NAG Library Routine Document

F07FBF (DPOSVX)

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

F07FBF (DPOSVX) uses the Cholesky factorization

$$A = U^{\mathrm{T}}U \qquad \text{or} \qquad A = LL^{\mathrm{T}}$$

to compute the solution to a real system of linear equations

$$AX = B,$$

where A is an n by n symmetric positive-definite matrix and X and B are n by r matrices. Error bounds on the solution and a condition estimate are also provided.

2 Specification

```
 SUBROUTINE F07FBF(FACT, UPLO, N, NRHS, A, LDA, AF, LDAF, EQUED, S, B,
1                  LDB, X, LDX, RCOND, FERR, BERR, WORK, IWORK, INFO)
 INTEGER           N, NRHS, LDA, LDAF, LDB, LDX, IWORK(N), INFO
 double precision  A(LDA,*), AF(LDAF,*), S(*), B(LDB,*), X(LDX,*), RCOND,
1                  FERR(NRHS), BERR(NRHS), WORK(3*N)
 CHARACTER*1       FACT, UPLO, EQUED
```

The routine may be called by its LAPACK name ***dposvx***.

3 Description

F07FBF (DPOSVX) performs the following steps:

1. If FACT = 'E', real diagonal scaling factors, D_S, are computed to equilibrate the system:

   $$(D_S A D_S)\left(D_S^{-1} X\right) = D_S B.$$

   Whether or not the system will be equilibrated depends on the scaling of the matrix A, but if equilibration is used, A is overwritten by $D_S A D_S$ and B by $D_S B$.

2. If FACT = 'N' or 'E', the Cholesky decomposition is used to factor the matrix A (after equilibration if FACT = 'E') as $A = U^{\mathrm{T}}U$ if UPLO = 'U' or $A = LL^{\mathrm{T}}$ if UPLO = 'L', where U is an upper triangular matrix and L is a lower triangular matrix.

3. If the leading i by i principal minor is not positive-definite, then the routine returns with INFO $= i$. Otherwise, the factored form of A is used to estimate the condition number of the matrix A. If the reciprocal of the condition number is less than ***machine precision***, INFO $= \mathrm{N}+1$ is returned as a warning, but the routine still goes on to solve for X and compute error bounds as described below.

4. The system of equations is solved for X using the factored form of A.

5. Iterative refinement is applied to improve the computed solution matrix and to calculate error bounds and backward error estimates for it.

6. If equilibration was used, the matrix X is premultiplied by D_S so that it solves the original system before equilibration.

4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphiahttp://www.netlib.org/lapack/lug

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Higham N J (2002) *Accuracy and Stability of Numerical Algorithms* (2nd Edition) SIAM, Philadelphia

5 Parameters

1: FACT – CHARACTER*1 *Input*

On entry: specifies whether or not the factorized form of the matrix A is supplied on entry, and if not, whether the matrix A should be equilibrated before it is factorized.

FACT = 'F'
AF contains the factorized form of A. If EQUED = 'Y', the matrix A has been equilibrated with scaling factors given by S. A and AF will not be modified.

FACT = 'N'
The matrix A will be copied to AF and factorized.

FACT = 'E'
The matrix A will be equilibrated if necessary, then copied to AF and factorized.

Constraint: FACT = 'F', 'N' or 'E'.

2: UPLO – CHARACTER*1 *Input*

On entry: if UPLO = 'U', the upper triangle of A is stored.

If UPLO = 'L', the lower triangle of A is stored.

Constraint: UPLO = 'U' or 'L'.

3: N – INTEGER *Input*

On entry: n, the number of linear equations, i.e., the order of the matrix A.

Constraint: $\mathrm{N} \geq 0$.

4: NRHS – INTEGER *Input*

On entry: r, the number of right-hand sides, i.e., the number of columns of the matrix B.

Constraint: $\mathrm{NRHS} \geq 0$.

5: A(LDA,$*$) – ***double precision*** array *Input/Output*

Note: the second dimension of the array A must be at least $\max(1,\mathrm{N})$.

On entry: the n by n symmetric matrix A.

If FACT = 'F' and EQUED = 'Y', A must have been equilibrated by the scaling factor in S as D_SAD_S.

If UPLO = 'U', the upper triangular part of A must be stored and the elements of the array below the diagonal are not referenced.

If UPLO = 'L', the lower triangular part of A must be stored and the elements of the array above the diagonal are not referenced.

On exit: if FACT = 'F' or 'N', or if FACT = 'E' and EQUED = 'N', A is not modified.

If FACT = 'E' and EQUED = 'Y', A is overwritten by D_SAD_S.

6: LDA – INTEGER *Input*

On entry: the first dimension of the array A as declared in the (sub)program from which F07FBF (DPOSVX) is called.

Constraint: LDA $\geq \max(1, \mathrm{N})$.

7: AF(LDAF,$*$) – ***double precision*** array *Input/Output*

Note: the second dimension of the array AF must be at least $\max(1, \mathrm{N})$.

On entry: if FACT = 'F', AF contains the triangular factor U or L from the Cholesky factorization $A = U^{\mathrm{T}}U$ or $A = LL^{\mathrm{T}}$, in the same storage format as A. If EQUED $\neq$ 'N', AF is the factorized form of the equilibrated matrix D_SAD_S.

On exit: if FACT = 'N', AF returns the triangular factor U or L from the Cholesky factorization $A = U^{\mathrm{T}}U$ or $A = LL^{\mathrm{T}}$ of the original matrix A.

If FACT = 'E', AF returns the triangular factor U or L from the Cholesky factorization $A = U^{\mathrm{T}}U$ or $A = LL^{\mathrm{T}}$ of the equilibrated matrix A (see the description of A for the form of the equilibrated matrix).

8: LDAF – INTEGER *Input*

On entry: the first dimension of the array AF as declared in the (sub)program from which F07FBF (DPOSVX) is called.

Constraint: LDAF $\geq \max(1, \mathrm{N})$.

9: EQUED – CHARACTER*1 *Input/Output*

On entry: if FACT = 'N' or 'E', EQUED need not be set.

If FACT = 'F', EQUED must specify the form of the equilibration that was performed as follows:

if EQUED = 'N', no equilibration;

if EQUED = 'Y', equilibration was performed, i.e., A has been replaced by D_SAD_S.

On exit: if FACT = 'F', EQUED is unchanged from entry.

Otherwise, if INFO ≥ 0, EQUED specifies the form of the equilibration that was performed as specified above.

Constraint: if FACT = 'F', EQUED = 'N' or 'Y'.

10: S($*$) – ***double precision*** array *Input/Output*

Note: the dimension of the array S must be at least $\max(1, \mathrm{N})$.

On entry: if FACT = 'N' or 'E', S need not be set.

If FACT = 'F' and EQUED = 'Y', S must contain the scale factors, D_S, for A; each element of S must be positive.

On exit: if FACT = 'F', S is unchanged from entry.

Otherwise, if INFO ≥ 0 and EQUED = 'Y', S contains the scale factors, D_S, for A; each element of S is positive.

11: B(LDB,$*$) – ***double precision*** array *Input/Output*

Note: the second dimension of the array B must be at least $\max(1, \mathrm{NRHS})$.

On entry: the n by r right-hand side matrix B.

On exit: if EQUED = 'N', B is not modified.

If EQUED = 'Y', B is overwritten by D_SB.

12: LDB – INTEGER *Input*

On entry: the first dimension of the array B as declared in the (sub)program from which F07FBF (DPOSVX) is called.

Constraint: $\text{LDB} \geq \max(1, \text{N})$.

13: X(LDX,$*$) – ***double precision*** array *Output*

Note: the second dimension of the array X must be at least $\max(1, \text{NRHS})$.

On exit: if $\text{INFO} = 0$ or $\text{INFO} = \text{N} + 1$, the n by r solution matrix X to the original system of equations. Note that the arrays A and B are modified on exit if EQUED = 'Y', and the solution to the equilibrated system is $D_S^{-1}X$.

14: LDX – INTEGER *Input*

On entry: the first dimension of the array X as declared in the (sub)program from which F07FBF (DPOSVX) is called.

Constraint: $\text{LDX} \geq \max(1, \text{N})$.

15: RCOND – ***double precision*** *Output*

On exit: if no constraints are violated, an estimate of the reciprocal condition number of the matrix A (after equilibration if that is performed), computed as $\text{RCOND} = 1/\left(\|A\|_1\|A^{-1}\|_1\right)$.

16: FERR(NRHS) – ***double precision*** array *Output*

On exit: if $\text{INFO} = 0$ or $\text{N} + 1$, an estimate of the forward error bound for each computed solution vector, such that $\|\hat{x}_j - x_j\|_\infty / \|x_j\|_\infty \leq \text{FERR}(j)$ where $\hat{x}_j$ is the jth column of the computed solution returned in the array X and x_j is the corresponding column of the exact solution X. The estimate is as reliable as the estimate for RCOND, and is almost always a slight overestimate of the true error.

17: BERR(NRHS) – ***double precision*** array *Output*

On exit: if $\text{INFO} = 0$ or $\text{N} + 1$, an estimate of the component-wise relative backward error of each computed solution vector $\hat{x}_j$ (i.e., the smallest relative change in any element of A or B that makes $\hat{x}_j$ an exact solution).

18: WORK($3 \times \text{N}$) – ***double precision*** array *Workspace*

19: IWORK(N) – INTEGER array *Workspace*

20: INFO – INTEGER *Output*

On exit: $\text{INFO} = 0$ unless the routine detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the routine:

$\text{INFO} < 0$

If $\text{INFO} = -i$, the ith argument had an illegal value. An explanatory message is output, and execution of the program is terminated.

$\text{INFO} > 0$ and $\text{INFO} \leq \text{N}$

If $\text{INFO} = i$ and $i \leq \text{N}$, the leading minor of order i of A is not positive-definite, so the factorization could not be completed, and the solution has not been computed. $\text{RCOND} = 0$ is returned.

INFO $= \mathrm{N} + 1$

The triangular matrix U (or L) is nonsingular, but RCOND is less than ***machine precision***, meaning that the matrix is singular to working precision. Nevertheless, the solution and error bounds are computed because there are a number of situations where the computed solution can be more accurate than the value of RCOND would suggest.

7 Accuracy

For each right-hand side vector b, the computed solution x is the exact solution of a perturbed system of equations $(A+E)x = b$, where

if UPLO = 'U', $|E| \le c(n)\epsilon\left|U^{\mathrm{T}}\right||U|$;

if UPLO = 'L', $|E| \le c(n)\epsilon|L|\left|L^{\mathrm{T}}\right|$,

$c(n)$ is a modest linear function of n, and ϵ is the ***machine precision***. See Section 10.1 of Higham (2002) for further details.

If $\hat{x}$ is the true solution, then the computed solution x satisfies a forward error bound of the form

$$\frac{\|x - \hat{x}\|_\infty}{\|\hat{x}\|_\infty} \le w_c \operatorname{cond}(A, \hat{x}, b)$$

where $\operatorname{cond}(A, \hat{x}, b) = \left\|\left|A^{-1}\right|(|A||\hat{x}| + |b|)\right\|_\infty / \|\hat{x}\|_\infty \le \operatorname{cond}(A) = \left\|\left|A^{-1}\right||A|\right\|_\infty \le \kappa_\infty(A)$. If $\hat{x}$ is the jth column of X, then w_c is returned in BERR(j) and a bound on $\|x - \hat{x}\|_\infty / \|\hat{x}\|_\infty$ is returned in FERR(j). See Section 4.4 of Anderson *et al.* (1999) for further details.

8 Further Comments

The factorization of A requires approximately $\frac{1}{3}n^3$ floating-point operations.

For each right-hand side, computation of the backward error involves a minimum of $4n^2$ floating-point operations. Each step of iterative refinement involves an additional $6n^2$ operations. At most five steps of iterative refinement are performed, but usually only one or two steps are required. Estimating the forward error involves solving a number of systems of equations of the form $Ax = b$; the number is usually 4 or 5 and never more than 11. Each solution involves approximately $2n^2$ operations.

The complex analogue of this routine is F07FPF (ZPOSVX).

9 Example

This example solves the equations

$$AX = B,$$

where A is the symmetric positive-definite matrix

$$A = \begin{pmatrix} 4.16 & -3.12 & 0.56 & -0.10 \\ -3.12 & 5.03 & -0.83 & 1.18 \\ 0.56 & -0.83 & 0.76 & 0.34 \\ -0.10 & 1.18 & 0.34 & 1.18 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 8.70 & 8.30 \\ -13.35 & 2.13 \\ 1.89 & 1.61 \\ -4.14 & 5.00 \end{pmatrix}.$$

Error estimates for the solutions, information on equilibration and an estimate of the reciprocal of the condition number of the scaled matrix A are also output.

9.1 Program Text

```
*     F07FBF Example Program Text
*     Mark 21 Release. NAG Copyright 2004.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX
      PARAMETER        (NMAX=8)
      INTEGER          LDA, LDAF, LDB, LDX, NRHSMX
      PARAMETER        (LDA=NMAX,LDAF=NMAX,LDB=NMAX,LDX=NMAX,
     +                 NRHSMX=NMAX)
*     .. Local Scalars ..
      DOUBLE PRECISION RCOND
      INTEGER          I, IFAIL, INFO, J, N, NRHS
      CHARACTER        EQUED
*     .. Local Arrays ..
      DOUBLE PRECISION A(LDA,NMAX), AF(LDAF,NMAX), B(LDB,NRHSMX),
     +                 BERR(NRHSMX), FERR(NRHSMX), S(NMAX),
     +                 WORK(3*NMAX), X(LDX,NRHSMX)
      INTEGER          IWORK(NMAX)
*     .. External Subroutines ..
      EXTERNAL         DPOSVX, X04CAF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'F07FBF Example Program Results'
      WRITE (NOUT,*)
*     Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N, NRHS
      IF (N.LE.NMAX .AND. NRHS.LE.NRHSMX) THEN
*
*        Read the upper triangular part of A from data file
*
         READ (NIN,*) ((A(I,J),J=I,N),I=1,N)
*
*        Read B from data file
*
         READ (NIN,*) ((B(I,J),J=1,NRHS),I=1,N)
*
*        Solve the equations AX = B for X
*
         CALL DPOSVX('Equilibration','Upper',N,NRHS,A,LDA,AF,LDAF,EQUED,
     +               S,B,LDB,X,LDX,RCOND,FERR,BERR,WORK,IWORK,INFO)
*
         IF ((INFO.EQ.0) .OR. (INFO.EQ.N+1)) THEN
*
*           Print solution, error bounds, condition number and the form
*           of equilibration
*
            IFAIL = 0
            CALL X04CAF('General',' ',N,NRHS,X,LDX,'Solution(s)',IFAIL)
*
            WRITE (NOUT,*)
            WRITE (NOUT,*) 'Backward errors (machine-dependent)'
            WRITE (NOUT,99999) (BERR(J),J=1,NRHS)
            WRITE (NOUT,*)
            WRITE (NOUT,*)
     +        'Estimated forward error bounds (machine-dependent)'
            WRITE (NOUT,99999) (FERR(J),J=1,NRHS)
            WRITE (NOUT,*)
            WRITE (NOUT,*) 'Estimate of reciprocal condition number'
            WRITE (NOUT,99999) RCOND
            WRITE (NOUT,*)
            IF (EQUED.EQ.'N') THEN
               WRITE (NOUT,*) 'A has not been equilibrated'
            ELSE IF (EQUED.EQ.'Y') THEN
               WRITE (NOUT,*)
     +           'A has been row and column scaled as diag(S)*A*diag(S)'
            END IF
*
            IF (INFO.EQ.N+1) THEN
```

```
               WRITE (NOUT,*)
               WRITE (NOUT,*)
     +            'The matrix A is singular to working precision'
            END IF
         ELSE
            WRITE (NOUT,99998) 'The leading minor of order ', INFO,
     +        ' is not positive definite'
         END IF
      ELSE
         WRITE (NOUT,*) 'NMAX and/or NRHSMX too small'
      END IF
*
99999 FORMAT ((3X,1P,7E11.1))
99998 FORMAT (1X,A,I3,A)
      END
```

9.2 Program Data

```
F07FBF Example Program Data
  4      2                   :Values of N and NRHS
  4.16  -3.12   0.56  -0.10
         5.03  -0.83   1.18
                0.76   0.34
                       1.18 :End of matrix A
   8.70  8.30
 -13.35  2.13
   1.89  1.61
  -4.14  5.00               :End of matrix B
```

9.3 Program Results

```
 F07FBF Example Program Results

 Solution(s)
             1          2
 1      1.0000     4.0000
 2     -1.0000     3.0000
 3      2.0000     2.0000
 4     -3.0000     1.0000

 Backward errors (machine-dependent)
       6.7E-17    7.9E-17

 Estimated forward error bounds (machine-dependent)
       2.3E-14    2.3E-14

 Estimate of reciprocal condition number
       1.0E-02

 A has not been equilibrated
```