

# NAG Library Routine Document

## S21BBF

**Note:** before using this routine, please read the Users' Note for your implementation to check the interpretation of **bold italicised** terms and other implementation-dependent details.

### 1 Purpose

S21BBF returns a value of the symmetrised elliptic integral of the first kind, via the routine name.

### 2 Specification

**double precision** FUNCTION S21BBF(X, Y, Z, IFAIL)  
 INTEGER IFAIL  
**double precision** X, Y, Z

### 3 Description

S21BBF calculates an approximation to the integral

$$R_F(x, y, z) = \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}$$

where  $x, y, z \geq 0$  and at most one is zero.

The basic algorithm, which is due to Carlson (1979) and Carlson (1988), is to reduce the arguments recursively towards their mean by the rule:

$$\begin{aligned} x_0 &= \min(x, y, z), & z_0 &= \max(x, y, z), \\ y_0 &= \text{remaining third intermediate value argument.} \end{aligned}$$

(This ordering, which is possible because of the symmetry of the function, is done for technical reasons related to the avoidance of overflow and underflow.)

$$\begin{aligned} \mu_n &= (x_n + y_n + 3z_n)/3 \\ X_n &= (1 - x_n)/\mu_n \\ Y_n &= (1 - y_n)/\mu_n \\ Z_n &= (1 - z_n)/\mu_n \\ \lambda_n &= \sqrt{x_n y_n} + \sqrt{y_n z_n} + \sqrt{z_n x_n} \\ x_{n+1} &= (x_n + \lambda_n)/4 \\ y_{n+1} &= (y_n + \lambda_n)/4 \\ z_{n+1} &= (z_n + \lambda_n)/4 \end{aligned}$$

$\epsilon_n = \max(|X_n|, |Y_n|, |Z_n|)$  and the function may be approximated adequately by a fifth order power series:

$$R_F(x, y, z) = \frac{1}{\sqrt{\mu_n}} \left( 1 - \frac{E_2}{10} + \frac{E_2^2}{24} - \frac{3E_2 E_3}{44} + \frac{E_3}{14} \right)$$

where  $E_2 = X_n Y_n + Y_n Z_n + Z_n X_n$ ,  $E_3 = X_n Y_n Z_n$ .

The truncation error involved in using this approximation is bounded by  $\epsilon_n^6/4(1 - \epsilon_n)$  and the recursive process is stopped when this truncation error is negligible compared with the **machine precision**.

Within the domain of definition, the function value is itself representable for all representable values of its arguments. However, for values of the arguments near the extremes the above algorithm must be modified so as to avoid causing underflows or overflows in intermediate steps. In extreme regions arguments are prescaled away from the extremes and compensating scaling of the result is done before returning to the calling program.

## 4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Carlson B C (1979) Computing elliptic integrals by duplication *Numerische Mathematik* **33** 1–16

Carlson B C (1988) A table of elliptic integrals of the third kind *Math. Comput.* **51** 267–280

## 5 Parameters

- |    |                             |              |
|----|-----------------------------|--------------|
| 1: | X – <i>double precision</i> | <i>Input</i> |
| 2: | Y – <i>double precision</i> | <i>Input</i> |
| 3: | Z – <i>double precision</i> | <i>Input</i> |

*On entry:* the arguments  $x$ ,  $y$  and  $z$  of the function.

*Constraint:*  $X, Y, Z \geq 0.0$  and only one of  $X$ ,  $Y$  and  $Z$  may be zero.

- |    |                 |                     |
|----|-----------------|---------------------|
| 4: | IFAIL – INTEGER | <i>Input/Output</i> |
|----|-----------------|---------------------|

*On entry:* IFAIL must be set to 0, –1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

*On exit:* IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value –1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value –1 or 1 is used it is essential to test the value of IFAIL on exit.**

## 6 Error Indicators and Warnings

If on entry IFAIL = 0 or –1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, one or more of  $X$ ,  $Y$  and  $Z$  is negative; the function is undefined.

IFAIL = 2

On entry, two or more of  $X$ ,  $Y$  and  $Z$  are zero; the function is undefined. On soft failure, the routine returns zero.

## 7 Accuracy

In principle S21BBF is capable of producing full *machine precision*. However round-off errors in internal arithmetic will result in slight loss of accuracy. This loss should never be excessive as the algorithm does not involve any significant amplification of round-off error. It is reasonable to assume that the result is accurate to within a small multiple of the *machine precision*.

## 8 Further Comments

You should consult the S Chapter Introduction which shows the relationship of this function to the classical definitions of the elliptic integrals.

If two arguments are equal, the function reduces to the elementary integral  $R_C$ , computed by S21BAF.

## 9 Example

This example simply generates a small set of non-extreme arguments which are used with the routine to produce the table of low accuracy results.

### 9.1 Program Text

```
*      S21BBF Example Program Text
*      Mark 14 Revised. NAG Copyright 1989.
*      .. Parameters ..
      INTEGER          NOUT
      PARAMETER        (NOUT=6)
*      .. Local Scalars ..
      DOUBLE PRECISION RF, X, Y, Z
      INTEGER          IFAIL, IX
*      .. External Functions ..
      DOUBLE PRECISION S21BBF
      EXTERNAL          S21BBF
*      .. Executable Statements ..
      WRITE (NOUT,*) 'S21BBF Example Program Results'
      WRITE (NOUT,*)
      WRITE (NOUT,*) '      X      Y      Z      S21BBF  IFAIL'
      WRITE (NOUT,*)
      DO 20 IX = 1, 3
         X = IX*0.5D0
         Y = (IX+1)*0.5D0
         Z = (IX+2)*0.5D0
         IFAIL = 1

*
         RF = S21BBF(X,Y,Z,IFAIL)
*
         IF (IFAIL.GE.0) THEN
            WRITE (NOUT,99999) X, Y, Z, RF, IFAIL
         ELSE
            WRITE (NOUT,99998) IFAIL
            GO TO 40
         END IF
      20 CONTINUE
*
      40 CONTINUE
*
99999 FORMAT (1X,3F7.2,F12.4,I5)
99998 FORMAT (1X,' ** S21BBF returned with IFAIL = ',I5)
      END
```

### 9.2 Program Data

None.

### 9.3 Program Results

S21BBF Example Program Results

| X    | Y    | Z    | S21BBF | IFAIL |
|------|------|------|--------|-------|
| 0.50 | 1.00 | 1.50 | 1.0281 | 0     |
| 1.00 | 1.50 | 2.00 | 0.8260 | 0     |
| 1.50 | 2.00 | 2.50 | 0.7116 | 0     |

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