6.3 Dynamic Stabilization of the $\mathrm{m} = 1$ Instability in a Bumpy Theta Pinch

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6.3 DYNAMIC STABILIZATION OF THE m = 1 INSTABILITY IN A BUMPY THETA PINCH

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ABSTRACT

We investigate the possibility of dynamic stabilization of slightly bumpy axisymmetric systems, using a general theory which can be regarded as a generalization of the well-known energy principle. The equilibrium is taken to be one in which there is a sharp boundary, with a surface current but no current in the main body of the plasma. An oscillatory external magnetic field then sets up oscillatory motion in the plasma column. The problem is made soluable by assuming the sharp boundary to have a prescribed motion. This motion will in general cause induced currents to flow in the plasma. A sufficient condition for dynamic stabilization of this system is obtained.

I. INTRODUCTION

Theoretical (Haas and Wesson, 1966;1967b)and experimental† (Bodin, McCartan, Newton, and Wolf, 1969) investigations of a linear (as opposed to a toroidal) high-β bumpy theta pinch show that it is always unstable. One possibility for its stabilization is dynamic stabilization (Weibel, 1960; Haas and Wesson, 1967a;Troyon, 1967; Berge, 1969a, 1969b). Results from straight systems of this kind are relevant to large aspect ratio toroidal systems. In this connection the M and S systems (Meyer and Schmidt, 1958; Pfirsch and Wobig, 1966; Morse, Risenfeld, and Johnson, 1968; Lotz, Remy, and Wolf, 1964; Wolf, 1969b) are of special significance. An expansion of a toroidal system in the reciprocal aspect ratio results in a linear system in leading order. In addition, a linear system is much easier to handle mathematically than the toroidal one.

In the following analysis we first outline the model being studied. The m = 1 instability of this model is then investigated, and results are discussed.

II. MODEL. A. Equilibrium. We study a high-β axi-symmetric system where the plasma is confined to the region around the axis

† H.A.B. Bodin, A.A. Newton, G.H. Wolf, and J.A. Wesson, to be published in Physics of Fluids.

by a mainly axial magnetic field ($B_z \gg B_r$). The axis is straight and the plasma assumed to be separated from the vacuum region by a sharp boundary carrying a sheet current. The equilibrium is the same as that investigated by Haas and Wesson (1967b). We also make use of their stability calculations, neglecting the stabilizing effect of finite infinitely conducting boundaries in the vacuum region. The stabilizing effect from a wall is small unless it is close to the plasma-vacuum interface. For obvious reasons this situation is not realized for highly compressed theta pinch plasmas. Cylindrical coordinates and the usual notation are used.

B. The oscillatory state. It is assumed that by controlling the external magnetic field, an oscillatory motion is set up around the equilibrium state just described. Although this has to be done experimentally by oscillating currents in external circuits, we assume that by doing this properly we are able to give the interface between plasma and vacuum a prescribed motion. This interface is assumed to be given by:

$$F(r,z,t) \equiv r - R(z,t) = 0 \tag{1}$$

where

$$R(z,t) = \overline{R}(z)\left\{1 + \varepsilon \sin \omega_s t \cos \alpha z\right\} = \overline{R}(z) + \tilde{R}(z,t). \tag{2}$$

We assume that $\varepsilon \ll 1$, and ω_s is the frequency of the oscillations, $\alpha = 2\pi/L$ where L is the wavelength in the axial direction. The magnetic field is assumed to remain tangential to the plasma-vacuum interface during the course of motion. The following solutions are obtained for the oscillatory motion:

$$\underline{\tilde{v}} = \varepsilon\left\{\omega_s \overline{R} S \cos \alpha z,\ 0,\ C_s^2 \alpha A Q \omega_s^{-1} \sin \alpha z\right\} \cos \omega_s t + O(\varepsilon^2), \tag{3}$$

$$\tilde{\rho} = -\varepsilon \overline{\rho} A Q \cos \alpha z \sin \omega_s t + O(\varepsilon^2), \tag{4a}$$

$$\tilde{p} = C_s^2 \tilde{\rho} + O(\varepsilon^2), \tag{4b}$$

$$\underline{\tilde{B}} = -\varepsilon B_{izo}\left\{\alpha \overline{R} S \sin \alpha z,\ 0,\ Q \cos \alpha z\right\} \sin \omega_s t + O(\varepsilon^2). \tag{5}$$

The symbols are defined as follows: $Q = x_o J_o(x)/J_1(x_o)$, $S = J_1(x)/J_1(x_o)$, J_o and J_1 are the zero and first order Bessel functions, $A = (1-a^2)^{-1}$, $a = \alpha C_s \omega_s^{-1}$, $x_o = k\overline{R}$, $x = kr$,

$$k = \frac{\omega_s}{C_s}\left[\frac{1-ba^2}{A+b}\right]^{\frac{1}{2}}, \quad b = \frac{2}{\gamma}\frac{1-\beta_o}{\beta_o}, \quad \beta_o = \frac{2\bar{p}}{\bar{B}^2_{vzo}}, \quad C_s^2 = \frac{\gamma\bar{p}}{\bar{\rho}},$$

and the barred quantities refer to the equilibrium. Subscripts i and v refer to the internal plasma region and the vacuum region respectively.

III. STABILITY. The stability analysis of this model is based on results derived elsewhere (Berge, 1969a, 1968). The problem is to investigate the sign of †

$$\delta\overline{W} = \delta\overline{W}_0 + \delta\overline{W}_D + \delta\overline{W}_E \tag{6}$$

where

$$\delta\overline{W}_o = -\omega_s^{-2}\int \underline{\xi}_o^* \cdot (\varepsilon^{-2}\underline{F}_e + \tilde{F}_2)(\underline{\xi}_o)d\underline{r}, \tag{7a}$$

and
$$\delta W_D = \int \bar{\rho}^{-1}|F_1'(\underline{\xi}_o)|^2 d\underline{r}. \tag{7b}$$

According to the previous results (Berge, 1968) (Appendix H, Sec. b), $\delta W_E \geq 0$, and we can obtain sufficient conditions for stability with this term omitted. Analysis shows that for the case m = 1 (and $\partial/\partial r = 0$ for the perturbations) δW_D minimizes to zero for $\xi_z = 0$. Moreover the most unstable modes are those for which $\nabla \cdot \underline{\xi}_o = 0$ and $\partial\xi_r/\partial z$ is small or of the order ε ($\underline{\xi}_o = \{\xi_r, \xi_\theta, \xi_z\}$). After a considerable amount of algebra involving expansions in small parameters, we arrive at the following expression

$$\delta\overline{W} \geq 2\pi \int \left\{a\left|\frac{d\xi_r}{dz} + \frac{b}{a}\xi_r\right|^2 + \left(c - \frac{b^2}{a}\right)|\xi_r|^2\right\}\overline{B}^2_{vzo}\, dz, \tag{8}$$

where

$$\begin{aligned} a &= (2-\beta_o)\overline{R}^2, \quad b = 2(1-\beta_o)\overline{R}\,\frac{d\overline{R}}{dz}, \\ c &= 2(1-2\beta_o)(1-\beta_o)\left(\frac{d\overline{R}}{dz}\right)^2 + \left(\frac{\partial\tilde{R}}{\partial z} + \varepsilon\,\overline{R}\,\overline{B}^{-1}_{vzo}\frac{\partial}{\partial z}\tilde{B}_{vz1}\right)^2 \\ &\quad + \varepsilon\ 2\overline{R}\,\overline{B}^{-1}_{vzo}\,\tilde{B}_{vz1}\frac{\partial^2\tilde{R}}{\partial z^2}. \end{aligned} \tag{9}$$

† The notation used here is the same as that in Berge(1968).

From Eq. (8) we obtain the following sufficient condition for MHD stability

$$\langle\bar{c}\rangle - \langle\frac{\overline{b^2}}{c}\rangle > 0 \; . \tag{10}$$

where $\langle\rangle$ denotes averaging over z, $-$ averaging over t. From the pressure balance condition we obtain in leading order:

$$\tilde{B}_{vz1} = - \bar{B}_{vzo}\, Y \cos \alpha z \sin \omega_s t, \tag{11}$$

where

$$Y = Q\left[1-\beta_o + \frac{\gamma\beta_o}{2} A\right] .$$

Let $I = \left[-\frac{\pi}{\alpha}, \frac{\pi}{\alpha}\right]$ be an interval on the z-axis. We define:

$$\bar{R} = R_o\left[1 + \delta(1 + \cos \alpha z)\right], \; z \in I \;;$$

$$\bar{R} = R_o, \; z \notin I. \tag{12}$$

This gives us a single bulge with amplitude $2\delta R_o$. By averaging condition (10) over the bulged region we obtain the stability criterion:

$$\frac{\varepsilon^2}{4}\left[Y^2 + 1\right] - \left[\frac{\beta_o(1-\beta_o)(3-2\beta_o)}{2-\beta_o}\right]\delta^2 > 0 \; , \tag{13}$$

IV. DISCUSSION. For parameter values relevant to the experimental results† ($Y \approx 2$, $\beta_o \approx 0.5$) condition (13) becomes $\varepsilon > 0.5\,\delta$. This can explain the observed effect.

By considering a periodic structure rather than a single bulge one obtains the same condition (13). One would think that this case had some relevance to large-aspect ratio M and S systems. Realistic values of Y are of the order 2-3, and one can easily estimate the amount of dynamic stabilization required to stabilize a given δ (Morse, Risenfeld, and Johnson, 1968).

The present scheme may also be used to produce "dynamic equilibrium" where no equilibrium would otherwise exist (Wolf and Berge, 1969).

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†H.A.B. Bodin, E.P. Butt, J. McCartan, and G.H. Wolf to be published.

In the context of a fusion reactor the application of dynamic stabilization will become an economic problem. This has not yet been properly considered. Dynamic stabilization may, however, also prove useful in cases where one has to pass through an unstable situation in order to build up a configuration which finally is stable. A more detailed version of this work has been published recently (Berge, 1970).