

NAG Library Routine Document

D03NDF

Note: before using this routine, please read the Users' Note for your implementation to check the interpretation of ***bold italicised*** terms and other implementation-dependent details.

1 Purpose

D03NDF computes an analytic solution to the Black–Scholes equation for a certain set of option types.

2 Specification

```

SUBROUTINE D03NDF(KOPT, X, S, T, TMAT, TDPAR, R, Q, SIGMA, F, THETA,
1              DELTA, GAMMA, LAMBDA, RHO, IFAIL)
    INTEGER          KOPT, IFAIL
    double precision X, S, T, TMAT, R(*), Q(*), SIGMA(*), F, THETA, DELTA,
1              GAMMA, LAMBDA, RHO
    LOGICAL          TDPAR(3)

```

3 Description

D03NDF computes an analytic solution to the Black–Scholes equation (see Hull (1989) and Wilmott *et al.* (1995))

$$\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} = rf \quad (1)$$

$$S_{\min} < S < S_{\max}, \quad t_{\min} < t < t_{\max}, \quad (2)$$

for the value f of a European put or call option, or an American call option with zero dividend q . In equation (1) t is time, S is the stock price, X is the exercise price, r is the risk free interest rate, q is the continuous dividend, and σ is the stock volatility. The parameters r , q and σ may be either constant, or functions of time. In the latter case their average instantaneous values over the remaining life of the option should be provided to D03NDF. An auxiliary routine D03NEF is available to compute such averages from values at a set of discrete times.

D03NDF also returns values of the Greeks

$$\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial x}, \quad \Gamma = \frac{\partial^2 f}{\partial x^2}, \quad \Lambda = \frac{\partial f}{\partial \sigma}, \quad \rho = \frac{\partial f}{\partial r}.$$

Further details of the analytic solution returned are given in Section 8.1.

4 References

Hull J (1989) *Options, Futures and Other Derivative Securities* Prentice–Hall

Wilmott P, Howison S and Dewynne J (1995) *The Mathematics of Financial Derivatives* Cambridge University Press

5 Parameters

- 1: KOPT – INTEGER *Input*
On entry: specifies the kind of option to be valued:
 KOPT = 1
 A European call option.

KOPT = 2

An American call option.

KOPT = 3

A European put option.

Constraints:

KOPT = 1, 2 or 3;
if $q \neq 0$, KOPT $\neq$ 2.

- 2: **X – double precision** *Input*
On entry: the exercise price X .
Constraint: $X \geq 0.0$.
- 3: **S – double precision** *Input*
On entry: the stock price at which the option value and the Greeks should be evaluated.
Constraint: $S \geq 0.0$.
- 4: **T – double precision** *Input*
On entry: the time at which the option value and the Greeks should be evaluated.
Constraint: $T \geq 0.0$.
- 5: **TMAT – double precision** *Input*
On entry: the maturity time of the option.
Constraint: $\text{TMAT} \geq T$.
- 6: **TDPAR(3) – LOGICAL array** *Input*
On entry: specifies whether or not various parameters are time-dependent. More precisely, r is time-dependent if $\text{TDPAR}(1) = \text{.TRUE.}$ and constant otherwise. Similarly, $\text{TDPAR}(2)$ specifies whether q is time-dependent and $\text{TDPAR}(3)$ specifies whether σ is time-dependent.
- 7: **R(*) – double precision array** *Input*
Note: the dimension of the array R must be at least 3 if $\text{TDPAR}(1) = \text{.TRUE.}$, and at least 1 otherwise.
On entry: if $\text{TDPAR}(1) = \text{.FALSE.}$ then $R(1)$ must contain the constant value of r . The remaining elements need not be set.
 If $\text{TDPAR}(1) = \text{.TRUE.}$ then $R(1)$ must contain the value of r at time T and $R(2)$ must contain its average instantaneous value over the remaining life of the option:

$$\hat{r} = \int_T^{\text{TMAT}} r(\zeta) d\zeta.$$
 The auxiliary routine D03NEF may be used to construct R from a set of values of r at discrete times.
- 8: **Q(*) – double precision array** *Input*
Note: the dimension of the array Q must be at least 3 if $\text{TDPAR}(2) = \text{.TRUE.}$, and at least 1 otherwise.
On entry: if $\text{TDPAR}(2) = \text{.FALSE.}$ then $Q(1)$ must contain the constant value of q . The remaining elements need not be set.
 If $\text{TDPAR}(2) = \text{.TRUE.}$ then $Q(1)$ must contain the constant value of q and $Q(2)$ must contain its average instantaneous value over the remaining life of the option:

$$\hat{q} = \int_T^{\text{TMAT}} q(\zeta) d\zeta.$$

The auxiliary routine D03NEF may be used to construct Q from a set of values of q at discrete times.

9: SIGMA(*) – **double precision** array *Input*

Note: the dimension of the array SIGMA must be at least 3 if TDPAR(3) = .TRUE., and at least 1 otherwise.

On entry: if TDPAR(3) = .FALSE. then SIGMA(1) must contain the constant value of σ . The remaining elements need not be set.

If TDPAR(3) = .TRUE. then SIGMA(1) must contain the value of σ at time T, SIGMA(2) the average instantaneous value $\hat{\sigma}$, and SIGMA(3) the second-order average $\bar{\sigma}$, where:

$$\hat{\sigma} = \int_T^{\text{TMAT}} \sigma(\zeta) d\zeta,$$

$$\bar{\sigma} = \left(\int_T^{\text{TMAT}} \sigma^2(\zeta) d\zeta \right)^{1/2}.$$

The auxiliary routine D03NEF may be used to compute SIGMA from a set of values at discrete times.

Constraints:

if TDPAR(3) = .FALSE., SIGMA(1) > 0.0;
if TDPAR(3) = .TRUE., SIGMA(i) > 0.0, for $i = 1, 2, 3$.

10: F – **double precision** *Output*

On exit: the value f of the option at the stock price S and time T.

11: THETA – **double precision** *Output*

12: DELTA – **double precision** *Output*

13: GAMMA – **double precision** *Output*

14: LAMBDA – **double precision** *Output*

15: RHO – **double precision** *Output*

On exit: the values of various Greeks at the stock price S and time T, as follows:

$$\text{THETA} = \Theta = \frac{\partial f}{\partial t}, \quad \text{DELTA} = \Delta = \frac{\partial f}{\partial S}, \quad \text{GAMMA} = \Gamma = \frac{\partial^2 f}{\partial S^2},$$

$$\text{LAMBDA} = \Lambda = \frac{\partial f}{\partial \sigma}, \quad \text{RHO} = \rho = \frac{\partial f}{\partial r}.$$

16: IFAIL – INTEGER *Input/Output*

On entry: IFAIL must be set to 0, -1 or 1. If you are unfamiliar with this parameter you should refer to Section 3.3 in the Essential Introduction for details.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

For environments where it might be inappropriate to halt program execution when an error is detected, the value -1 or 1 is recommended. If the output of error messages is undesirable, then the value 1 is recommended. Otherwise, if you are not familiar with this parameter, the recommended value is 0. **When the value -1 or 1 is used it is essential to test the value of IFAIL on exit.**

6 Error Indicators and Warnings

If on entry IFAIL = 0 or -1, explanatory error messages are output on the current error message unit (as defined by X04AAF).

Errors or warnings detected by the routine:

IFAIL = 1

On entry, KOPT < 1,
 or KOPT > 3,
 or KOPT = 2 when $q \neq 0$,
 or $X < 0.0$,
 or $S < 0.0$,
 or $T < 0.0$,
 or $\text{TMAT} < T$,
 or $\text{SIGMA}(1) \leq 0.0$, with $\text{TDPAR}(3) = \text{.FALSE.}$,
 or $\text{SIGMA}(i) \leq 0.0$, with $\text{TDPAR}(3) = \text{.TRUE.}$, for some $i = 1, 2$ or 3 .

7 Accuracy

Given accurate values of R, Q and SIGMA no further approximations are made in the evaluation of the Black-Scholes analytic formulae, and the results should therefore be within machine accuracy. The values of R, Q and SIGMA returned from D03NEF are exact for polynomials of degree up to 3.

8 Further Comments

8.1 Algorithmic Details

The Black-Scholes analytic formulae are used to compute the solution. For a European call option these are as follows:

$$f = Se^{-\hat{q}(T-t)}N(d_1) - Xe^{-\hat{r}(T-t)}N(d_2)$$

where

$$d_1 = \frac{\log(S/X) + (\hat{r} - \hat{q} + \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}},$$

$$d_2 = \frac{\log(S/X) + (\hat{r} - \hat{q} - \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}} = d_1 - \bar{\sigma}\sqrt{T-t},$$

$N(x)$ is the cumulative Normal distribution function and $N'(x)$ is its derivative

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\zeta^2/2} d\zeta,$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The functions $\hat{q}$, $\hat{r}$, $\hat{\sigma}$ and $\bar{\sigma}$ are average values of q , r and σ over the time to maturity:

$$\hat{q} = \frac{1}{T-t} \int_t^T q(\zeta) d\zeta,$$

$$\hat{r} = \frac{1}{T-t} \int_t^T r(\zeta) d\zeta,$$

$$\hat{\sigma} = \frac{1}{T-t} \int_t^T \sigma(\zeta) d\zeta,$$

$$\bar{\sigma} = \left(\frac{1}{T-t} \int_t^T \sigma^2(\zeta) d\zeta \right)^{1/2}.$$

The Greeks are then calculated as follows:

$$\begin{aligned}\Delta &= \frac{\partial f}{\partial S} = e^{-\hat{q}(T-t)} N(d_1) + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S \sqrt{T-t}}, \\ \Gamma &= \frac{\partial^2 f}{\partial S^2} = \frac{S e^{-\hat{q}(T-t)} N'(d_1) + X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma} S^2 \sqrt{T-t}} + \frac{S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2)}{\bar{\sigma}^2 S^2 (T-t)}, \\ \Theta &= \frac{\partial f}{\partial t} = r f + (q - r) S \Delta - \frac{\sigma^2 S^2}{2} \Gamma, \\ \Lambda &= \frac{\partial f}{\partial \sigma} = \left(\frac{X d_1 e^{-\hat{r}(T-t)} N'(d_2) - S d_2 e^{-\hat{q}(T-t)} N'(d_1)}{\bar{\sigma}^2} \right) \hat{\sigma}, \\ \rho &= \frac{\partial f}{\partial r} = X(T-t) e^{-\hat{r}(T-t)} N(d_2) + \frac{\left(S e^{-\hat{q}(T-t)} N'(d_1) - X e^{-\hat{r}(T-t)} N'(d_2) \right) \sqrt{T-t}}{\bar{\sigma}}.\end{aligned}$$

Note: that Θ is obtained from substitution of other Greeks in the Black–Scholes partial differential equation, rather than differentiation of f . The values of q , r and σ appearing in its definition are the instantaneous values, not the averages. Note also that both the first-order average $\hat{\sigma}$ and the second-order average $\bar{\sigma}$ appear in the expression for Λ . This results from the fact that Λ is the derivative of f with respect to σ , not $\hat{\sigma}$.

For a European put option the equivalent equations are:

$$\begin{aligned}f &= X e^{-\hat{r}(T-t)} N(-d_2) - S e^{-\hat{q}(T-t)} N(-d_1), \\ \Delta &= \frac{\partial f}{\partial S} = -e^{-\hat{q}(T-t)} N(-d_1) + \frac{S e^{-\hat{q}(T-t)} N'(-d_1) - X e^{-\hat{r}(T-t)} N'(-d_2)}{\bar{\sigma} S \sqrt{T-t}}, \\ \Gamma &= \frac{\partial^2 f}{\partial S^2} = \frac{X e^{-\hat{r}(T-t)} N'(-d_2) + S e^{-\hat{q}(T-t)} N'(-d_1)}{\bar{\sigma} S^2 \sqrt{T-t}} + \frac{X e^{-\hat{r}(T-t)} N''(-d_2) - S e^{-\hat{q}(T-t)} N''(-d_1)}{\bar{\sigma}^2 S^2 (T-t)}, \\ \Theta &= \frac{\partial f}{\partial t} = r f + (q - r) S \Delta - \frac{\sigma^2 S^2}{2} \Gamma, \\ \Lambda &= \frac{\partial f}{\partial \sigma} = \left(\frac{X d_1 e^{-\hat{r}(T-t)} N'(-d_2) - S d_2 e^{-\hat{q}(T-t)} N'(-d_1)}{\bar{\sigma}^2} \right) \hat{\sigma}, \\ \rho &= \frac{\partial f}{\partial r} = -X(T-t) e^{-\hat{r}(T-t)} N(-d_2) + \frac{\left(S e^{-\hat{q}(T-t)} N'(-d_1) - X e^{-\hat{r}(T-t)} N'(-d_2) \right) \sqrt{T-t}}{\bar{\sigma}}.\end{aligned}$$

The analytic solution for an American call option with $q = 0$ is identical to that for a European call, since early exercise is never optimal in this case. For all other cases no analytic solution is known.

9 Example

This example solves the Black–Scholes equation for valuation of a 5-month American call option on a non-dividend-paying stock with an exercise price of \$50. The risk-free interest rate is 10% per annum, and the stock volatility is 40% per annum.

The option is valued at a range of times and stock prices.

9.1 Program Text

```

*      D03NDF Example Program Text
*      Mark 20 Release. NAG Copyright 2001.
*      .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
      INTEGER          NSMAX, NTMAX
      PARAMETER        (NSMAX=100,NTMAX=100)
*      .. Local Scalars ..
      DOUBLE PRECISION DS, DT, TMAT, X
      INTEGER          I, IFAIL, IGREEK, J, KOPT, NS, NT
*      .. Local Arrays ..
      DOUBLE PRECISION DELTA(NSMAX,NTMAX), F(NSMAX,NTMAX),
+      GAMMA(NSMAX,NTMAX), LAMBDA(NSMAX,NTMAX), Q(3),
+      R(3), RHO(NSMAX,NTMAX), S(NSMAX), SIGMA(3),
+      T(NTMAX), THETA(NSMAX,NTMAX)
      LOGICAL          GPRNT(5), TDPAR(3)
      CHARACTER*6      GNAME(5)
*      .. External Subroutines ..
      EXTERNAL         D03NDF
*      .. Intrinsic Functions ..
      INTRINSIC        DBLE
*      .. Data statements ..
      DATA            GNAME/'Theta ', 'Delta ', 'Gamma ', 'Lambda',
+      'Rho ' /
      DATA            GPRNT/5*.TRUE./
*      .. Executable Statements ..
      WRITE (NOUT,*) 'D03NDF Example Program Results'
      WRITE (NOUT,*)

*      Skip heading in data file
*
      READ (NIN,*)

*      Read problem parameters
*
      READ (NIN,*) KOPT
      READ (NIN,*) X
      READ (NIN,*) TMAT
      READ (NIN,*) R(1)
      READ (NIN,*) Q(1)
      READ (NIN,*) SIGMA(1)
      READ (NIN,*) NS, NT
      READ (NIN,*) S(1), S(NS)
      READ (NIN,*) T(1), T(NT)

*      TDPAR(1) = .FALSE.
      TDPAR(2) = .FALSE.
      TDPAR(3) = .FALSE.

*      IF (NS.LT.2 .OR. NS.GT.NSMAX) THEN
        WRITE (NOUT,*) 'NS invalid.'
      ELSE IF (NT.LT.2 .OR. NT.GT.NTMAX) THEN
        WRITE (NOUT,*) 'NT invalid.'
      ELSE

*          DS = (S(NS)-S(1))/DBLE(NS-1)
          DT = (T(NT)-T(1))/DBLE(NT-1)

*
*      Loop over times
*
      DO 40 J = 1, NT
        T(J) = T(1) + (J-1)*DT

*
*      Loop over stock prices
*
      DO 20 I = 1, NS
        S(I) = S(1) + (I-1)*DS

*
*      Call Black-Scholes solver

```

```

*
      IFAIL = 1
      CALL D03NDF(KOPT,X,S(I),T(J),TMAT,TDPAR,R,Q,SIGMA,F(I,J),
+              THETA(I,J),DELTA(I,J),GAMMA(I,J),LAMBDA(I,J),
+              RHO(I,J),IFAIL)
      IF (IFAIL.NE.0) THEN
        WRITE (NOUT,99997) IFAIL
        GO TO 120
      END IF
*
20      CONTINUE
40      CONTINUE
*
*      Output option values.
*
      WRITE (NOUT,*)
      WRITE (NOUT,*) 'Option Values'
      WRITE (NOUT,*) '-----'
      WRITE (NOUT,*) '  Stock Price |   Time to Maturity (months)'
      WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
      WRITE (NOUT,'(11A)') '-----',
+      ('-----',J=1,NT)
      DO 60 I = 1, NS
        WRITE (NOUT,99998) S(I), '|', (F(I,J),J=1,NT)
60      CONTINUE
*
      DO 100 IGREEK = 1, 5
*
      IF (GPRNT(IGREEK)) THEN
        WRITE (NOUT,*)
        WRITE (NOUT,*) GNAME(IGREEK)
        WRITE (NOUT,*) '-----'
        WRITE (NOUT,*)
+        '  Stock Price |   Time to Maturity (months)'
        WRITE (NOUT,99999) '|', (12*(TMAT-T(J)),J=1,NT)
        WRITE (NOUT,'(11A)') '-----',
+        ('-----',J=1,NT)
        DO 80 I = 1, NS
          IF (IGREEK.EQ.1) THEN
            WRITE (NOUT,99998) S(I), '|', (THETA(I,J),J=1,NT)
          ELSE IF (IGREEK.EQ.2) THEN
            WRITE (NOUT,99998) S(I), '|', (DELTA(I,J),J=1,NT)
          ELSE IF (IGREEK.EQ.3) THEN
            WRITE (NOUT,99998) S(I), '|', (GAMMA(I,J),J=1,NT)
          ELSE IF (IGREEK.EQ.4) THEN
            WRITE (NOUT,99998) S(I), '|', (LAMBDA(I,J),J=1,NT)
          ELSE IF (IGREEK.EQ.5) THEN
            WRITE (NOUT,99998) S(I), '|', (RHO(I,J),J=1,NT)
          END IF
80          CONTINUE
        END IF
*
100      CONTINUE
*
      END IF
120      CONTINUE
*
99999 FORMAT (16X,A,1X,12(1P,E12.4))
99998 FORMAT (1X,1P,E12.4,3X,A,1X,12(1P,E12.4))
99997 FORMAT (1X,/1X,' ** D03NDF returned with IFAIL = ',I5)
      END

```

9.2 Program Data

D03NDF Example Program Data

2	KOPT
50.	X
0.4166667	TMAT
0.1	R(1)

```

0.0          Q(1)
0.4          SIGMA(1)
21  4        NS, NT
0.0 100.     S(1), S(NS)
0.0 0.125    T(1), T(NT)

```

9.3 Program Results

D03NDF Example Program Results

Option Values

```

-----
      Stock Price |      Time to Maturity (months)
                  |      5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----
0.0000E+00 | 0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
5.0000E+00 | 4.4491E-19  4.5989E-21  1.5461E-23  1.0478E-26
1.0000E+01 | 5.5566E-10  5.5129E-11  3.1298E-12  8.0281E-14
1.5000E+01 | 4.7337E-06  1.2187E-06  2.2774E-07  2.7003E-08
2.0000E+01 | 7.2236E-04  3.1054E-04  1.1005E-04  2.9678E-05
2.5000E+01 | 1.6557E-02  9.6610E-03  5.0099E-03  2.2012E-03
3.0000E+01 | 1.3307E-01  9.4037E-02  6.1869E-02  3.6848E-02
3.5000E+01 | 5.6631E-01  4.5257E-01  3.4667E-01  2.5053E-01
4.0000E+01 | 1.6004E+00  1.3850E+00  1.1699E+00  9.5640E-01
4.5000E+01 | 3.4384E+00  3.1328E+00  2.8168E+00  2.4891E+00
5.0000E+01 | 6.1165E+00  5.7600E+00  5.3874E+00  4.9960E+00
5.5000E+01 | 9.5300E+00  9.1645E+00  8.7846E+00  8.3882E+00
6.0000E+01 | 1.3509E+01  1.3163E+01  1.2808E+01  1.2445E+01
6.5000E+01 | 1.7883E+01  1.7568E+01  1.7251E+01  1.6932E+01
7.0000E+01 | 2.2513E+01  2.2230E+01  2.1949E+01  2.1671E+01
7.5000E+01 | 2.7301E+01  2.7045E+01  2.6792E+01  2.6544E+01
8.0000E+01 | 3.2182E+01  3.1946E+01  3.1713E+01  3.1485E+01
8.5000E+01 | 3.7117E+01  3.6894E+01  3.6674E+01  3.6458E+01
9.0000E+01 | 4.2081E+01  4.1868E+01  4.1656E+01  4.1446E+01
9.5000E+01 | 4.7062E+01  4.6854E+01  4.6647E+01  4.6441E+01
1.0000E+02 | 5.2052E+01  5.1847E+01  5.1643E+01  5.1439E+01

```

Theta

```

-----
      Stock Price |      Time to Maturity (months)
                  |      5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----
0.0000E+00 | 0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00
5.0000E+00 | -4.4017E-17 -5.5977E-19 -2.3735E-21 -2.0936E-24
1.0000E+01 | -2.7827E-08 -3.3857E-09 -2.4163E-10 -8.0398E-12
1.5000E+01 | -1.3953E-04 -4.3864E-05 -1.0258E-05 -1.5706E-06
2.0000E+01 | -1.3287E-02 -6.9342E-03 -3.0567E-03 -1.0576E-03
2.5000E+01 | -1.9512E-01 -1.3714E-01 -8.7730E-02 -4.9018E-02
3.0000E+01 | -1.0161E+00 -8.5596E-01 -6.8695E-01 -5.1395E-01
3.5000E+01 | -2.8112E+00 -2.6426E+00 -2.4328E+00 -2.1723E+00
4.0000E+01 | -5.1662E+00 -5.1709E+00 -5.1500E+00 -5.0892E+00
4.5000E+01 | -7.2196E+00 -7.4540E+00 -7.7180E+00 -8.0183E+00
5.0000E+01 | -8.3848E+00 -8.7388E+00 -9.1543E+00 -9.6525E+00
5.5000E+01 | -8.6152E+00 -8.9372E+00 -9.3056E+00 -9.7329E+00
6.0000E+01 | -8.2058E+00 -8.4077E+00 -8.6186E+00 -8.8343E+00
6.5000E+01 | -7.5116E+00 -7.5845E+00 -7.6368E+00 -7.6553E+00
7.0000E+01 | -6.7905E+00 -6.7711E+00 -6.7202E+00 -6.6262E+00
7.5000E+01 | -6.1758E+00 -6.1099E+00 -6.0160E+00 -5.8893E+00
8.0000E+01 | -5.7084E+00 -5.6310E+00 -5.5359E+00 -5.4234E+00
8.5000E+01 | -5.3786E+00 -5.3103E+00 -5.2340E+00 -5.1533E+00
9.0000E+01 | -5.1582E+00 -5.1071E+00 -5.0551E+00 -5.0062E+00
9.5000E+01 | -5.0165E+00 -4.9835E+00 -4.9536E+00 -4.9298E+00
1.0000E+02 | -4.9281E+00 -4.9107E+00 -4.8979E+00 -4.8916E+00

```

Delta

```

-----
      Stock Price |      Time to Maturity (months)
                  |      5.0000E+00  4.5000E+00  4.0000E+00  3.5000E+00
-----

```

0.0000E+00		0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00		3.1381E-18	3.5969E-20	1.3576E-22	1.0494E-25
1.0000E+01		1.4005E-09	1.5376E-10	9.7805E-12	2.8553E-13
1.5000E+01		6.1418E-06	1.7452E-06	3.6436E-07	4.9030E-08
2.0000E+01		5.6040E-04	2.6494E-04	1.0451E-04	3.1863E-05
2.5000E+01		8.3312E-03	5.3217E-03	3.0570E-03	1.5104E-03
3.0000E+01		4.5711E-02	3.5158E-02	2.5461E-02	1.6934E-02
3.5000E+01		1.3765E-01	1.1889E-01	9.9459E-02	7.9557E-02
4.0000E+01		2.8307E-01	2.6258E-01	2.3996E-01	2.1479E-01
4.5000E+01		4.5320E-01	4.3858E-01	4.2214E-01	4.0335E-01
5.0000E+01		6.1427E-01	6.0856E-01	6.0249E-01	5.9601E-01
5.5000E+01		7.4525E-01	7.4687E-01	7.4937E-01	7.5308E-01
6.0000E+01		8.4052E-01	8.4611E-01	8.5298E-01	8.6148E-01
6.5000E+01		9.0433E-01	9.1096E-01	9.1862E-01	9.2752E-01
7.0000E+01		9.4449E-01	9.5045E-01	9.5699E-01	9.6412E-01
7.5000E+01		9.6862E-01	9.7325E-01	9.7808E-01	9.8300E-01
8.0000E+01		9.8260E-01	9.8589E-01	9.8913E-01	9.9221E-01
8.5000E+01		9.9050E-01	9.9269E-01	9.9473E-01	9.9653E-01
9.0000E+01		9.9487E-01	9.9627E-01	9.9748E-01	9.9848E-01
9.5000E+01		9.9725E-01	9.9811E-01	9.9881E-01	9.9935E-01
1.0000E+02		9.9854E-01	9.9905E-01	9.9945E-01	9.9972E-01

Gamma

Stock Price	Time to Maturity (months)			
	5.0000E+00	4.5000E+00	4.0000E+00	3.5000E+00
0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00	2.1246E-17	2.7112E-19	1.1536E-21	1.0211E-24
1.0000E+01	3.3102E-09	4.0468E-10	2.9020E-11	9.7029E-13
1.5000E+01	7.2660E-06	2.2982E-06	5.4080E-07	8.3319E-08
2.0000E+01	3.8245E-04	2.0111E-04	8.9333E-05	3.1153E-05
2.5000E+01	3.5190E-03	2.4960E-03	1.6118E-03	9.0924E-04
3.0000E+01	1.2392E-02	1.0554E-02	8.5660E-03	6.4838E-03
3.5000E+01	2.4348E-02	2.3181E-02	2.1626E-02	1.9580E-02
4.0000E+01	3.2765E-02	3.3274E-02	3.3650E-02	3.3795E-02
4.5000E+01	3.4099E-02	3.5763E-02	3.7655E-02	3.9828E-02
5.0000E+01	2.9625E-02	3.1360E-02	3.3403E-02	3.5860E-02
5.5000E+01	2.2600E-02	2.3743E-02	2.5052E-02	2.6569E-02
6.0000E+01	1.5672E-02	1.6137E-02	1.6603E-02	1.7048E-02
6.5000E+01	1.0123E-02	1.0119E-02	1.0032E-02	9.8216E-03
7.0000E+01	6.1999E-03	5.9720E-03	5.6534E-03	5.2154E-03
7.5000E+01	3.6474E-03	3.3666E-03	3.0215E-03	2.6027E-03
8.0000E+01	2.0815E-03	1.8329E-03	1.5510E-03	1.2387E-03
8.5000E+01	1.1610E-03	9.7196E-04	7.7211E-04	5.6851E-04
9.0000E+01	6.3660E-04	5.0529E-04	3.7553E-04	2.5382E-04
9.5000E+01	3.4468E-04	2.5884E-04	1.7950E-04	1.1099E-04
1.0000E+02	1.8494E-04	1.3118E-04	8.4708E-05	4.7786E-05

Lambda

Stock Price		Time to Maturity (months)			
		5.0000E+00	4.5000E+00	4.0000E+00	3.5000E+00

0.0000E+00		0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00		8.8525E-17	1.0167E-18	3.8453E-21	2.9781E-24
1.0000E+01		5.5171E-08	6.0702E-09	3.8694E-10	1.1320E-11
1.5000E+01		2.7247E-04	7.7565E-05	1.6224E-05	2.1871E-06
2.0000E+01		2.5496E-02	1.2066E-02	4.7644E-03	1.4538E-03
2.5000E+01		3.6656E-01	2.3400E-01	1.3431E-01	6.6299E-02
3.0000E+01		1.8588E+00	1.4248E+00	1.0279E+00	6.8080E-01
3.5000E+01		4.9710E+00	4.2595E+00	3.5323E+00	2.7983E+00
4.0000E+01		8.7374E+00	7.9857E+00	7.1787E+00	6.3084E+00
4.5000E+01		1.1508E+01	1.0863E+01	1.0167E+01	9.4094E+00
5.0000E+01		1.2344E+01	1.1760E+01	1.1134E+01	1.0459E+01
5.5000E+01		1.1394E+01	1.0773E+01	1.0104E+01	9.3768E+00
6.0000E+01		9.4033E+00	8.7137E+00	7.9693E+00	7.1602E+00
6.5000E+01		7.1285E+00	6.4127E+00	5.6514E+00	4.8412E+00
7.0000E+01		5.0632E+00	4.3894E+00	3.6936E+00	2.9815E+00
7.5000E+01		3.4194E+00	2.8406E+00	2.2661E+00	1.7080E+00
8.0000E+01		2.2203E+00	1.7596E+00	1.3235E+00	9.2488E-01

8.5000E+01		1.3981E+00	1.0534E+00	7.4380E-01	4.7920E-01
9.0000E+01		8.5941E-01	6.1393E-01	4.0558E-01	2.3986E-01
9.5000E+01		5.1846E-01	3.5040E-01	2.1600E-01	1.1686E-01
1.0000E+02		3.0824E-01	1.9677E-01	1.1294E-01	5.5750E-02

Rho

Stock Price	Time to Maturity (months)			
	5.0000E+00	4.5000E+00	4.0000E+00	3.5000E+00
0.0000E+00		0.0000E+00	0.0000E+00	0.0000E+00
5.0000E+00		6.3524E-18	6.5717E-20	2.2112E-22
1.0000E+01		5.6040E-09	5.5594E-10	3.1558E-11
1.5000E+01		3.6414E-05	9.3595E-06	1.7459E-06
2.0000E+01		4.3690E-03	1.8706E-03	6.6008E-04
2.5000E+01		7.9884E-02	4.6268E-02	2.3805E-02
3.0000E+01		5.1594E-01	3.6026E-01	2.3399E-01
3.5000E+01		1.7715E+00	1.3907E+00	1.0448E+00
4.0000E+01		4.0509E+00	3.4193E+00	2.8095E+00
4.5000E+01		7.0648E+00	6.2263E+00	5.3932E+00
5.0000E+01		1.0249E+01	9.2505E+00	8.2458E+00
5.5000E+01		1.3108E+01	1.1967E+01	1.0810E+01
6.0000E+01		1.5384E+01	1.4101E+01	1.2790E+01
6.5000E+01		1.7041E+01	1.5617E+01	1.4153E+01
7.0000E+01		1.8167E+01	1.6613E+01	1.5013E+01
7.5000E+01		1.8894E+01	1.7231E+01	1.5521E+01
8.0000E+01		1.9344E+01	1.7597E+01	1.5806E+01
8.5000E+01		1.9615E+01	1.7807E+01	1.5959E+01
9.0000E+01		1.9774E+01	1.7924E+01	1.6039E+01
9.5000E+01		1.9865E+01	1.7987E+01	1.6080E+01
1.0000E+02		1.9917E+01	1.8022E+01	1.6101E+01



