dE \frac{dE}{dt}.
\]

(A6)

After integrating over the gyrophase \( \vartheta \) and replacing the quantity \( D_{z \rightarrow \tilde{z}} F_{H} \) by its discrete form, Eq. (A1), we get

\[
-\frac{d\mathcal{E}_{H}}{dt} = -\frac{2\pi}{m_{H}} \int d^{3}Z \sum_{l} \Delta_{l} F_{H}(Z_{l}) \left( \frac{dE}{dt} \right) \delta^{(5)}(Z - Z_{l}). \tag{A7}
\]

In the following of this appendix, we will indicate by \( Z \) the coordinates \((r, \theta, \phi, M, U)\), neglecting the gyrophase.

The power-transfer density \( P(r, M, U) \), defined by Eq. (2) then reads

\[
P(r, M, U, t) \simeq -2\pi \sum_{l} \Delta_{l} F_{H}(Z_{l}) \left( \frac{dE}{dt} \right) \delta^{(5)}(U - U_{l}). \tag{A8}
\]

Let us now consider a different coordinate system, \( \tilde{Z} \). We can write the power transfer rate as

\[
-\frac{d\mathcal{E}_{H}}{dt} = -\frac{2\pi}{m_{H}} \int d^{3}Z \sum_{l} \Delta_{l} F_{H}(Z_{l}) \left( \frac{dE}{dt} \right) \delta^{(5)}(Z - Z_{l})
\]

\[
= -\frac{2\pi}{m_{H}} \int d^{3}Z \sum_{l} \Delta_{l} F_{H}(Z_{l}) \left( \frac{dE}{dt} \right) \delta^{(5)}(\tilde{Z} - Z_{l})
\]

\[
= -\frac{2\pi}{m_{H}} \int d^{3}Z \sum_{l} \Delta_{l} F_{H}(Z_{l}) \left( \frac{dE}{dt} \right) \delta^{(5)}(\tilde{Z} - \tilde{Z}_{l}). \tag{A9}
\]

Here, we have used the \( \delta \)-function property

\[
\int dx \delta(x - x) = \int dy \delta(y - y(x)) = \int dy \delta(y - y(x)). \tag{A10}
\]

Note that the transformation \( Z \rightarrow \tilde{Z} \) could be very complicated and possibly a time dependent one. Equation (A9), however, shows that, if we choose the new coordinates to perform the computation instead of the old ones, the only further quantities we need to know are the coordinates \( Z_{l} \) of each marker: we do not need to compute the Jacobian of the transformation (which could be, as we will see in the following, a difficult or even impossible task).

If the coordinate system \( Z \) includes, as \( Z \) does, the angles \( \theta \) and \( \phi \) (that is, \( Z_{2} = \theta \) and \( Z_{3} = \phi \)), we can define a power transfer density \( \tilde{P} \) in the reduced space \((\tilde{Z}_{1}, \tilde{Z}_{2}, \tilde{Z}_{3})\) as

\[
-\frac{d\mathcal{E}_{H}}{dt} = \frac{1}{m_{H}} \int d\tilde{Z}_{1} d\tilde{Z}_{2} d\tilde{Z}_{3} \tilde{P}(\tilde{Z}_{1}, \tilde{Z}_{2}, \tilde{Z}_{3}, t) \tag{A11}
\]
\[ \dot{p}(z_1, z_2, z_3, t) \equiv -2\pi \int d\theta d\phi D_{z,-z} D_{z,-z} F_U \frac{dE}{dt} \]
\[ \approx -2\pi \sum_{l} A_l F_U(z_l) \frac{dE}{dt} \int_{l} \times \delta(z_l - z_l^0) \delta(z_l^0 - 2z_l^0) \delta(z_l^0 - 2z_l^0). \] (A12)

Among all the possible choices of the phase-space coordinate system, some are particularly suited to yield meaningful information about particle dynamics. So, it can be worth replacing \( r \) and \( U \) by the toroidal angular momentum \( P_t \) and the kinetic energy \( E \). They are invariants of the unperturbed motion, and this allows to immediately obtain evidence, in the evolution of particle coordinates, of the effects of the mode–particle interaction.

A useful alternative is to choose the equatorial coordinates \( r_{eq} \) and \( U_{eq} \). These coordinates, as defined in Sec. II, can be numerically computed, as functions of the \( Z \) coordinates, from the conservation of \( P_t, M, \) and \( E \) in the unperturbed motion. They represent only an approximation, of course, of the actual coordinates that a particle will present when effectively crossing the equatorial plane, as its motion is perturbed by fluctuating electromagnetic fields; however, it is a satisfactory approximation until the perturbation of particle orbits remains relatively small during a transit or bounce period. One of the merits of this choice is that the equatorial coordinates, along with the magnetic moment \( M \), are able to immediately identify unperturbed particle orbits, clearly separating passing particle contributions from trapped particle ones. Here, the concept of trapping refers to the particle motion in the equilibrium magnetic field. Trapped-particle coordinates satisfy the following condition:

\[ \psi_{eq}(t) = \frac{m_i c}{4\pi e R_0} R_0 U \] for \( R_0 \equiv R_0 + r_{eq} \).

REFERENCES

34. Here and in the Appendix, we use this representation of the \( Z \) coordinate system to emphasise that the second (\( \theta \)), third (\( \phi \)) and sixth coordinate (\( \theta' \)) of the original \( Z \) system are left unmodified, while the other three coordinates (\( \theta'' \) and \( \phi'' \)) can be chosen in any way, even without a one-to-one correspondence to \( r, M \) and \( U \).
37. In the following, we focus on the energetic-ion population. Any reference to particles or related quantities must be understood, unless otherwise specified, as referring to that population.
40. Using the expression “density flattening” we only mean that the negative density gradient is reduced; not necessarily that it vanishes. In certain cases, however, it can vanish or even invert its sign.
41. Note that here we are analysing the dynamics of particles belonging to the same phase-space slice, different from what presented in Figs. 8 and 12.
43. The expression “density flattening” is usually meant to mean that the negative density gradient is reduced; not necessarily that it vanishes. In certain cases, however, it can vanish or even invert its sign.
44. Particle trapping also occurs, to a smaller extent, on the outer side of the island, because of the mode-amplitude growth.
45. In this case, the width of the resonance region is smaller than the mode width.