

o NAG Library Routine Document

F04BBF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of *bold italicised* terms and other implementation-dependent details.

1 Purpose

F04BBF computes the solution to a real system of linear equations $AX = B$, where A is an n by n band matrix, with k_l subdiagonals and k_u superdiagonals, and X and B are n by r matrices. An estimate of the condition number of A and an error bound for the computed solution are also returned.

2 Specification

```
SUBROUTINE F04BBF(N, KL, KU, NRHS, AB, LDAB, IPIV, B, LDB, RCOND,
1          ERRBND, IFAIL)
      INTEGER          N, KL, KU, NRHS, LDAB, IPIV(N), LDB, IFAIL
      double precision AB(LDAB,*), B(LDB,*), RCOND, ERRBND
```

3 Description

The LU decomposition with partial pivoting and row interchanges is used to factor A as $A = PLU$, where P is a permutation matrix, L is the product of permutation matrices and unit lower triangular matrices with k_l subdiagonals, and U is upper triangular with $(k_l + k_u)$ superdiagonals. The factored form of A is then used to solve the system of equations $AX = B$.

4 References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J J, Du Croz J J, Greenbaum A, Hammarling S, McKenney A and Sorensen D (1999) *LAPACK Users' Guide* (3rd Edition) SIAM, Philadelphia <http://www.netlib.org/lapack/lug>

Higham N J (2002) *Accuracy and Stability of Numerical Algorithms* (2nd Edition) SIAM, Philadelphia

5 Parameters

- 1: N – INTEGER *Input*
On entry: the number of linear equations n , i.e., the order of the matrix A .
Constraint: $N \geq 0$.
- 2: KL – INTEGER *Input*
On entry: the number of subdiagonals k_l , within the band of A .
Constraint: $KL \geq 0$.
- 3: KU – INTEGER *Input*
On entry: the number of superdiagonals k_u , within the band of A .
Constraint: $KU \geq 0$.
- 4: NRHS – INTEGER *Input*
On entry: the number of right-hand sides r , i.e., the number of columns of the matrix B .
Constraint: $NRHS \geq 0$.

- 5: AB(LDAB,*) – **double precision** array *Input/Output*
Note: the second dimension of the array AB must be at least $\max(1, N)$.
On entry: the n by n matrix A .
The matrix is stored in rows $k_l + 1$ to $2k_l + k_u + 1$; the first k_l rows need not be set, more precisely, the element A_{ij} must be stored in
$$AB(k_l + k_u + 1 + i - j, j) = A_{ij} \quad \text{for } \max(1, j - k_u) \leq i \leq \min(n, j + k_l).$$
See Section 8 for further details.
On exit: if IFAIL ≥ 0 , AB is overwritten by details of the factorization.
The upper triangular band matrix U , with $k_l + k_u$ superdiagonals, is stored in rows 1 to $k_l + k_u + 1$ of the array, and the multipliers used to form the matrix L are stored in rows $k_l + k_u + 2$ to $2k_l + k_u + 1$.
- 6: LDAB – INTEGER *Input*
On entry: the first dimension of the array AB as declared in the (sub)program from which F04BBF is called.
Constraint: $LDAB \geq 2 \times KL + KU + 1$.
- 7: IPIV(N) – INTEGER array *Output*
On exit: if IFAIL ≥ 0 , the pivot indices that define the permutation matrix P ; at the i th step row i of the matrix was interchanged with row IPIV(i). IPIV(i) = i indicates a row interchange was not required.
- 8: B(LDB,*) – **double precision** array *Input/Output*
Note: the second dimension of the array B must be at least $\max(1, NRHS)$. To solve the equations $Ax = b$, where b is a single right-hand side, B may be supplied as a one-dimensional array with length $LDB = \max(1, N)$.
On entry: the n by r matrix of right-hand sides B .
On exit: if IFAIL = 0 or $N + 1$, the n by r solution matrix X .
- 9: LDB – INTEGER *Input*
On entry: the first dimension of the array B as declared in the (sub)program from which F04BBF is called.
Constraint: $LDB \geq \max(1, N)$.
- 10: RCOND – **double precision** *Output*
On exit: if IFAIL ≥ 0 , an estimate of the reciprocal of the condition number of the matrix A , computed as $RCOND = 1 / (\|A\|_1 \|A^{-1}\|_1)$.
- 11: ERRBND – **double precision** *Output*
On exit: if IFAIL = 0 or $N + 1$, an estimate of the forward error bound for a computed solution $\hat{x}$, such that $\|\hat{x} - x\|_1 / \|x\|_1 \leq ERRBND$, where $\hat{x}$ is a column of the computed solution returned in the array B and x is the corresponding column of the exact solution X . If RCOND is less than **machine precision**, then ERRBND is returned as unity.
- 12: IFAIL – INTEGER *Input/Output*
On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.
On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0 . **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry $IFAIL = 0$ or -1 , explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

$IFAIL < 0$ and $IFAIL \neq -999$

If $IFAIL = -i$, the i th argument had an illegal value.

$IFAIL = -999$

Allocation of memory failed. The INTEGER allocatable memory required is N , and the **double precision** allocatable memory required is $3 \times N$. In this case the factorization and the solution X have been computed, but $RCOND$ and $ERRBND$ have not been computed.

$IFAIL > 0$ and $IFAIL \leq N$

If $IFAIL = i$, u_{ii} is exactly zero. The factorization has been completed, but the factor U is exactly singular, so the solution could not be computed.

$IFAIL = N + 1$

$RCOND$ is less than **machine precision**, so that the matrix A is numerically singular. A solution to the equations $AX = B$ has nevertheless been computed.

7 Accuracy

The computed solution for a single right-hand side, $\hat{x}$, satisfies an equation of the form

$$(A + E)\hat{x} = b,$$

where

$$\|E\|_1 = O(\epsilon)\|A\|_1$$

and ϵ is the **machine precision**. An approximate error bound for the computed solution is given by

$$\frac{\|\hat{x} - x\|_1}{\|x\|_1} \leq \kappa(A) \frac{\|E\|_1}{\|A\|_1},$$

where $\kappa(A) = \|A^{-1}\|_1\|A\|_1$, the condition number of A with respect to the solution of the linear equations. F04BBF uses the approximation $\|E\|_1 = \epsilon\|A\|_1$ to estimate $ERRBND$. See Section 4.4 of Anderson *et al.* (1999) for further details.

8 Further Comments

The band storage scheme for the array AB is illustrated by the following example, when $n = 6$, $k_l = 1$, and $k_u = 2$. Storage of the band matrix A in the array AB:

*	*	*	+	+	+
*	*	a_{13}	a_{24}	a_{35}	a_{46}
*	a_{12}	a_{23}	a_{34}	a_{45}	a_{56}
a_{11}	a_{22}	a_{33}	a_{44}	a_{55}	a_{66}
a_{21}	a_{32}	a_{43}	a_{54}	a_{65}	*

Array elements marked * need not be set and are not referenced by the routine. Array elements marked + need not be set, but are defined on exit from the routine and contain the elements u_{14} , u_{25} and u_{36} .

The total number of floating-point operations required to solve the equations $AX = B$ depends upon the pivoting required, but if $n \gg k_l + k_u$ then it is approximately bounded by $O(nk_l(k_l + k_u))$ for the factorization and $O(n(2k_l + k_u)r)$ for the solution following the factorization. The condition number estimation typically requires between four and five solves and never more than eleven solves, following the factorization.

In practice the condition number estimator is very reliable, but it can underestimate the true condition number; see Section 15.3 of Higham (2002) for further details.

The complex analogue of F04BBF is F04CBF.

9 Example

This example solves the equations

$$AX = B,$$

where A is the band matrix

$$A = \begin{pmatrix} -0.23 & 2.54 & -3.66 & 0 \\ -6.98 & 2.46 & -2.73 & -2.13 \\ 0 & 2.56 & 2.46 & 4.07 \\ 0 & 0 & -4.78 & -3.82 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4.42 & -36.01 \\ 27.13 & -31.67 \\ -6.14 & -1.16 \\ 10.50 & -25.82 \end{pmatrix}.$$

An estimate of the condition number of A and an approximate error bound for the computed solutions are also printed.

9.1 Program Text

```
*      F04BBF Example Program Text
*      Mark 21 Release. NAG Copyright 2004.
*      .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX, KLMAX, KUMAX, NRHSMX
      PARAMETER        (NMAX=8, KLMAX=4, KUMAX=4, NRHSMX=2)
      INTEGER          LDAB, LDB
      PARAMETER        (LDAB=2*KLMAX+KUMAX+1, LDB=NMAX)
*      .. Local Scalars ..
      DOUBLE PRECISION ERBND, RCOND
      INTEGER          I, IERR, IFAIL, J, K, KL, KU, N, NRHS
*      .. Local Arrays ..
      DOUBLE PRECISION AB(LDAB,NMAX), B(LDB,NRHSMX)
      INTEGER          IPIV(NMAX)
*      .. External Subroutines ..
      EXTERNAL         F04BBF, X04CAF, X04CEF
*      .. Intrinsic Functions ..
      INTRINSIC        MAX, MIN
*      .. Executable Statements ..
      WRITE (NOUT,*) 'F04BBF Example Program Results'
      WRITE (NOUT,*)
*      Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N, KL, KU, NRHS
      IF (N.LE.NMAX .AND. KL.LE.KLMAX .AND. KU.LE.KUMAX .AND. NRHS.LE.
+       NRHSMX) THEN
*
*      Read A and B from data file
*
*      K = KL + KU + 1
      READ (NIN,*) ((AB(K+I-J,J), J=MAX(I-KL,1), MIN(I+KU,N)), I=1,N)
      READ (NIN,*) ((B(I,J), J=1, NRHS), I=1,N)
*
*      Solve the equations AX = B for X
*
```

```

      IFAIL = 1
      CALL F04BBF(N,KL,KU,NRHS,AB,LDAB,IPIV,B,LDB,RCOND,ERRBND,IFAIL)
*
      IF (IFAIL.EQ.0) THEN
*
*       Print solution, estimate of condition number and approximate
*       error bound
*
*       IERR = 0
*       CALL X04CAF('General',' ',N,NRHS,B,LDB,'Solution',IERR)
*
*       WRITE (NOUT,*)
*       WRITE (NOUT,*) 'Estimate of condition number'
*       WRITE (NOUT,99999) 1.0D0/RCOND
*       WRITE (NOUT,*)
*       WRITE (NOUT,*)
+       'Estimate of error bound for computed solutions'
*       WRITE (NOUT,99999) ERBND
      ELSE IF (IFAIL.EQ.N+1) THEN
*
*       Matrix A is numerically singular. Print estimate of
*       reciprocal of condition number and solution
*
*       WRITE (NOUT,*)
*       WRITE (NOUT,*) 'Estimate of reciprocal of condition number'
*       WRITE (NOUT,99999) RCOND
*
*       WRITE (NOUT,*)
*       IERR = 0
*       CALL X04CAF('General',' ',N,NRHS,B,LDB,'Solution',IERR)
*
      ELSE IF (IFAIL.GT.0 .AND. IFAIL.LE.N) THEN
*
*       The upper triangular matrix U is exactly singular. Print
*       details of factorization
*
*       WRITE (NOUT,*)
*       IERR = 0
*       CALL X04CEF(N,N,KL,KL+KU,AB,LDAB,'Details of factorization',
+       IERR)
*
*       Print pivot indices
*
*       WRITE (NOUT,*)
*       WRITE (NOUT,*) 'Pivot indices'
*       WRITE (NOUT,99998) (IPIV(I),I=1,N)
      ELSE
*       WRITE (NOUT,99997) IFAIL
      END IF
      ELSE
*       WRITE (NOUT,*)
+       'One or more of NMAX, KLMAX, KUMAX or NRHSMX is too small'
      END IF
*
99999 FORMAT (6X,1P,E9.1)
99998 FORMAT ((1X,7I11))
99997 FORMAT (1X,' ** F04BBF returned with IFAIL = ',I5)
      END

```

9.2 Program Data

F04BBF Example Program Data

4	1	2	2	:Values of N, KL, KU and NRHS
-0.23	2.54	-3.66		
-6.98	2.46	-2.73	-2.13	
	2.56	2.46	4.07	
		-4.78	-3.82	:End of matrix A

```
4.42 -36.01
27.13 -31.67
-6.14 -1.16
10.50 -25.82          :End of matrix B
```

9.3 Program Results

F04BBF Example Program Results

Solution

	1	2
1	-2.0000	1.0000
2	3.0000	-4.0000
3	1.0000	7.0000
4	-4.0000	-2.0000

Estimate of condition number

5.6E+01

Estimate of error bound for computed solutions

6.3E-15
