J. Plasma Physics (1981), *vol.* 25, *part* 3, *pp.* 459–464
Printed in Great Britain

The bumpy Z-pinch

By TORKIL H. JENSEN AND MING SHENG CHU
General Atomic Company, San Diego, California, U.S.A.

(Received 24 May 1980 and in revised form 19 August 1980)

The 'bumpy Z-pinch' is a magnetic configuration with potential usefulness for fusion reactors. A conceptually simple version of the configuration is axisymmetric. It contains regions of closed and open field lines. In the region of closed field lines, the field line topology is much like that of a tokamak; these regions link the region of open field lines around the axis of symmetry. Assuming that the plasma spontaneously maintains an equilibrium as described by J. B. Taylor, it is possible to maintain indefinitely the regions of closed field lines by driving an axial current through the plasma in the region of open field lines. The ratio between the total axial driven current and the total poloidal current in each of the tokamak-like regions can, in principle, be made arbitrarily small, which means that the load impedance can be arbitrarily large. In addition, the configuration has the inherent virtue similar to that of the spheromak that the tokamak-like part of the plasma does not link any material coils.

1. Introduction

It was found experimentally by Robinson & King (1968) that the axial magnetic field of a Z-pinch spontaneously reverses at the plasma surface (and becomes a reversed field pinch) whenever the parameter $IR_1\mu_0/\phi$ exceeds a certain value; here I is the total axial current, ϕ the total axial magnetic flux, R_1 the radius of the pinch and μ_0 the vacuum permeability. This result was explained theoretically by Taylor (1974) through a remarkably simple assumption, namely that the plasma, presumably owing to resistive instabilities, will reach a state for which the current density is proportional to the magnetic field strength.

Consider a pinch arrangement for which the total axial current, I, and axial magnetic flux, ϕ, are constant but where the radius of the pinch, R_1, is dependent upon the distance, Z, along the axis. From the knowledge of spontaneous reversal of magnetic field in Z-pinches, one can imagine circumstances under which reversal will take place in regions where $R_1(Z)$ is large but not in regions where $R_1(Z)$ is small. Under such circumstances there must exist closed field lines in the sense that the excursions in Z of such closed field lines are limited. Such a configuration may be useful for a fusion reactor since one can imagine maintaining a configuration with closed field lines, populated by hot plasma, by driving an axial current through the region of open field lines; these open field lines may terminate on electrodes and may be populated by relatively cold plasma. One can, of course, also imagine the whole configuration bent into a 'skinny' torus to avoid the electrodes; in this configuration the 'open' field lines are really closed and linked by the regions of 'closed' field lines.

In the following section, a description is given of such equilibria in the straight approximation where the configuration is axisymmetric. The last section contains a brief discussion of the stability and dynamics of such configurations.

2. Equilibrium

The axisymmetric MHD equilibrium satisfies the Grad–Shafranov equation for a pressureless plasma

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}+\frac{d}{d\psi}\left(\frac{R^2B_\theta^2}{2}\right)=0, \tag{1}$$

where R, θ and Z are the usual cylindrical co-ordinates, ψ the usual flux function and where, as indicated, the product of the radius and the θ component of the magnetic field, B_θ, is a function of ψ only. Since the Taylor condition leads to a pressureless plasma, no pressure term is included in (1). The Taylor condition is

$$\bar{\jmath}=(k/\mu_0)\bar{B}, \tag{2}$$

where $\bar{\jmath}$ is the current density and k is a constant independent of space. It is easy to see that (1) and (2) lead to

$$R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{\partial^2\psi}{\partial Z^2}+k^2\psi=0. \tag{3}$$

We may assume the boundary condition established by a conducting wall (corrugated tube), the radius of which may reasonably be parametrized by

$$R_1(Z)=R_0[1-b(1-\cos\kappa Z)], \tag{4}$$

where R_0 is the maximum radius, κ the 'axial wavenumber' and b the 'bumpiness'. We assume that the axial flux through the tube, ϕ, is independent of Z so that the flux function value at the wall, ψ_1, is given by $\psi_1=\phi/(2\pi)$. Since the magnetic field at the wall is parallel to the wall, the total axial plasma current through the tube, I, is independent of Z. It is easy to find that

$$I=k\phi/\mu_0. \tag{5}$$

Thus one finds that an equilibrium is characterized by three parameters only, namely kR_0, κR_0 and b. In the two extremes, $\kappa R_0=0$ and $\kappa R_0=\infty$, the equilibrium can be treated analytically.

In the limit $\kappa R_0=0$ one finds readily from (3) that

$$\left.\begin{aligned}\psi(R)&=\psi_1\frac{RJ_1(kR)}{R_0J_1(kR_0)}\quad\text{for}\quad\kappa Z=0,\\ \psi(R)&=\psi_1\frac{RJ_1(kR)}{R_0(1-2b)J_1(kR_0(1-2b))}\quad\text{for}\quad\kappa Z=\pi,\end{aligned}\right\} \tag{6}$$

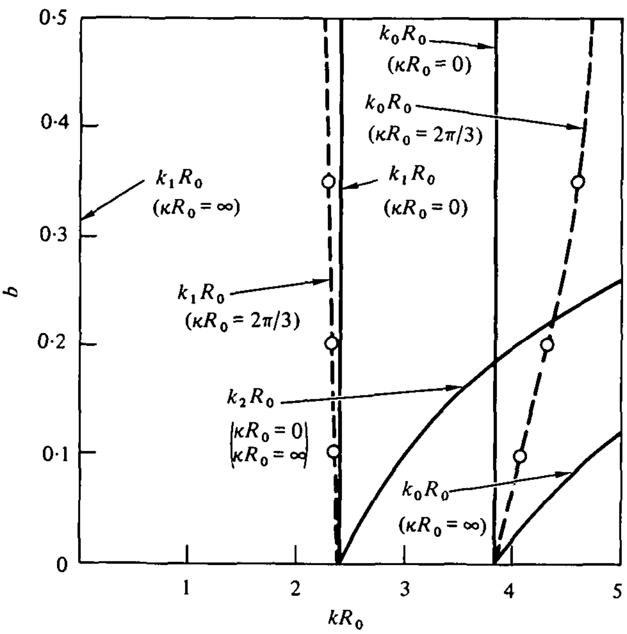

FIGURE 1. Location of the lines k_0R_0, k_1R_0 and k_2R_0 in the (kR_0, b) plane for various values of κR_0.

where J_1 is the Bessel function of the first kind. One notices that, for

$$kR_0 < k_1R_0 = 2{\cdot}404,$$

no regions of closed field lines exist since the maximum value of ψ is ψ_1, independently of b. One also notices that when $kR_0 \to k_0R_0 = 3{\cdot}83$, then the maximum value of ψ approaches infinity. Thus, for $0 < b < 0{\cdot}5$ and $k_1R_0 < kR_0 < k_0R_0$, we have regions of closed field lines. Furthermore, we see from (6) that, if $kR_0 > k_2R_0 = 2{\cdot}404/(1\text{–}2b)$, the field is reversed also for $\kappa Z = \pi$, meaning that the separatrix between the closed and open field lines has a value of ψ larger than ψ_1 or that the separatrix is detached from the wall.

In the limit $\kappa R_0 \to \infty$, it is clear that there exist closed field lines for any value of kR_0 whenever $b > 0$, so that for this case $k_1R_0 \to 0$. One also finds that the separatrix is detached from the wall for $kR_0 > k_2R_0 = 2{\cdot}404/(1-2b)$, and that the maximum value of ψ approaches infinity whenever $kR_0 \to k_0R_0 = 3{\cdot}83/(1-2b)$. These features are illustrated in figure 1, where the lines k_0R_0, k_1R_0 and k_2R_0 are shown in the (kR_0, b) plane.

The bumpy Z-pinch is, of course, most interesting in the regime $\kappa R_0 \simeq 1$. It has been pointed out by Taylor (1980, private communication) that

$$\psi = R\{J_1(kR) + \epsilon J_1([k^2-\kappa^2]^{\frac{1}{2}}R) \times \cos \kappa Z\} \tag{7}$$

satisfies (3) and a boundary condition similar to (4), and therefore in general illustrates the properties of the solutions.

As an example we have numerically found solutions for $\kappa R_0 = 2\pi/3$ which

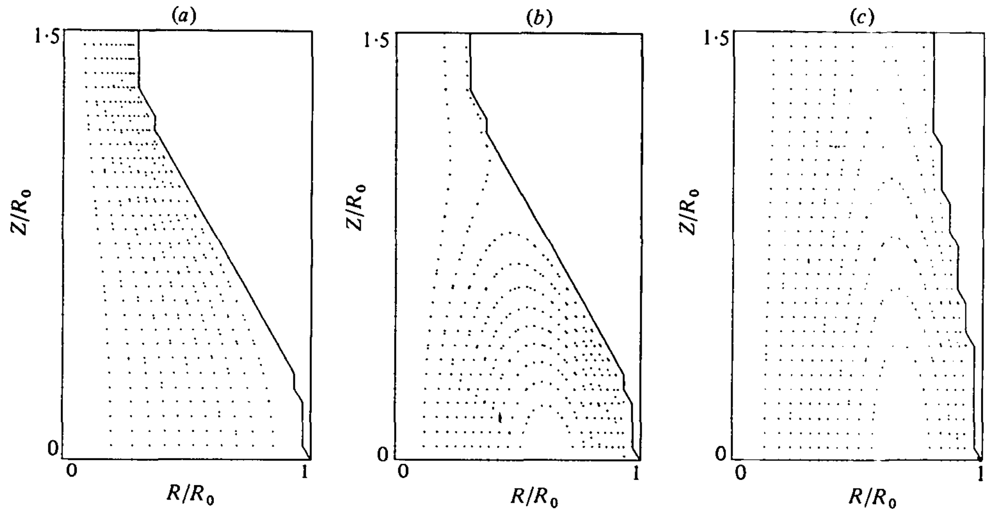

FIGURE 2. Contour plots of the flux function for $\kappa R_0 = 2\pi/3$ and various values of kR_0 and b. The three possible field line topologies are illustrated: (*a*) no closed field lines, $b = 0{\cdot}35$, $kR_0 = 2$; (*b*) closed field lines, separatrix at the wall, $b = 0{\cdot}35$, $kR_0 = 4{\cdot}5$; (*c*) closed field lines, separatrix away from the wall, $b = 0{\cdot}1$, $kR_0 = 3{\cdot}8$. The step-like shape of the wall is due to the finite grid used in the calculations.

satisfy (3) and (4). Points from these calculations are also indicated on figure 1. In figure 2 are shown contour plots of the flux function in the $(Z/R_0, R/R_0)$ plane; the three contour plots illustrate the three possible flux line topologies.

In this it was assumed that the state of lowest energy is an axisymmetric state. As pointed out by Taylor (1974) this is not always the case; for the reversed field pinch case, $\kappa R_0 = 0$, $b = 0$, he found that the state of lowest energy is a helical solution whenever $kR_0 > 3{\cdot}11$. It would have been more satisfactory in figure 1 to be able to show the location of $k_3 R_0$ $(b, \kappa R_0)$ to illustrate the transition from cases for which the state of lowest energy is axisymmetric to ones for which it is non-axisymmetric. Unfortunately, it seems to be a rather complicated numerical problem to calculate the location of the $k_3 R_0$ lines. We may, however, using qualitative arguments, determine some of the properties of such lines. First, in the limit of $\kappa R_0 = 0$, $k_3 R_0 = 3{\cdot}11$ independently of b; this follows from Taylor's result which also gives the same value for $b = 0$, independent of κR_0. For finite values of κR_0 we may expect $k_3 R_0$ to increase with increasing b. One interesting question is whether, for any values of κR_0 and b, $k_3 R_0$ can be larger than $k_0 R_0$. For the special case of $b = 0{\cdot}5$, $kR_0 = k_0 R_0$, the configuration considered is very similar to that of the spheromak. For the spheromak, it was found by Rosenbluth & Bussac (1979) that the state of lowest energy is axisymmetric when the surrounding conducting shell is flattened relative to a sphere and non-axisymmetric when the shell is elongated. We conjecture here that flatness is increasing with κR_0, and that therefore, for $b = 0{\cdot}5$ and sufficiently large values of κR_0, $k_3 R_0 > k_0 R_0$, or that, for sufficiently large values of κR_0, the $k_3 R_0$ line will intersect the $k_0 R_0$ line in the (kR_0, b) plane.

One of the purposes of figure 1 is to illustrate the relationship of the bumpy

Z-pinch to familiar configurations. The Z-pinches are located at $b = 0$; the reversed field pinch has $kR_0 > 2{\cdot}404$. The mirror machines are located at $kR_0 = 0$; the mirror ratio is zero for $b = 0$ and infinity at $b = 0{\cdot}5$. As mentioned, the spheromaks (Rosenbluth & Bussac 1979) are located at $b = 0{\cdot}5$ and $kR_0 = k_0 R_0$; their flatness depends on κR_0.

For $kR_0 > k_1 R_0$, the bumpy Z-pinches have an elliptic axis and one can define a safety factor, q, in the region of closed field lines. One sees immediately that $q \to \infty$ at the separatrix. It is easy to find the safety factor at the elliptic axis, q_E, for the case when the flux surfaces are circular at the elliptic axis. One gets $q_E = 2/(kR_E)$, where R_E is the radius of the elliptic axis. For the case $\kappa R_0 = 2\pi/3$, $b = 0{\cdot}35$ and $kR_0 = 4{\cdot}5$ illustrated in figure 2, we find $q_E \simeq 0{\cdot}7$; numerical calculations show that, for this case, the flux surface of $q = 1$ is very close to the separatrix. The configuration of this example resembles a spheromak; it may, however, be significant that q vanishes at the spheromak surface while it approaches infinity at the separatrix for the bumpy Z-pinch.

3. Discussion

The most attractive feature of the bumpy Z-pinch seems to be the potential for a practical way of establishing and maintaining the configuration. This feature depends on the assumption that the equilibrium will always be of the Taylor type. If at one time the plasma configuration is of the Taylor type, microscopic transport such as resistivity will cause the configuration to deviate from the Taylor equilibrium. When such deviations have reached a certain threshold, instabilities may set in, causing a relaxation to a state closer to a Taylor equilibrium. It is not known how large such thresholds are and how violent the relaxation phenomena are. Optimistically one could hope that only the flux surfaces near the separatrix would be brought into contact with the wall during this relaxation process; pessimistically one may fear that all parts of the plasma will contact the wall during the relaxation.

In general, one expects a Taylor equilibrium to be stable against kink and tearing modes. This requires, however, that the Taylor condition holds clear to the conducting wall; in a real experiment one expects the current density to approach zero at a wall, so the plasma might be unstable to certain kink or tearing modes. Also strict Taylor equilibria are pressureless, which in principle makes them useless for fusion applications. One expects, however, that there exist nearby stable equilibria with finite pressure or beta. One may suspect the beta limit to be relatively large because q is small.

Finally, it is helpful to consider a gedanken experiment in which the axial current, I, (and thereby kR_0) is slowly increased in time while the axial flux ϕ, the bumpiness and the axial wavenumber are kept fixed as they would be in a real experiment. The interesting effect occurs when kR_0 is approaching $k_0 R_0$; according to the Taylor model the maximum value of ψ, and thereby the magnetic fields, approaches infinity in this limit. Therefore the voltage required by the source which drives the axial current must approach infinity. Thus the load appears as an inductor with a current-dependent inductance which approaches infinity as the current approaches a certain limiting current. This assumes, as mentioned in § 2, that the state of lowest energy for $k = k_0$ is axisymmetrical.

The conditions for this assumption to be valid are not determined quantitatively, but, from the arguments given in § 2, it is believed that this condition holds true for $\kappa R_0 \gtrsim 1$ for $b \simeq 0{\cdot}5$.

The authors wish to express their appreciation to F. W. McClain for the numerical calculations he expertly performed. Also, the authors wish to acknowledge the significance for this paper of several discussions with T. Ohkawa.

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