

NAG Library Routine Document

F02FCF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

F02FCF computes selected eigenvalues, and optionally the corresponding eigenvectors, of a real symmetric matrix.

2 Specification

```
SUBROUTINE F02FCF(JOB, RANGE, UPLO, N, A, LDA, WL, WU, IL, IU, MEST, M,
1                W, Z, LDZ, WORK, LWORK, IWORK, IFAIL)
      INTEGER      N, LDA, IL, IU, MEST, M, LDZ, LWORK, IWORK(5*N), IFAIL
      double precision A(LDA,*), WL, WU, W(N), Z(LDZ,MEST), WORK(LWORK)
      CHARACTER*1  JOB, RANGE, UPLO
```

3 Description

F02FCF computes selected eigenvalues, and optionally the corresponding eigenvectors, of a real symmetric matrix A :

$$Az_i = \lambda_i z_i.$$

The eigenvalues λ_i are selected either by *value* (all the eigenvalues in a half-open interval):

$$w_l \leq \lambda_i < w_u$$

or by *index*, assuming that the eigenvalues are indexed in *ascending* order:

$$i_l \leq i \leq i_u, \quad \text{where} \quad \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n.$$

4 References

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Parlett B N (1998) *The Symmetric Eigenvalue Problem* SIAM, Philadelphia

5 Parameters

- 1: JOB – CHARACTER*1 *Input*
On entry: indicates whether eigenvectors are to be computed.
 JOB = 'N'
 Only eigenvalues are computed.
 JOB = 'V'
 Eigenvalues and eigenvectors are computed.
Constraint: JOB = 'N' or 'V'.
- 2: RANGE – CHARACTER*1 *Input*
On entry: indicates how eigenvalues are to be selected.
 RANGE = 'V'
 Eigenvalues are selected by value (see WL and WU);

- RANGE = 'I'
Eigenvalues are selected by index (see IL and IU).
Constraint: RANGE = 'V' or 'I'.
- 3: UPLO – CHARACTER*1 *Input*
On entry: indicates whether the upper or lower triangular part of A is stored.
UPLO = 'U'
The upper triangular part of A is stored.
UPLO = 'L'
The lower triangular part of A is stored.
Constraint: UPLO = 'U' or 'L'.
- 4: N – INTEGER *Input*
On entry: n , the order of the matrix A .
Constraint: $N \geq 0$.
- 5: A(LDA,*) – **double precision** array *Input/Output*
Note: the second dimension of the array A must be at least $\max(1, N)$.
On entry: the n by n symmetric matrix A .
If UPLO = 'U', the upper triangle of A must be stored and the elements of the array below the diagonal need not be set.
If UPLO = 'L', the lower triangle of A must be stored and the elements of the array above the diagonal need not be set.
On exit: the contents of A are overwritten. The diagonal and first off-diagonal contain the upper or lower triangle of the symmetric tridiagonal matrix T (see Section 8).
- 6: LDA – INTEGER *Input*
On entry: the first dimension of the array A as declared in the (sub)program from which F02FCF is called.
Constraint: $LDA \geq \max(1, N)$.
- 7: WL – **double precision** *Input*
8: WU – **double precision** *Input*
On entry: w_l and w_u , the lower and upper bounds of the interval in which eigenvalues are selected if RANGE = 'V'. Not referenced if RANGE = 'I'.
Constraint: $WU > WL$.
- 9: IL – INTEGER *Input*
10: IU – INTEGER *Input*
On entry: i_l and i_u , the lower and upper bounds of the indices of the eigenvalues which are selected if RANGE = 'I'. Not referenced if RANGE = 'V'.
Constraint: if $N > 0$, $1 \leq IL \leq IU \leq N$.
- 11: MEST – INTEGER *Input*
On entry: the second dimension of the array Z as declared in the (sub)program from which F02FCF is called.
If JOB = 'V', MEST must be an upper bound on m , the number of eigenvalues and eigenvectors selected. No eigenvectors are computed if $MEST < m$.

If JOB = 'N', MEST is not referenced.

Constraint: $MEST \geq \max(1, m)$; $MEST \geq IU - IL + 1$ if RANGE = 'I'.

12: M – INTEGER

Output

On exit: m , the number of eigenvalues actually selected.

If RANGE = 'I', $m = i_u - i_l + 1$.

13: W(N) – **double precision** array

Output

On exit: the first M elements hold the selected eigenvalues in ascending order; elements M + 1 to N are used as workspace.

14: Z(LDZ,MEST) – **double precision** array

Output

On exit: if JOB = 'V', the first M columns of Z contain the selected eigenvectors, with the i th column holding the eigenvector z_i associated with the eigenvalue λ_i (stored in $W(i)$).

If JOB = 'N', Z is not referenced.

15: LDZ – INTEGER

Input

On entry: the first dimension of the array Z as declared in the (sub)program from which F02FCF is called.

Constraint: $LDZ \geq \max(1, N)$ if JOB = 'V'; $LDZ \geq 1$ otherwise.

16: WORK(LWORK) – **double precision** array

Workspace

17: LWORK – INTEGER

Input

On entry: the dimension of the array WORK as declared in the (sub)program from which F02FCF is called. On some high-performance computers, increasing the dimension of WORK will enable the routine to run faster; a value of $64 \times N$ should allow near-optimal performance on almost all machines.

Constraint: $LWORK \geq \max(1, 8 \times N)$.

18: IWORK($5 \times N$) – INTEGER array

Workspace

19: IFAIL – INTEGER

Input/Output

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, JOB $\neq$ 'N' or 'V',
or RANGE $\neq$ 'V' or 'I',
or UPLO $\neq$ 'U' or 'L',

or $N < 0$,
 or $LDA < \max(1, N)$,
 or $WU \leq WL$ when $RANGE = 'V'$,
 or $IL < 1$ when $RANGE = 'I'$,
 or $IU > N$, or $IL > IU$ and $N > 0$, when $RANGE = 'I'$,
 or $MEST < 1$,
 or $LDZ < 1$, or $LDZ < N$ when $JOB = 'V'$,
 or $LWORK < \max(1, 8 \times N)$.

IFAIL = 2

The bisection algorithm failed to compute all the eigenvalues. No eigenvectors have been computed.

IFAIL = 3

There are more than MEST eigenvalues in the specified range. The actual number of eigenvalues in the range is returned in M. No eigenvectors have been computed. Rerun with the second dimension of $Z = MEST \geq M$.

IFAIL = 4

Inverse iteration failed to compute all the specified eigenvectors. If an eigenvector failed to converge, the corresponding column of Z is set to zero.

7 Accuracy

If λ_i is an exact eigenvalue, and $\tilde{\lambda}_i$ is the corresponding computed value, then

$$|\tilde{\lambda}_i - \lambda_i| \leq c(n)\epsilon\|A\|_2,$$

where $c(n)$ is a modestly increasing function of n , and ϵ is the *machine precision*.

If z_i is the corresponding exact eigenvector, and $\tilde{z}_i$ is the corresponding computed eigenvector, then the angle $\theta(\tilde{z}_i, z_i)$ between them is bounded as follows:

$$\theta(\tilde{z}_i, z_i) \leq \frac{c(n)\epsilon\|A\|_2}{\min_{i \neq j} |\lambda_i - \lambda_j|}.$$

Thus the accuracy of a computed eigenvector depends on the gap between its eigenvalue and all the other eigenvalues.

8 Further Comments

F02FCF calls routines from LAPACK in Chapter F08. It first reduces A to tridiagonal form T , using an orthogonal similarity transformation: $A = QTQ^T$. Eigenvalues of T are computed by bisection. If eigenvectors are required, eigenvectors of T are computed by inverse iteration, and are transformed to eigenvectors of A by premultiplying them with the orthogonal matrix Q that was used in the reduction to tridiagonal form.

Each eigenvector z is normalized so that $\|z\|_2 = 1$ and the element of largest absolute value is positive.

The time taken by the routine is approximately proportional to n^3 .

The routine can be used to compute *all* eigenvalues, and optionally *all* eigenvectors, by setting $RANGE = 'I'$, $IL = 1$ and $IU = N$. In some circumstances it may do this more efficiently than F08FAF (DSYEV), but this depends on the machine, the size of the problem, and the distribution of eigenvalues. Eigenvectors computed by F02FCF may not be orthogonal to as high a degree of accuracy as those computed by F02FAF.

9 Example

This example computes the two smallest eigenvalues of the matrix A , and their corresponding eigenvectors, where

$$A = \begin{pmatrix} 4.16 & -3.12 & 0.56 & -0.10 \\ -3.12 & 5.03 & -0.83 & 1.18 \\ 0.56 & -0.83 & 0.76 & 0.34 \\ -0.10 & 1.18 & 0.34 & 1.18 \end{pmatrix}.$$

9.1 Program Text

```
*      F02FCF Example Program Text
*      Mark 17 Release. NAG Copyright 1995.
*      .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NMAX, MEST, LDA, LDZ, LWORK
      PARAMETER        (NMAX=8,MEST=3,LDA=NMAX,LDZ=NMAX,LWORK=64*NMAX)
*      .. Local Scalars ..
      DOUBLE PRECISION WL, WU
      INTEGER          I, IFAIL, IL, IU, J, M, N
      CHARACTER        UPLO
*      .. Local Arrays ..
      DOUBLE PRECISION A(LDA,NMAX), W(NMAX), WORK(LWORK), Z(LDZ,MEST)
      INTEGER          IWORK(5*NMAX)
*      .. External Subroutines ..
      EXTERNAL         F02FCF, X04CAF
*      .. Executable Statements ..
      WRITE (NOUT,*) 'F02FCF Example Program Results'
*      Skip heading in data file
      READ (NIN,*)
      READ (NIN,*) N, IL, IU
      IF (N.LE.NMAX) THEN
*
*          Read A from data file
*
*          READ (NIN,*) UPLO
*          IF (UPLO.EQ.'U') THEN
*              READ (NIN,*) ((A(I,J),J=I,N),I=1,N)
*          ELSE IF (UPLO.EQ.'L') THEN
*              READ (NIN,*) ((A(I,J),J=1,I),I=1,N)
*          END IF
*
*          Compute selected eigenvalues and eigenvectors
*
*          IFAIL = 1
*
*          CALL F02FCF('Vectors','Index',UPLO,N,A,LDA,WL,WU,IL,IU,MEST,M,
+              W,Z,LDZ,WORK,LWORK,IWORK,IFAIL)
*
*          IF (IFAIL.EQ.0) THEN
*              WRITE (NOUT,*)
*              WRITE (NOUT,*) 'Eigenvalues'
*              WRITE (NOUT,99999) (W(I),I=1,M)
*              WRITE (NOUT,*)
*
*              IFAIL = 0
*              CALL X04CAF('General',' ',N,M,Z,LDZ,'Eigenvectors',IFAIL)
*          ELSE
*              WRITE (NOUT,99998) IFAIL
*          END IF
*
*          END IF
*
*          99999 FORMAT (3X,(8F8.4))
*          99998 FORMAT (1X,/1X,' ** F02FCF returned with IFAIL = ',I5)
*          END
```

9.2 Program Data

F02FCF Example Program Data

```
4 1 2 :Values of N, IL, IU
'L' :Value of UPLO
4.16
-3.12 5.03
0.56 -0.83 0.76
-0.10 1.18 0.34 1.18 :End of matrix A
```

9.3 Program Results

F02FCF Example Program Results

Eigenvalues

```
0.1239 1.0051
```

Eigenvectors

```
1 2
1 0.1859 -0.4209
2 0.3791 -0.3108
3 0.6621 0.7210
4 -0.6192 0.4543
```
