LIBRA: A New Free-Boundary MHD Equilibrium Code for the Analysis of PROTO-SPHERA Plasma Configurations

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Abstract

PROTO-SPHERA [1,2] is an innovative magnetic-confinement experiment operating at the ENEA Frascati Research Centre. In this configuration, the metallic centerpost of a conventional spherical tokamak is replaced by an electrode-driven screw-pinch (SP), whose current generates the toroidal magnetic field required to confine a surrounding spherical torus (ST). The toroidal plasma is expected to form and be sustained through magnetic reconnection and helicity injection [1,2]. Predictive equilibrium calculations are therefore essential for interpreting present experiments, identifying suitable operating scenarios, and supporting the future development of the machine. To address these objectives, the axisymmetric free-boundary magnetohydrodynamic (MHD) equilibrium code LIBRA has been developed. LIBRA solves the nonlinear Grad–Shafranov equation [3] using a finite-element formulation on an unstructured triangular mesh [4], with distinct source profiles for the screw-pinch and spherical-torus regions [1]. The nonlinear problem is treated through a relaxed Picard iteration, while Green-function boundary conditions [5,6] incorporate the magnetic contributions of the plasma currents and of the complete poloidal-field coil system, including the coils located near the electrodes. The equilibrium calculation itself is restricted to the core region, approximately between the two PF2 mirror coils, which contains the central screw-pinch and the surrounding spherical torus; the plasma extensions towards the electrodes are excluded from the present computational domain. Within this core region, the plasma boundaries are determined self-consistently, and the numerical formulation does not impose up–down symmetry. The numerical implementation has been assessed through a direct numerical benchmark: a coupled SP–ST equilibrium calculated with LIBRA was compared with the corresponding solution obtained using the previous PROTO-SPHERA equilibrium code [1], showing satisfactory agreement in magnetic topology and global plasma parameters. More recently, in order to model the plasma rotation observed in experiments [2], LIBRA equilibrium model has been extended to stationary equilibria with toroidal plasma rotation [6,7]. The model includes a rigidly rotating, isothermal screw-pinch in which the radial pressure gradient balances the centrifugal force while preserving the force-free character of the magnetic configuration [8]. In the spherical torus, the plasma pressure, angular velocity, and temperature are prescribed as magnetic-flux functions. At the common SP–ST boundary, continuity is imposed only on the physical pressure; no matching condition is applied independently to the angular velocity or temperature, which may therefore be discontinuous across the interface. Representative numerical equilibria are presented to assess how rotation modifies the pressure distribution, magnetic-axis position, and flux-surface geometry. In conclusion, the validated LIBRA framework provides a useful tool for selecting experimental scenarios, optimizing the magnetic configuration, and assessing possible scaled developments of PROTO-SPHERA. It can also provide consistent equilibria for subsequent stability, transport, and plasma-formation studies.

Methods

LIBRA - code overview

LIBRA is an axisymmetric free-boundary code developed to compute coupled screw-pinch and spherical-torus equilibria in the core region of PROTO-SPHERA [1,2], approximately between the two PF2 mirror coils. It solves the static ideal-MHD equilibrium using a Galerkin finite-element formulation on an unstructured triangular mesh. The SP and ST plasma boundaries are updated self-consistently during the nonlinear iteration. Green-function Dirichlet boundary conditions include the magnetic contributions of the plasma currents and of the complete poloidal-field coil system, including the coils located in the electrode regions. No up-down symmetry is imposed.

Grad Shafranov equation

Introducing the poloidal flux function $\psi = \psi_p/(2\pi)$ and the toroidal-field function

$$g(\psi) = RB_\phi,$$

the axisymmetric equilibrium is governed by [3]

$$\boxed{\Delta^*\psi = -\mu_0 R^2 \frac{dp}{d\psi} - g\frac{dg}{d\psi} = -\mu_0 R J_\phi}$$

with

$$\Delta^*\psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial Z^2},$$

and

$$J_\phi = R\frac{dp}{d\psi} + \frac{g}{\mu_0 R}\frac{dg}{d\psi}.$$

Free-boundary conditions

At each Picard iteration, the Dirichlet values of the poloidal flux on the computational boundary Γ are recomputed by superposing the flux generated by the current plasma distribution and by all poloidal-field coils, including those located in the electrode regions [5]:

$$\psi_\Gamma(R,Z) = \int_{V_p} G\,(R,Z;R',Z')J_\phi(R',Z')\,dR'\,dZ' + \sum_{i=1}^{N_c} G\,(R,Z;R_{i,c},Z_{i,c})I_i.$$

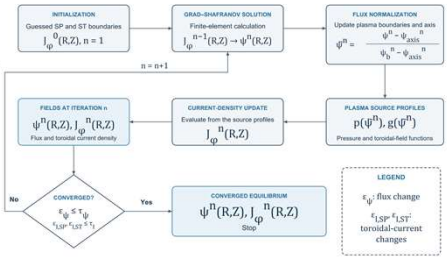
Here, N_c is the number of coil-current filaments, I_i is the corresponding ampere-turn current at $(R_{i,c}, Z_{i,c})$, and V_p denotes the plasma region in the poloidal plane. The axisymmetric Green function associated with the Grad–Shafranov operator is

$$G(R,Z;R',Z') = \frac{\mu_0\sqrt{RR'}}{2\pi k}\left[(2-k^2)K(k^2) - 2E(k^2)\right],$$

where

$$k^2 = \frac{4RR'}{(R+R')^2 + (Z-Z')^2}.$$

K and E are the complete elliptic integrals of the first and second kind, respectively. Thus, although the computational boundary is fixed, its Dirichlet data are updated during the nonlinear iteration, while the SP and ST plasma boundaries are determined self-consistently.



Flowchart of the iterative algorithm implemented in LIBRA to solve the linearized Grad–Shafranov equation numerically.

PROTO SPHERA source functions

For the static equilibrium model, two normalized flux coordinates are introduced [1]:

$$\tilde\psi_{SP} = \frac{\psi}{\psi_B}, \qquad \tilde\psi_{ST} = \frac{\psi - \psi_b}{\psi_B - \psi_b},$$

where $\psi = 0$ on the symmetry axis of the SP, ψ_b is the flux at the ST magnetic axis, and ψ_B is the common SP-ST boundary flux.

The screw pinch is described as a homogeneous force-free plasma [8]:

$$\boxed{p_{SP}(\psi) = p_0, \qquad g_{SP}(\psi) = g_0\tilde\psi_{SP}, \qquad g_0 = \frac{\mu_0 I_c}{2\pi}}$$

so that $dp_{SP}/d\psi = 0$.

In the spherical torus, the standard PROTO-SPHERA profiles are

$$\boxed{p_{ST}(\psi) = p_0 + (p_{max} - p_b)\left(1 - \tilde\psi_{ST}\right)^{\alpha_p}}$$

and

$$\boxed{g_{ST}^2(\psi) = g_0^2\left[1 + \alpha_g\left(1 - \tilde\psi_{ST}^{n_2}\right)^{n_1}\right]}$$

with

$$\alpha_p = 1.1, \qquad n_1 = 1, \qquad n_2 = 1.1.$$

Here p_{max} is the pressure at the ST magnetic axis, while α_g is updated so that the prescribed toroidal plasma current is recovered:

$$I_p = \int_{\Omega_{ST}} J_\phi\, dR\, dZ.$$

At $\tilde\psi_{ST} = 1$, the profiles satisfy $p_{ST} = p_0$ and $g_{ST}^2 = g_0^2$, ensuring matching with the screw-pinch profiles at the common boundary.

Linearized equation and relaxed Picard iteration

At iteration n, the source current density is evaluated from the previous flux distribution:

$$J_{\phi,*}^{(n)} = R\left.\frac{dp}{d\psi}\right|_{\psi^{(n)}} + \frac{1}{\mu_0 R}g\left.\frac{dg}{d\psi}\right|_{\psi^{(n)}}.$$

In the current implementation, this source map is under-relaxed according to

$$J_\phi^{(n)} = \alpha_{rel}J_{\phi,*}^{(n)} + (1-\alpha_{rel})J_\phi^{(n-1)}.$$

The source is then held fixed while the following linear finite-element problem is solved:

$$\boxed{\Delta^*\hat\psi^{(n+1)} = -\mu_0 R J_\phi^{(n)}}$$

or, equivalently, in the form implemented in the MATLAB PDE solver [4],

$$\boxed{-\nabla_{RZ}\cdot\left(\frac{1}{R}\nabla_{RZ}\hat\psi^{(n+1)}\right) = \mu_0 J_\phi^{(n)}}$$

with $c = 1/R$, $a = 0$, and $f = \mu_0 J_\phi^{(n)}$.

The unrelaxed FEM solution $\hat\psi^{(n+1)}$ is combined with the previous iterate through

$$\boxed{\psi^{(n+1)} = \alpha_{rel}\hat\psi^{(n+1)} + (1-\alpha_{rel})\psi^{(n)}, \qquad 0 < \alpha_{rel} < 1}$$

using $\alpha_{rel} = 0.5$ in the present calculations. At every iteration, the plasma boundaries, source functions, α_g, and Green-function boundary values are updated consistently with the relaxed flux distribution.

Toroidal plasma rotation model

For a stationary axisymmetric equilibrium with purely toroidal flow [6,7],

$$\mathbf{u} = R\Omega(\psi)\mathbf{e}_\phi,$$

the radial pressure gradient balances the centrifugal force, while the Grad–Shafranov equation contains the derivative of the physical pressure at fixed R:

$$\frac{\partial p}{\partial R} = \rho R\Omega^2, \qquad \Delta^*\psi = -\mu_0 R^2\left(\frac{\partial p}{\partial\psi}\right)_R - g\frac{dg}{d\psi}.$$

Force-free screw pinch. The SP is assumed to rotate rigidly and to be isothermal while retaining the magnetic force-free condition [6,8]:

$$\mathbf{J}_{SP}\times\mathbf{B}_{SP} = 0, \qquad \Omega_{SP} = \Omega_{0,SP}, \qquad T_{SP} = T_{0,SP}.$$

The physical pressure and centrifugal coefficient are:

$$p_{SP}(R) = p_0\exp(C_{SP}R^2), \qquad C_{SP} = \frac{m_i\Omega_{0,SP}^2}{4q_eT_{0,SP}}.$$

Because $(\partial p_{SP}/\partial\psi)_R = 0$, the radial pressure gradient is balanced entirely by the centrifugal term and does not alter the force-free magnetic character of the pinch.

Spherical torus. Let R_0 be the major radius of the ST magnetic axis and define the boundary pressure P_b, pressure amplitude p_{01}, and normalized pressure profile $\hat p(\psi)$ as:

$$P_b = p_0\exp(C_{SP}R_0^2), \quad p_{01}(\psi) = P_b + \Delta p\left(1-\tilde\psi_{ST}\right)^{\alpha_p}, \quad \hat p(\psi) = \frac{p_{01}(\psi)}{p_{0x}}$$

where $\Delta p = p_{0x} - P_b$. The ST physical pressure profile used in LIBRA is [6,7]:

$$\boxed{p_{ST}(R,\psi) = p_{01}(\psi)\exp[C_{ST}\hat p(\psi)(R^2 - R_0^2)]}$$

while the temperature, angular velocity profiles and centrifugal coefficient are:

$$T_{ST} = T_{0,ST}\,\hat p(\psi), \qquad \Omega_{ST}(\psi) = \Omega_{0,ST}\,\hat p(\psi), \qquad C_{ST} = \frac{m_i\Omega_{0,ST}^2}{4q_eT_{0,ST}}.$$

The pressure source entering in the rotating Grad–Shafranov equation is:

$$\left(\frac{\partial p_{ST}}{\partial\hat p}\right)_R = \left[\frac{dp_{01}}{d\psi} + C_{ST}p_{01}(R^2 - R_0^2)\frac{d\hat p}{d\psi}\right]\exp[C_{ST}\hat p(R^2 - R_0^2)],$$

where

$$\frac{dp_{01}}{d\psi} = -\frac{\alpha_p\Delta p}{\Delta\psi}\left(1-\tilde\psi_{ST}\right)^{\alpha_p-1}, \qquad \frac{d\hat p}{d\psi} = \frac{1}{p_{0x}}\frac{dp_{01}}{d\psi}.$$

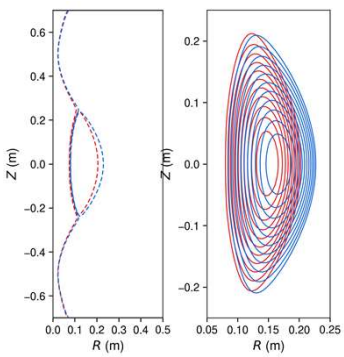
The pressure continuity along the common SP–ST boundary is obtained by choosing:

$$C_{ST}\hat p_b = C_{SP}, \qquad \Omega_{0,ST} = \Omega_{0,SP}\sqrt{\frac{T_{0,ST}}{T_{0,SP}\hat p_b}}$$

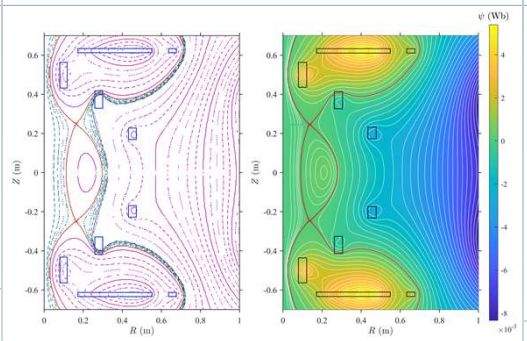
where $\hat p_b = P_b/p_{0x}$.

This is a pressure-matching condition only: no independent continuity condition is imposed on Ω or T, which may therefore be discontinuous across the interface. Here q_e is the elementary charge and the temperatures are expressed in electronvolts as $T_0 = (T_e + T_i)/2$.

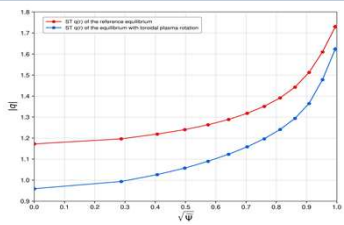
Results



Effect of toroidal plasma rotation on the coupled SP–ST equilibrium. The non-rotating reference equilibrium is shown in red, whereas the equilibrium with toroidal plasma rotation is shown in blue. The left panel compares the SP and ST plasma boundaries within the modeled core region, while the right panel shows an enlarged view of the ST magnetic-flux surfaces. Toroidal rotation produces an outward displacement of the ST equilibrium and modifies its flux-surface geometry, while the SP boundary remains nearly unchanged. Simulation parameters: I_A = 1.1 kA, I_B = 1.875 kA, p_0 = 10 Pa, p_{max} = 200 Pa, I_c = 15 kA, I_p = 20 kA, f_{SP} = 1 kHz, argon plasma, $T_{0,SP}$ = 5 eV, and $T_{0,ST}$ = 200 eV.



Numerical benchmark. Comparison of a coupled SP–ST equilibrium computed with the previous PROTO-SPHERA equilibrium code [1,2] (left) and with LIBRA code (right); see Table 1 for a quantitative comparison. The comparison has been made for I_A = 1.1 kA (Group-A coils), I_B = 1.875 kA (Group-B and External coils), I_e = 11.2 kA, ρ_0 = 10 Pa (SP), I_p = 20 kA and β_p = 1.494 (p_{max} = 734.26 Pa in ST). Note that the lower half of the reference equilibrium was reconstructed by reflecting the solution about the midplane (Z=0).



Safety-factor profile for the reference equilibrium and effect of toroidal plasma rotation on the spherical-torus (ST) safety-factor profile. The red line and symbols represent the non-rotating reference equilibrium, whereas the blue line and symbols represent the equilibrium with toroidal plasma rotation. Both profiles increase monotonically from the magnetic axis to the plasma edge. Rotation lowers q over the entire ST region, reducing the magnetic-axis value from q_{axis} = 1.17 to q_{axis} = 0.96 and the edge value from approximately 1.73 to 1.62. The reference equilibrium therefore remains above q = 1 throughout the ST, whereas the rotating equilibrium develops a central region with q < 1 and crosses the q = 1 surface at $\sqrt{\Psi}$ = 0.31. This suggests that the rotating equilibrium may be more susceptible to an $m = n = 1$ internal-kink mode; however, the q-profile alone does not establish stability, which must be assessed through a dedicated MHD stability analysis. Both equilibria were obtained using the same simulation parameters as those reported in the left figure.

Table 1. Comparison of equilibrium parameters obtained with the previous PROTO-SPHERA code (Reference) and LIBRA.

Parameter	Reference	LIBRA	Relative difference (%)
Upper X-point: $R_{X,+}$ (m)	0.162	0.151	6.8
Upper X-point: $Z_{X,+}$ (m)	0.249	0.246	1.3
Lower X-point: $R_{X,-}$ (m)†	0.162	0.151	6.8
Lower X-point: $Z_{X,-}$ (m)†	-0.249	-0.246	1.3
Magnetic-axis major radius R_0 (m)	0.215	0.211	2.0
Aspect ratio $A = R_0/a$	2.340	2.244	4.0
Elongation κ	2.710	2.558	5.6
Triangularity δ	0.450	0.476	6.0
Normalized internal inductance $l_i/2$	0.260	0.222	14.7

Relative differences are computed as $100|Q_{LIBRA} - Q_{Reference}|/|Q_{Reference}|$, using unrounded LIBRA values.
†Lower X-point coordinates of the Reference are inferred by reflection about $Z = 0$.

Discussion & Conclusions

LIBRA proved to be a flexible framework for computing free-boundary SP–ST equilibria for the actual configuration of the PROTO-SPHERA experiment. The numerical benchmark against the previous PROTO-SPHERA equilibrium code [1,2] showed satisfactory agreement in magnetic topology and global plasma parameters, supporting the reliability of the implemented formulation. The present equilibrium calculation is restricted to the core region, approximately between the two PF2 mirror coils, which contains the central screw-pinch and the surrounding spherical torus. This approach reduces both the geometrical complexity and computational cost while retaining the magnetic contribution of the complete poloidal-field coil system, including the coils located near the electrodes. As a future development, the model could be extended to the electrode-connected plasma regions by treating them as additional finite-element subdomains coupled to the core through suitable interface boundary conditions and updated magnetic contributions.

LIBRA has also enabled the evaluation of the safety-factor profile of the current machine configuration. For the equilibrium considered here, the edge value of q is markedly lower than the values typically encountered in conventional tokamaks. This result cannot yet be generalized, since it refers to a single equilibrium, and its implications for plasma stability cannot be inferred from the q-profile alone. A dedicated MHD stability analysis is therefore required.

More recently, the model implemented in LIBRA has been extended to compute stationary equilibria with toroidal plasma flow [6,7], enabling the resulting centrifugal effects on the pressure distribution and magnetic configuration to be investigated. Taken together, these capabilities make LIBRA a promising tool for designing future PROTO-SPHERA configurations, selecting suitable operating scenarios, interpreting new experiments, and providing consistent equilibrium inputs for subsequent stability and transport studies.

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