# Dynamic Stabilization of the m=1 Instability in a Bumpy Theta Pinch

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# Dynamic Stabilization of the $m = 1$ Instability in a Bumpy Theta Pinch

G. BERGE*
*Culham Laboratory, United Kingdom Atomic Energy Authority, Abingdon, Berkshire, England*
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Both theoretically and experimentally a bumpy theta pinch is known to be hydromagnetically unstable. The possibility of dynamic stabilization of this configuration is discussed. The results show that stability can be achieved in this way. The plasma column is assumed to be separated from a vacuum magnetic field by a sharp boundary. The analysis is based on a generalization of the hydromagnetic energy principle. Within the framework of this theory a sufficient condition for stability is obtained for the $m = 1$ mode. This condition contains the amplitude of the unstable bulge, the frequency and amplitude of the forced oscillations, and the characteristic lengths and speeds in the system. The condition shows that stability is obtained for sufficiently large amplitude in the oscillatory magnetic field. The case of $\beta = 1$ and "arbitrary" $m$ numbers is also briefly reviewed.

## I. INTRODUCTION

The stability of a linear (as opposed to a toroidal) high-$\beta$ bumpy theta pinch has been investigated both theoretically[1,2] and experimentally.[3] The results reveal that such systems are always hydromagnetically unstable. It is, therefore, of great interest to see whether there exist any means by which they can be made stable. One possibility is dynamic stabilization.[4-9] Our main interest in the straight system is its relevance to large aspect ratio toroidal systems. In this connection the bumpy toroidal systems[10-14] are of special significance. An expansion of a toroidal system in the reciprocal aspect ratio results in a linear system to leading order. In addition, a straight system is much easier to handle mathematically than the corresponding toroidal system.

Previously a similar system was discussed,[7] but restricted to the limit $\beta = 1$; that is, no magnetic field in the plasma region. In the subsequent analysis this restriction is removed, and the results previously obtained are improved. We first outline the model being used. The stability of this model is then studied. Although the first part of the discussion is general, the final results are limited to the $m = 1$ instability for $\beta \leq 1$, and "arbitrary" $m$ for $\beta = 1$, where $m$ as usual refers to the azimuthal wave-number. The results are discussed and comparison made with experiment. The possibility of stabilizing bumpy toroidal systems by this scheme is outlined briefly, and also its possible application to the problem of lack of equilibrium.

## II. THE MODEL

### A. Equilibrium

We shall study a high-$\beta$ axisymmetric system where the plasma is confined to the region around the axis by a mainly axial magnetic field ($B_z \gg B_r$). The axis is taken to be straight. The plasma is assumed to be separated from the vacuum region by a sharp boundary carrying a sheet current. First, the equilibrium (unstable) is considered. By assumption there is no current flowing in the plasma region; accordingly, there is no pressure gradient (except at the surface). Using the usual cylindrical coordinates $r$, $\theta$, and $z$, the surface of the plasma column (infinitely long) is assumed to be given by

$$\bar{F}(r, z) \equiv r - \bar{R}(z) = 0. \tag{1}$$

The static system is specified by the density $\bar{\rho}$, the pressure $\bar{p}$, the internal magnetic field $\bar{\mathbf{B}}_i$, and the vacuum magnetic field $\bar{\mathbf{B}}_v$. Following Haas and Wesson[2] we shall assume that all quantities are weakly dependent on $z$, so that we can take $\bar{R}(\partial/\partial z) \sim \delta$ as a parameter of smallness in terms of which this equilibrium can be expanded. $\bar{R}$ is some average value of $\bar{R}(z)$, yet to be specified.

The equilibrium problem and its stability has been investigated by Haas and Wesson[2] and for completeness we shall list their results. Introducing the parameter of smallness $\delta$ (already mentioned), we have the following leading order solution to the equilibrium problem:

$$\bar{p} = \text{const}, \qquad \bar{\rho} = \text{const}, \tag{2}$$

$$\bar{\mathbf{B}}_i = (\delta\bar{B}_{ir1}, 0, \bar{B}_{iz0} + \delta\bar{B}_{iz1}) + O(\delta^2), \tag{3}$$

$$\bar{\mathbf{B}}_v = (\delta\bar{B}_{vr1}, 0, \bar{B}_{vz0} + \delta\bar{B}_{vz1}) + O(\delta^2), \tag{4}$$

where

$$\delta\bar{B}_{ir1} = -\frac{1}{2} r \frac{d\bar{B}_{iz0}}{dz}, \tag{5}$$

$$\delta\bar{B}_{vr1} = r(1 - \beta_0)\frac{\bar{B}_{vz0}}{\bar{R}}\frac{d\bar{R}(z)}{dz} + \frac{\beta_0}{r}\bar{B}_{vz0}\bar{R}\frac{d\bar{R}(z)}{dz}, \tag{6}$$

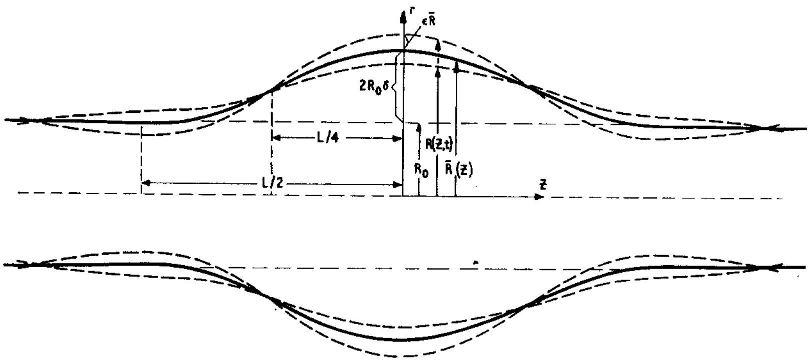


FIG. 1. Plasma column with bulge and superimposed standing wave.

$$\bar{B}_{iz0} = B_i(z), \qquad \bar{B}_{iz1} = \bar{B}_{iz1}(z), \tag{7}$$

$$\bar{B}_{vz0} = B_v(z), \qquad \bar{B}_{vz1} = \bar{B}_{vz1}(z). \tag{8}$$

By definition we have

$$\beta_0 \equiv \frac{2\bar{p}}{\bar{B}_{vz0}^2} = 1 - \frac{\bar{B}_{i0}^2}{\bar{B}_{v0}^2} = 1 - \frac{B_i^2}{B_v^2}. \tag{9}$$

The magnetic field lines have to be parallel to the interface between plasma and vacuum, that is,

$$\left.\frac{\bar{B}_{ir}}{\bar{B}_{iz}}\right|_{r=\bar{R}(z)} = \left.\frac{\bar{B}_{vr}}{\bar{B}_{vz}}\right|_{r=\bar{R}(z)} = \frac{d\bar{R}(z)}{dz}. \tag{10}$$

The expansion of the vacuum magnetic field is assumed to be valid in the vicinity of the plasma region, which is all we require for our problem. For simplicity, we shall neglect the effect of finite boundaries in the vacuum region. We know, however, that infinitely conducting walls bounding the vacuum region must have a stabilizing effect as they will always limit the number of solutions otherwise possible. This effect has been calculated for the present case by Haas and Wesson.[2] The stabilizing effect of the wall is, however, small unless the wall is very close to the plasma vacuum interface. For highly compressed theta-pinch plasmas, wall stabilization is, therefore, insignificant.

### B. The Oscillatory State

We now assume that by controlling the external magnetic field, an oscillatory motion is set up around the equilibrium just described. Although this has to be done experimentally by oscillating currents in external circuits, we assume that by doing this properly we are able to give the interface between plasma and vacuum a prescribed motion. In the following we shall assume that the interface between plasma and vacuum is given by (see Fig. 1):

$$F(r, z, t) \equiv r - R(z, t) = 0, \tag{11}$$

where

$$\begin{aligned} R(z, t) &= \bar{R}(z)(1 + \epsilon \sin \omega_s t \cos \alpha z) \\ &= \bar{R}(z) + \tilde{R}(z, t). \end{aligned} \tag{12}$$

We assume that $\epsilon \ll 1$. The condition that the surface Eq. (11) is moving with the fluid can be expressed as

$$\frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F = 0. \tag{13}$$

In the context of the present model this is to be regarded as a boundary condition on the velocity field in the fluid. Moreover, the magnetic field lines must be tangential to the surface Eq. (11), this requires that the following relation is satisfied:

$$\left.\frac{B_r}{B_z}\right|_{r=R(z,t)} = \frac{\partial R(z, t)}{\partial z}, \tag{14}$$

where $R$ is given by Eq. (12) and Eq. (14) applies both to the internal and the external magnetic field. We can now solve the equations of motion [see Ref. 7 Eqs. (1)–(7)]. The following solution is obtained for the forced oscillatory motion:

$$\tilde{v}_r = \epsilon\omega_s \bar{R} S \cos \alpha z \cos \omega_s t + O(\epsilon^2), \tag{15}$$

$$\tilde{v}_z = \epsilon \frac{C_s^2 \alpha A Q}{\omega_s} \sin \alpha z \cos \omega_s t + O(\epsilon^2), \tag{16}$$

$$\tilde{\rho} = -\epsilon\bar{\rho} A Q \cos \alpha z \sin \omega_s t + O(\epsilon^2), \tag{17}$$

$$\tilde{p} = C_s^2 \tilde{\rho} + O(\epsilon^2), \tag{18}$$

$$\tilde{B}_{ir} = -\epsilon \bar{B}_{iz0} \alpha \bar{R} S \sin \alpha z \sin \omega_s t + O(\epsilon^2), \tag{19}$$

$$\tilde{B}_{iz} = -\epsilon \bar{B}_{iz0} Q \cos \alpha z \sin \omega_s t + O(\epsilon^2). \tag{20}$$

The following notation is used:

$$S = S(x) = \frac{J_1(x)}{J_1(x_0)}, \qquad Q = Q(x) = \frac{x_0 J_0(x)}{J_1(x_0)},$$

$$x_0 = k\bar{R}, \qquad x = kr,$$

$$k^2 = \frac{\omega_s^2 - V_{Ai}^2 \alpha^2}{V_{Ai}^2 + A C_s^2} = \frac{\omega_s^2}{C_s^2} \frac{1 - ba^2}{A + b}, \tag{21}$$

$$A = \frac{1}{1 - a^2}, \qquad a = \frac{\alpha C_s}{\omega_s}, \qquad C_s^2 = \frac{\gamma \bar{p}}{\bar{\rho}},$$

$$\alpha = \frac{2\pi}{L}, \qquad b = \frac{2}{\gamma} \frac{1 - \beta_0}{\beta_0}, \qquad V_{Ai}^2 = \frac{\bar{B}_{iz0}^2}{\bar{\rho}}.$$

$J_0$ and $J_1$ are the zero- and first-order Bessel functions with the usual notation, $L$ is the wavelength in the axial direction [see Eq. (12)], $\gamma$ is the adiabatic exponent, and otherwise the symbols are self-explanatory. We shall later require a relation between the vacuum magnetic field and the internal magnetic field at the plasma vacuum interface. This is obtained from the pressure balance condition [Ref. 7 Eq. (8)].

## III. STABILITY

The stability of the model established in Sec. II is now to be discussed. It will be based on results derived elsewhere.[7,8] Thus, the problem of stability is to be solved by examining the sign of $\delta\bar{W}$ given by Ref. 8, Eq. (4.12), and the system is stable if $\delta\bar{W} > 0$. We have

$$\delta\bar{W} = \delta\bar{W}_O + \delta\bar{W}_D + \delta\bar{W}_E, \tag{22}$$

where

$$\delta\bar{W}_O = -\omega_s^{-2} \overline{\int} \boldsymbol{\xi}_0^* \cdot (\epsilon^{-2}\mathbf{F}_e + \tilde{\mathbf{F}}_2)(\boldsymbol{\xi}_0)\, d\mathbf{r}, \tag{23}$$

$$\delta\bar{W}_D = \overline{\int} \bar{\rho}^{-1}\, |\mathbf{F}_1'(\boldsymbol{\xi}_0)|^2\, d\mathbf{r}, \tag{24}$$

$$\delta\bar{W}_E \geq 0, \tag{25}$$

$$\mathbf{F}(\boldsymbol{\xi}) = \nabla(\gamma p \nabla\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) - \mathbf{B}\times(\nabla\times\mathbf{Q}) - \mathbf{Q}\times(\nabla\times\mathbf{B}) + \nabla\cdot\left(\boldsymbol{\xi}\rho\frac{d\mathbf{v}}{dt} - \rho\mathbf{v}\mathbf{v}\cdot\nabla\boldsymbol{\xi}\right),$$

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}). \tag{26}$$

Here, $\boldsymbol{\xi} = \boldsymbol{\xi}_0 + \epsilon\boldsymbol{\xi}_1 + \cdots$ is a Lagrangian displacement of a fluid element, and $\overline{\int}$ means a time average over the fast oscillations, of the integral. The procedure is now to minimize the potential energy $\delta\bar{W}$ with respect to $\boldsymbol{\xi}$. Moreover, we have $\mathbf{F}_e(\boldsymbol{\xi}) = \lim_{\epsilon\to 0} \mathbf{F}(\boldsymbol{\xi})$, and $\mathbf{F}_n(\boldsymbol{\xi})$ is the contribution to $\mathbf{F}(\boldsymbol{\xi})$ which is of order $\epsilon^n$, and $\mathbf{F}_1'(\boldsymbol{\xi})$ is given explicitly by Eq. (41). Notice also that we have assumed the unperturbed quantities to be expanded as $q = \bar{q} + \epsilon\tilde{q}_1 + \epsilon^2\tilde{q}_2 + \cdots$, where the $\tilde{q}_n$'s are due to the oscillatory motion imposed on the system. Equation (25) is discussed in Ref. 8, Eq. (5.8) and in Ref. 8, Appendix H, Sec. b. Since $\delta\bar{W}_E$ is very difficult to evaluate, in this discussion we shall retain only the first two terms in the expression for $\delta\bar{W}$, Eq. (22). The results obtained by omitting this term lead to sufficient conditions for stability within the framework of our theory.

(a) We shall start by discussing the term $\delta\bar{W}_O$ given by Eq. (23). Since our problem is one where we have a sharp boundary, we shall also make use of the results given in Ref. 8, Eq. (7.2), which reads

$$\begin{aligned}
-\int \boldsymbol{\xi}^*\cdot\mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r}
&= \int_{V_v} |\nabla\times\mathbf{A}|^2\, d\mathbf{r} + \int_S |\mathbf{n}\cdot\boldsymbol{\xi}|^2\, \mathbf{n}\cdot\left\langle\nabla\left(p + \frac{B^2}{2}\right)\right\rangle d\sigma \\
&+ \int_{V_p} \left\{ |\mathbf{B}\cdot\nabla\boldsymbol{\xi}|^2 - \rho\,|\mathbf{v}\cdot\nabla\boldsymbol{\xi}|^2 \right. \\
&+ \boldsymbol{\xi}^*\boldsymbol{\xi} : \nabla\nabla\left(p + \frac{B^2}{2}\right) + |\nabla\cdot\boldsymbol{\xi}|^2\left(\gamma p + \frac{B^2}{2}\right) \\
&+ \nabla\cdot\boldsymbol{\xi}^*\left[\boldsymbol{\xi}\cdot\nabla\left(p + \frac{B^2}{2}\right) - \mathbf{B}\mathbf{B} : \nabla\boldsymbol{\xi}\right] \\
&\left. + \nabla\cdot\boldsymbol{\xi}\left[\boldsymbol{\xi}^*\cdot\nabla\left(p + \frac{B^2}{2}\right) - \mathbf{B}\mathbf{B} : \nabla\boldsymbol{\xi}^*\right]\right\} d\mathbf{r}.
\end{aligned} \tag{27}$$

According to Eq. (23), we require the expression above with $\mathbf{F}$ replaced by $\mathbf{F}_e + \epsilon^2\mathbf{F}_2$ and $\boldsymbol{\xi}$ by $\boldsymbol{\xi}_0$. This is most easily achieved by considering each term of $\delta\bar{W}_O$ separately.

(i) We shall start with the contribution from the plasma volume integral. In order to get the ordering right we shall first consider the term $|\mathbf{B}\cdot\nabla\boldsymbol{\xi}|^2$. We now write $\mathbf{B} = \bar{\mathbf{B}} + \tilde{\mathbf{B}}$ where $\bar{\mathbf{B}}$ refers to the equilibrium and $\tilde{\mathbf{B}}$ is the oscillatory component in the field. (We shall make use of the same notation for other quantities when it is convenient.) We shall formally expand the equilibrium problem in terms of a small parameter $\delta$ and the oscillatory problem is to be expanded in terms of $\epsilon$, where we assume that $\delta \sim \epsilon$. Now omitting terms which are linearly dependent on the oscillatory components we obtain

$$|\mathbf{B}\cdot\nabla\boldsymbol{\xi}_0|^2_{(2)} = |\bar{\mathbf{B}}\cdot\nabla\boldsymbol{\xi}|^2_{(2)} + |\tilde{\mathbf{B}}_1\cdot\nabla\boldsymbol{\xi}_0|^2, \tag{28}$$

where

$$|\bar{\mathbf{B}}\cdot\nabla\boldsymbol{\xi}_0|^2_{(2)} = \sum_k \left|\left(\bar{B}_{iz0}\frac{\partial}{\partial z} + \delta\bar{B}_{ir1}\frac{\partial}{\partial r}\right)\xi_k\right|^2, \tag{29}$$

$$|\tilde{\mathbf{B}}_1\cdot\nabla\boldsymbol{\xi}_0|^2 = \epsilon^2 \sum_k \left|\left(\tilde{B}_{iz1}\frac{\partial}{\partial z} + \tilde{B}_{ir1}\frac{\partial}{\partial r}\right)\xi_k\right|^2. \tag{30}$$

The summation is over $k = r, \theta, z$. The most important feature of Eq. (29) is that $B_{iz0}(\partial/\partial z)\xi_k$ must be a small quantity of order $\epsilon$ for the unstable modes. For definition of unstable modes see Ref. 8, page 11. Having established this ordering we now look at the term

$$|\nabla\cdot\boldsymbol{\xi}_0|^2\left(\gamma p + \frac{B^2}{2}\right).$$

This term which is positive definite must also be small, of order $\epsilon^2$ for the unstable modes. Thus, since

$\gamma p + (B^2/2)$ is of zero order this requires that $\nabla \cdot \boldsymbol{\xi}_0$ be small of order $\epsilon$. It is now convenient to change to new variables. We shall take $\xi_r$, $\xi_z$, and $\eta = \nabla \cdot \boldsymbol{\xi}_0$ as new variables instead of $\xi_r, \xi_\theta, \xi_z$. Since there is no explicit $\theta$ dependence in the problem, this is a simple transformation. By Fourier analyzing in the $\theta$ coordinate, i.e., taking $\boldsymbol{\xi}_0 \propto \exp(im\theta)$, it follows that

$$i\xi_\theta = -\frac{1}{m}\frac{\partial}{\partial r}(r\xi_r) - \frac{r}{m}\frac{\partial \xi_z}{\partial z} + \frac{r}{m}\eta. \tag{31}$$

By this equation, $\xi_\theta$ can be eliminated from the problem, and it is easy to see that $\delta\bar{W}_O$ is properly minimized by $\eta = \nabla \cdot \boldsymbol{\xi}_0 = 0$. One can easily prove the following relation:

$$\boldsymbol{\xi}_0^*\boldsymbol{\xi}_0 : \nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)} = \nabla \cdot \left[\boldsymbol{\xi}_0^*\boldsymbol{\xi}_0 \cdot \nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)}\right]$$
$$- \nabla \cdot \boldsymbol{\xi}^*\boldsymbol{\xi} \cdot \nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)} - \boldsymbol{\xi}^* \cdot \nabla \boldsymbol{\xi} \cdot \nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)}. \tag{32}$$

Now the term

$$\boldsymbol{\xi}_0^*\boldsymbol{\xi}_0 : \nabla\left(p + \frac{B^2}{2}\right)$$

has one contribution associated with the equilibrium and another which is associated with the oscillatory motion. In Eq. (32) we only consider the contribution which comes from the oscillatory motion. We can integrate this contribution by parts using Eq. (32). Then remembering that $\nabla \cdot \boldsymbol{\xi}_0^* = 0$, we are left with the term

$$-\boldsymbol{\xi}_0^* \cdot \nabla \boldsymbol{\xi}_0 \cdot \nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)}$$

integrated over the volume, and a surface integral which is cancelled by terms coming from the surface integral in Eq. (27). After some algebra we obtain (see Appendix A for details),

$$\boldsymbol{\xi}_0^* \cdot \nabla \boldsymbol{\xi}_0 = \left[\left(1 - \frac{1}{m^2}\right)\frac{1}{r}\frac{\partial}{\partial r}(r|\xi_r|^2) - \frac{r}{m^2}\left|\frac{\partial \xi_r}{\partial r}\right|^2\right]\mathbf{e}_r$$
$$+ a\mathbf{e}_\theta + \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r^*\xi_z)\mathbf{e}_z \tag{33}$$

(we do not have to specify $a$). Making use of the equations of motion we obtain

$$\overline{\nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)}}$$
$$= \overline{\left[-\frac{\bar{p}}{r}\frac{\partial}{\partial r}(r\tilde{v}_{r1}'^2) + \frac{\partial}{\partial r}\frac{\tilde{B}_{ir1}^2}{2} + \tilde{B}_{iz1}\frac{\partial \tilde{B}_{ir1}}{\partial z}\right]}\mathbf{e}_r$$
$$\cdot \frac{\partial f_1(r, z)}{\partial z}\mathbf{e}_r + \frac{\partial f_2(r, z)}{\partial z}\mathbf{e}_z, \qquad \tilde{\mathbf{v}}_1' = \omega_s \tilde{\mathbf{v}}_1, \tag{34}$$

where we do not need to specify $f_1$ and $f_2$ [see Appendix B for details of Eqs. (34)]. From Eq. (33) and (34) we now obtain

$$\overline{\boldsymbol{\xi}^* \cdot \nabla \boldsymbol{\xi} \cdot \nabla\left(p + \frac{B^2}{2}\right)_{(2)}}$$
$$= \frac{\partial}{\partial z}f(r, z) + \left[\left(1 - \frac{1}{m^2}\right)\frac{1}{r}\frac{\partial}{\partial r}(r|\xi_r|^2) - \frac{r}{m^2}\left|\frac{\partial \xi_r}{\partial r}\right|^2\right]$$
$$\cdot \left[-\frac{\bar{p}}{r}\frac{\partial}{\partial r}\overline{(r\tilde{v}_{r1}'^2)} + \frac{\partial}{\partial r}\frac{\overline{\tilde{B}_{ir1}^2}}{2} + \overline{\tilde{B}_{iz1}\frac{\partial \tilde{B}_{ir1}}{\partial z}}\right]. \tag{35}$$

(ii) Taking into account the cancellation due to an integration by parts coming from the volume integral of the plasma region, the surface integral can be written in the following form:

$$\overline{\int} |\mathbf{n} \cdot \boldsymbol{\xi}|^2 \mathbf{n} \cdot \left\langle \nabla\left(p + \frac{B^2}{2}\right)\right\rangle d\sigma + T_1$$
$$= \overline{\int} |\mathbf{n} \cdot \boldsymbol{\xi}|^2 \left(\beta_0 B_{vz0}^2 \frac{d^2\bar{R}}{dz^2} + 2\epsilon \bar{\mathbf{B}}_v \cdot \tilde{\mathbf{B}}_{v1}\frac{\partial^2 \tilde{R}}{\partial z^2}\right)\bar{R}\, d\theta\, dz. \tag{36}$$

The details of this derivation are given in Appendix C. Note that $T_1$ is the contribution coming from the integration by parts, Eq. (32).

(iii) The contribution to $\delta\bar{W}_O$ coming from the vacuum region can be expressed as

$$\overline{\int} |\nabla \times \mathbf{A}|^2\, d\mathbf{r} = \frac{2\pi}{m}\int (\bar{G}^2 + \overline{\tilde{G}^2})\, dz, \tag{37}$$

where

$$|\bar{G}|^2 = \left|\bar{B}_{vz0}\bar{R}\frac{d\xi_r}{dz} - \xi_r\bar{B}_{vz0}\frac{d\bar{R}}{dz}(1 - 2\beta_0)\right|^2, \tag{38}$$

$$|\tilde{G}|^2 = \left|\bar{B}_{vz0}\frac{\partial \tilde{R}}{\partial z}\left(\bar{R}\frac{\partial \xi_r}{\partial r} + \xi_r\right) + \epsilon\xi_r\bar{R}\frac{\partial \tilde{B}_{vz1}}{\partial z}\right|^2$$
$$+ \epsilon^2\bar{R}^2|\xi_z|^2\overline{\left(\frac{\partial \tilde{B}_{vr1}}{\partial z}\right)^2} + \frac{d}{dz}f_2(z), \tag{39}$$

and

$$\frac{d\xi_r}{dz} = \frac{\partial \xi_r}{\partial z} + \frac{\partial \bar{R}}{\partial z}\frac{\partial \xi_r}{\partial r}.$$

(For details see Appendix D.)

(b) We shall now discuss the "dynamic stabilization term" $\delta\bar{W}_D$ Eq. (22). From Ref. 8, Eqs. (4.16) and (4.18) we know that

$$\delta\bar{W}_D = \int \frac{1}{\bar{\rho}}|\mathbf{F}_1'(\boldsymbol{\xi}_0)|^2\, d\mathbf{r}, \tag{40}$$

where

$$\mathbf{F}_1'(\boldsymbol{\xi}_0) = \frac{1}{\omega_s^2}\tilde{\mathbf{F}}_1(\boldsymbol{\xi}_0) - \bar{\rho}\frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau} \cdot \nabla \boldsymbol{\xi}_0. \tag{41}$$

We can, therefore, write

$$\delta \bar{W}_D = \omega_s^{-4} \overline{\int \frac{1}{\bar{\rho}} |\tilde{\mathbf{F}}_1(\xi_0)|^2 \, d\mathbf{r}} + \overline{\int \bar{\rho} \left| \frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau} \cdot \nabla \xi_0 \right|^2 d\mathbf{r}}$$

$$- \frac{2}{\omega_s^2} \operatorname{Re} \left[ \overline{\int \tilde{\mathbf{F}}_1(\xi_0) \frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau} : \nabla \xi_0^* \, d\mathbf{r}} \right], \quad \tau = \omega_s t. \quad (42)$$

Written in this way, the second term above cancels the second term in the last integral of Eq. (27) when averaged over the time $\tau$. Re in front of the last term above means the real part. From Ref. 8, Eq. (4.5), we get

$$\begin{aligned} \tilde{\mathbf{F}}_1(\xi_0) = {} & \nabla(\gamma \tilde{p}_1 \nabla \cdot \xi_0) + \nabla \cdot \xi_0 \bar{\rho} \omega_s^2 \frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau} + \nabla \xi_0 \cdot \nabla \tilde{p}_1 \\ & + \xi_0 \cdot \nabla \left( \nabla \tilde{p} + \bar{\rho} \omega_s^2 \frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau_0} \right) \\ & - \bar{\mathbf{B}} \times (\nabla \times \tilde{\mathbf{Q}}_1) - \tilde{\mathbf{B}}_1 \times (\nabla \times \bar{\mathbf{Q}}) \\ & - \bar{\mathbf{Q}} \times (\nabla \times \tilde{\mathbf{B}}_1) - \bar{\mathbf{Q}}_1 \times (\nabla \times \bar{\mathbf{B}}). \end{aligned} \quad (43)$$

By assumption we have $\nabla \times \bar{\mathbf{B}} = 0$ to leading order. Moreover, it is easy to see that for the unstable modes we must have $\bar{\mathbf{Q}}$ of order $\epsilon$. We have already proved that $\nabla \cdot \xi_0 = \eta$ is of order $\epsilon$. By keeping terms to leading order we obtain

$$\tilde{\mathbf{F}}_1(\xi_0) = \nabla \xi_0 \cdot \nabla \tilde{p}_1 + \xi_0 \cdot \nabla [(\nabla \times \tilde{\mathbf{B}}_1) \times \bar{\mathbf{B}}] + (\nabla \times \tilde{\mathbf{Q}}_1) \times \bar{\mathbf{B}}. \quad (44)$$

Now $\tilde{\mathbf{Q}}_1$ is given by

$$\tilde{\mathbf{Q}}_1 = \nabla \times (\xi_0 \times \tilde{\mathbf{B}}_1). \quad (45)$$

Making use of Eq. (45) we can rewrite Eq. (44)

$$\begin{aligned} \tilde{\mathbf{F}}_1(\xi_0) = {} & \nabla \xi_0 \cdot \nabla \tilde{p}_1 + \bar{B}_{iz0} \left[ \frac{\partial \xi_r}{\partial r} \left( \frac{\partial \tilde{B}_{ir1}}{\partial z} + \frac{\partial \tilde{B}_{iz1}}{\partial r} \right) \right. \\ & \left. - \frac{\partial}{\partial r} \left( \tilde{B}_{ir1} \frac{\partial \xi_z}{\partial r} \right) + \frac{\partial \xi_z}{\partial r} \frac{\partial \tilde{B}_{iz1}}{\partial z} \right] \mathbf{e}_r \\ & + \frac{i}{m} \bar{B}_{iz0} \left[ \frac{m^2 - 1}{r} \xi_r \frac{\partial \tilde{B}_{iz1}}{\partial r} + \frac{\partial \xi_r}{\partial r} \left( 2 \frac{\partial \tilde{B}_{ir1}}{\partial z} - \frac{\partial \tilde{B}_{iz1}}{\partial r} \right) \right. \\ & \left. + r \frac{\partial^2 \xi_r}{\partial r^2} \frac{\partial \tilde{B}_{ir1}}{\partial z} + \frac{m^2}{r} \left( \xi_z \frac{\partial \tilde{B}_{iz1}}{\partial z} - \tilde{B}_{ir1} \frac{\partial \xi_z}{\partial r} \right) \right] \mathbf{e}_\theta. \end{aligned} \quad (46)$$

From the results obtained so far, the general case turns out to be rather complicated, we shall, therefore, have to limit ourselves to the special case $m = 1$.

### IV. THE $m = 1$ CASE

The expressions obtained so far simplify considerably for the case $m = 1$ and $\partial \xi / \partial r = 0$. Moreover, the experimental results[3,15] indicate that this is the most interesting case.

Using Eqs. (41) and (46) as well as the expression for $\nabla \xi_0$ given in Appendix A, we now obtain from Eq. (40) that

$$\delta \bar{W}_D = \omega_s^{-4} \overline{\int \frac{1}{\bar{\rho}} \left( \frac{\partial \tilde{p}_1}{\partial z} + \bar{B}_{iz0} \frac{\partial \tilde{B}_{iz1}}{\partial z} \right)^2 \frac{|\xi_z|^2}{r^2} \, d\mathbf{r}}. \quad (47)$$

From Eq. (35) we obtain

$$\xi_0^* \cdot \nabla \xi_0 \cdot \nabla \overline{\left( p + \frac{B^2}{2} \right)}_{(2)} = \frac{\partial}{\partial z} f'(r, z). \quad (48)$$

By assuming either periodic boundary conditions (to simulate a toroidal system) or that the perturbations are vanishingly small for large $|z|$, all terms of the form $(\partial / \partial z) f(r, z)$ integrate to zero in leading order (see Appendix E).

We now integrate the term given by Eq. (32) by parts. We find that the only contribution to $\delta \bar{W}_O$ coming from the plasma volume is associated with the equilibrium state.

Now $\xi_z$ occurs in Eqs. (47), (39), and the terms associated with the equilibrium problem [see Haas and Wesson,[2] Eqs. (3.11)–(3.13)]. Since all these terms are positive definite, we see that the "effective potential energy" $\delta \bar{W}$ is properly minimized by taking $\xi_z = 0$. From Eq. (47), we then obtain

$$\delta \bar{W}_D = 0 \qquad (\xi_z = 0). \quad (49)$$

We now write

$$\epsilon^2 \omega_s^2 \, \delta \bar{W}_O = \delta W_v + \delta W_p + \bar{S} + \tilde{S}. \quad (50)$$

Here, $\delta W_v$ is an expression for the perturbation in the "effective potential energy" in the vacuum region. $\delta W_p$ is the contribution from the plasma region. $\bar{S}$ and $\tilde{S}$ can be interpreted as perturbations in the surface potential. $\bar{S}$ originates from the associated equilibrium problem, and $\tilde{S}$ is due to the oscillatory motion, i.e., the dynamic stabilization.

Now, $\delta W_v$ is given by

$$\delta W_v = \overline{\int |\nabla \times \mathbf{A}|^2 \, d\mathbf{r}} = 2\pi \overline{\int (|\bar{G}|^2 + |\tilde{G}|^2) \, dz}, \quad (51)$$

where from Eqs. (38) and (39) we have that

$$|\bar{G}|^2 = \bar{B}_{vz0}^2 \left| \bar{R} \frac{d\xi_r}{dz} - \xi_r \frac{d\bar{R}}{dz} (1 - 2\beta_0) \right|^2 \quad (52)$$

and

$$|\tilde{G}|^2 = \bar{B}_{vz0}^2 \, |\xi_r|^2 \left( \frac{\partial \tilde{R}}{\partial z} + \epsilon \frac{\bar{R}}{\bar{B}_{vz0}} \frac{\partial}{\partial z} \tilde{B}_{vz1} \right)^2. \quad (53)$$

Furthermore, we obtain

$$\delta W_p = 2\pi \overline{\int \bar{B}_{iz0}^2 \left| \bar{R} \frac{d\xi_r}{dz} - \xi_r \frac{d\bar{R}}{dz} \right|^2 dz}, \quad (54)$$

$$\bar{S} = 2\pi \overline{\int |\xi_r|^2 \beta_0 \bar{B}_{vz0}^2 \frac{d^2\bar{R}}{dz^2} \bar{R}\, dz}, \tag{55}$$

and

$$\tilde{S} = 4\pi \overline{\int \epsilon |\xi_r|^2 \bar{B}_{vz0} \tilde{B}_{vz1} \frac{\partial^2 \tilde{R}}{\partial z^2} \tilde{R}\, dz}. \tag{56}$$

The only term giving a negative contribution to the potential energy is $\bar{S}$, namely, the source term of the instability in the equilibrium problem. This term can also be interpreted in terms of unfavorable curvature of the magnetic field lines at the surface of the plasma. The expression for $\bar{S}$, Eq. (55), can be integrated by parts and we obtain

$$\begin{aligned}\bar{S} = 2\pi \overline{\int} \bigg[\frac{d}{dz}\left(|\xi_r|^2 \beta_0 \bar{B}_{vz0}^2 \frac{d\bar{R}}{dz}\bar{R}\right) \\ - |\xi_r|^2 \beta_0 \bar{B}_{vz0}^2 \left(\frac{d\bar{R}}{dz}\right)^2\bigg] dz \\ - 2\pi \overline{\int \left(\xi_r^* \frac{d\xi_r}{dz} + \xi_r \frac{d\xi_r^*}{dz}\right) \beta_0 \bar{B}_{vz0}^2 \bar{R} \frac{d\bar{R}}{dz}\, dz}. \end{aligned}\tag{57}$$

By using the boundary conditions we have chosen, the first term in this expression integrates to zero. Notice also that $(d/dz)(\beta_0 B_{vz0}^2) = 0$. After some rearrangement we can now write

$$\begin{aligned}\epsilon^2\omega_s^2\, \delta\bar{W} \geq 2\pi \overline{\int} \bigg[a \left|\frac{d\xi_r}{dz}\right|^2 + 2b \\ \cdot \mathrm{Re}\left(\xi_r^* \frac{d\xi_r}{dz}\right) + c\, |\xi_r|^2\bigg] \bar{B}_{vz0}^2\, dz, \end{aligned}\tag{58}$$

where

$$a = (2 - \beta_0)\bar{R}^2, \tag{59}$$

$$b = -2(1 - \beta_0)\bar{R}\frac{d\bar{R}}{dz}, \tag{60}$$

$$\begin{aligned}c = 2(1 - 2\beta_0)(1 - \beta_0)\left(\frac{d\bar{R}}{dz}\right)^2 \\ + \overline{\left(\frac{\partial\tilde{R}}{\partial z} + \epsilon \frac{\bar{R}}{\bar{B}_{vz0}} \frac{\partial}{\partial z}\tilde{B}_{vz1}\right)^2} + \epsilon \frac{2\bar{R}}{\bar{B}_{vz0}} \overline{\tilde{B}_{vz1} \frac{\partial^2 \tilde{R}}{\partial z^2}}. \end{aligned}\tag{61}$$

Moreover, by completing the square containing $d\xi_r/dz$, we can rewrite Eq. (58) as

$$\begin{aligned}\epsilon^2\omega_s^2\, \delta\bar{W} \geq 2\pi \overline{\int} \bigg[a \left|\frac{d\xi_r}{dz} + \frac{b}{a}\xi_r\right|^2 \\ + \left(c - \frac{b^2}{a}\right) |\xi_r|^2\bigg] \bar{B}_{vz0}^2\, dz. \end{aligned}\tag{62}$$

From this expression we now obtain the following sufficient condition for stability:

$$\langle\bar{c}\rangle - \left\langle\frac{\overline{b^2}}{a}\right\rangle > 0. \tag{63}$$

Here, the bars indicate an average over time and the angled brackets an average over $z$. Explicitly, this condition becomes

$$\begin{aligned}\left\langle\overline{\left(\frac{\partial\tilde{R}}{\partial z} + \epsilon \frac{\bar{R}}{\bar{B}_{vz0}} \frac{\partial}{\partial z}\tilde{B}_{vz1}\right)^2 + \epsilon \frac{2\bar{R}}{\bar{B}_{vz0}} \tilde{B}_{vz1} \frac{\partial^2 \tilde{R}}{\partial z^2}}\right\rangle \\ - \left\langle\left[\frac{4(1 - \beta_0)^2}{2 - \beta_0} - 2(1 - 2\beta_0)(1 - \beta_0)\right]\left(\frac{d\bar{R}}{dz}\right)^2\right\rangle > 0. \end{aligned}\tag{64}$$

From the pressure balance condition, we obtain, in leading order after having substituted for $\tilde{B}_{iz1}$ and $\tilde{p}_1$ from Eqs. (20) and (18),

$$\frac{\tilde{B}_{vz1}}{\bar{B}_{vz0}} = -Y \cos \alpha z \sin \omega_s t \tag{65}$$

and

$$\frac{\bar{R}}{\bar{B}_{vz0}} \frac{\partial}{\partial z}\tilde{B}_{vz1} = \bar{R}\alpha Y \sin \alpha z \sin \omega_s t, \tag{66}$$

where

$$Y \equiv \frac{x_0 J_0(x_0)}{J_1(x_0)}\left(1 - \beta_0 + \frac{\gamma\beta_0}{2}A\right) \tag{67}$$

[see Eqs. (21) for definitions].

Note that the pressure balance condition can be differentiated along, but not across, the interface. Moreover, we have (see Fig. 1),

$$\tilde{R} = \epsilon\bar{R} \sin \omega_s t \cos \alpha z. \tag{68}$$

By taking $\bar{R}(z)$ to be given by (see Bodin *et al.*[3])

$$\bar{R} = R_0[1 + \delta(1 + \cos \alpha z)], \tag{69}$$

we can write condition (64) as

$$\tfrac{1}{4}\epsilon^2(Y^2 + 1) - Z(\beta_0)\,\delta^2 > 0, \tag{70}$$

where

$$\begin{aligned}Z(\beta_0) &= (2\beta_0 - 1)(1 - \beta_0) + \frac{2(1 - \beta_0)^2}{2 - \beta_0} \\ &= \frac{\beta_0(1 - \beta_0)(3 - 2\beta_0)}{2 - \beta_0}. \end{aligned}\tag{71}$$

In the case previously considered[7] the results were only valid for $\beta = 1$ and $Y^2 \gg 1$ in which case $Y^2 + 1 \rightarrow Y^2$ in Eq. (70). However, this case was treated for arbitrary $m$, long-wavelength instabilities. It is easily seen that the present results reduce to those previously obtained in the limits where they overlap. In conditions (64) and (70) the results are valid for arbitrary $Y$, but restricted to the case $m = 1$. $Y$ must, of course, not be so large that the linearized solution to the unperturbed oscillatory motion problem is no longer valid.

In the next section we are going to reconsider the case $\beta = 1$ with some improvement on the calculation previously done.

## V. THE CASE $\beta = 1$ ("ARBITRARY" $m$)

We shall briefly reconsider the case $\beta = 1$ with arbitrary $m$, subjected to the general limitation of long-wavelength perturbations. This treatment is an improvement and extension of results previously obtained.[7]

In this case the fluid motion is governed by the equations of ordinary hydrodynamics. From Cauchy's integral of Helmholtz's equation it follows that

$$\nabla \times \xi_0 = 0, \tag{72}$$

where we have made use of the expression for the perturbed velocity, Ref. 8, Eq. (2.17)

From Eq. (33) it follows that

$$\tilde{\mathbf{F}}_1(\xi_0) = \nabla\xi_0 \cdot \nabla\tilde{p}_1. \tag{73}$$

Moreover, we have that

$$\frac{\partial \tilde{\mathbf{v}}_1}{\partial \tau_0} = -\frac{1}{\rho\omega_s^2} \nabla\tilde{p}_1. \tag{74}$$

From Eq. (72) it follows that

$$\nabla\tilde{p}_1 \cdot \nabla\xi_0 = \nabla\xi_0 \cdot \nabla\tilde{p}_1. \tag{75}$$

Equation (42) can now be written as

$$\delta\bar{W}_D = \frac{3}{\omega_s^4} \overline{\int} \frac{1}{\bar{p}} |\nabla\xi \cdot \nabla\tilde{p}_1|^2 + \overline{\int} \bar{p} \left|\frac{\partial\tilde{\mathbf{v}}_1}{\partial\tau_0} \cdot \nabla\xi_0\right|^2 d\mathbf{r}. \tag{76}$$

Since $\delta\bar{W}$ is minimized by $\nabla \cdot \xi_0 = 0$, the proper solution is

$$\xi \propto r^{m-1}, \tag{77}$$

and with $\xi_z$ small of order $\epsilon\xi_r$. From Eq. (A4) we obtain

$$\nabla\xi_0 \cdot \nabla\tilde{p}_1 = (m-1)\frac{\partial\tilde{p}_1}{\partial r}\frac{\xi_r}{r}[\mathbf{e}_r + i\mathbf{e}_\theta]. \tag{78}$$

Thus, it follows that

$$|\nabla\xi_0 \cdot \nabla\tilde{p}_1|^2 = 2(m-1)^2\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2 \frac{|\xi_r|^2}{r^2}. \tag{79}$$

From Eq. (35) we have

$$-\xi_0^* \cdot \nabla\xi_0 \cdot \overline{\nabla\tilde{p}_2} = \frac{\partial}{\partial z} f(r, z) + 2(m-1)\frac{|\xi_r|^2}{r^2}\omega_s^2\bar{p}\frac{\partial}{\partial r}\overline{(r\tilde{v}_{r1}^2)}. \tag{80}$$

Substituting from Eq. (74) we can now write the contribution from the plasma region as

$$\begin{aligned}
\frac{1}{\epsilon^2\omega_s^2}\,\delta W_p + \delta\bar{W}_D
&= \frac{2(m-1)}{\omega_s^4}\int\left[(3m-2)\frac{|\xi_r|^2}{\bar{p}r^2}\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2} + \frac{|\xi_r|^2}{\bar{p}r}\frac{\partial}{\partial r}\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2}\right]d\mathbf{r} \\
&= \frac{2(m-1)}{\omega_s^4}\int\left\{m\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2}\frac{|\xi_r|^2}{\bar{p}r} + \frac{\partial}{\partial r}\left[\frac{|\xi_r|^2}{\bar{p}}\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2}\right]\right\}dr\,d\theta\,dz,
\end{aligned} \tag{81}$$

where we have omitted the term which integrates to zero in leading order.

From Eqs. (36)–(39), and (57) we obtain $\delta W_v$, $\bar{S}$, and $\tilde{S}$ with $\beta_0 = 1$. Together with Eq. (81) this results in the following expression for $\delta\bar{W}$:

$$\begin{aligned}
\epsilon^2\omega_s^2\,\delta\bar{W} &= 2\pi\overline{\int}\left(\bar{R}^2\left|\frac{d\xi_r}{dz}\right|^2 + c\,|\xi_r|^2\right)\bar{B}_{vz0}^2\,dz \\
&\quad + 2\pi\epsilon^2 m(m-1)\int\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2}\frac{|\xi_r|^2}{\bar{p}r}\,dr\,dz,
\end{aligned} \tag{82}$$

where

$$\begin{aligned}
c &= -\frac{m-1}{m}\bar{B}_{vz0}^2\left(\frac{d\bar{R}}{dz}\right)^2 + \epsilon^2\frac{2(m-1)}{\bar{p}\omega_s^2}\overline{\left(\frac{\partial\tilde{p}_1}{\partial r}\right)^2} \\
&\quad + 2\epsilon\bar{B}_{vz0}\tilde{B}_{vz1}\frac{\partial^2\tilde{R}}{\partial z^2}\bar{R} \\
&\quad + \frac{1}{m}\bar{B}_{vz0}^2\left(m\frac{\partial\tilde{R}}{\partial z} + \frac{\epsilon\bar{R}}{\bar{B}_{vz0}}\frac{\partial\tilde{B}_{vz1}}{\partial z}\right)^2.
\end{aligned} \tag{83}$$

As before, letting angle brackets indicate average over $z$ and overbar average over time, we have the following sufficient condition for stability:

$$\langle\bar{c}\rangle > 0. \tag{84}$$

Making use of Eqs. (65)–(69) with $\beta_0 = 1$ this condition can be written as

$$\epsilon^2\frac{(\bar{R}\alpha)^2}{4m}\left[m^2 + \left(\frac{\gamma AQ}{2}\right)^2\right] + \epsilon^2\frac{m-1}{2}\left(\frac{\omega_s\bar{R}}{V_A}\right)^2 - \frac{m-1}{2m}\delta^2(\alpha\bar{R})^2 > 0. \tag{85}$$

Since $(\alpha\bar{R})^2 = (2\pi\bar{R}/L)^2$ must be a small number for most practical cases, the second term will be the dominant term for $m > 1$. Thus, if the low $m$ numbers ($m > 1$) are stabilized, all higher $m$ numbers will be stable. Notice, however, that our long-wave-

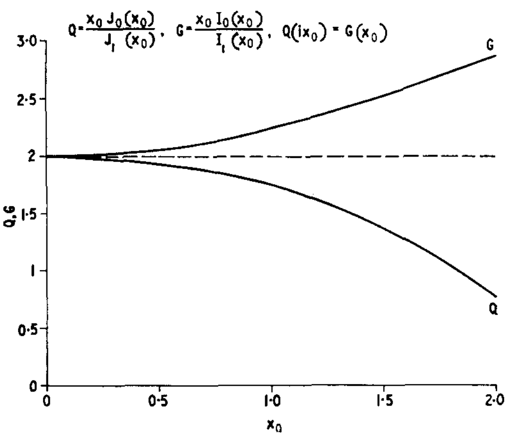


FIG. 2. Plot of $Q$ versus $x_0$, see Eq. (21).

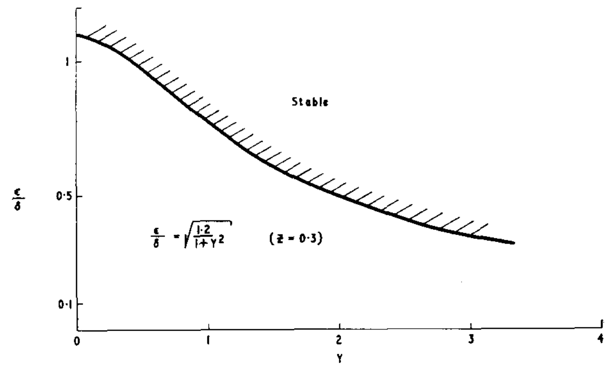


FIG. 4. Stability diagram.

length approximation puts limits on $m$. For $m = 1$ the static system is marginally stable and condition (85) is always satisfied.

## VI. DISCUSSION AND CONCLUSIONS

The results derived here show that the $m = 1$ mode can be dynamically stabilized. The dangerous instability present in the static system is associated with the flutelike modes. A flutelike mode is a displacement of the fluid transverse to the magnetic field which involves little or no bending of the magnetic field lines.

The stabilization of the $m = 1$ mode is associated with the change of direction of the field lines. One part, $(Y - 1)^2$, is due to the stabilizing effect of the vacuum magnetic field; another part, $2Y$, is due to the effect of the surface potential. These two effects together add up to give the factor $Y^2 + 1$ in condition (70).

With the present scheme of dynamic stabilization one expects the flute-type modes associated with the higher $m$ numbers to be suppressed by: (a) the fact that the direction of the field lines are changing in time, and (b) the fact that any radial dependence of the displacement $\xi_r$ will probably have a stabilizing effect. This is because a perturbation which is a flute-type perturbation (matches the field lines) at one instant of time will not continue to be a flute-type perturbation during the course of motion of the forced oscillations. Notice that since the instabilities present, by assumption, are slowly growing compared with the time scale of the forced oscillations, the perturbations will not have time to adjust themselves to flutelike modes. For $m > 1$, the effect of pulsation will also be significant for the same reason.

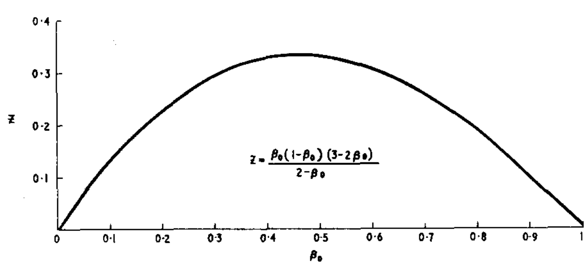


FIG. 3. Plot of the variation in the "effective" $\beta$, see Eq. (71).

The actual analysis is very difficult to carry through for $m > 1$ and $\beta_0 < 1$ and this still remains to be done. However, the $\beta_0 = 1$ case and the physical picture outlined here support the supposition that the $m = 1$ case is the most important one. Moreover, experiments[3,15] also seem to support this view.

In Figs. 2 and 3 we plot the parameters $Q(x_0)$ and $Z(\beta_0)$, which could be useful for evaluation purposes in connection with experiments. Figure 2 contains curves which make it easy to estimate the order of magnitude of $Y$. In Fig. 3 the finite $\beta$ effect, $Z(\beta_0)$, is plotted versus $\beta_0$. Figure 4 is a stability diagram for $Z(\beta_0) = 0.3$. Although at first glance it seems desirable to work in a parameter range for which $k^2 < 0$, Eq. (21), i.e., $x_0 \to ix_0$, one should be aware of cancellations in the expression for $Y$ Eq. (67), if $A < 0$. The effect of this can be a substantial reduction in $|Y|$ and, therefore, in dynamic stabilization.

Recent experiments by Bodin *et al.*[15] support the theoretical results derived here. For this particular experiment $Y \approx 2$ and $\beta_0 \approx 0.5$, in which case, the stability condition (70) becomes $\epsilon > 0.5\delta$.

The present scheme of dynamic stabilization should also have some relevance to a certain class of bumpy toroidal equilibria. Following Johnson *et al.*[12] we see that this is the case when $\delta_0 \gg \delta_l, l = 1, 2, 3, \cdots$ [see Ref. 12 Eqs. (13) and (21)]. The minimum $\delta$ required for a bumpy toroidal system of given aspect ratio can also be estimated from this theory. Our stability condition (70) provides us then with the answer to how much dynamic stabilization field is required to make a given equilibrium stable with the present scheme of dynamic stabilization.

The possibility also exists that the present scheme can be used for producing dynamic equilibrium[16] under circumstances where no equilibrium would otherwise exist.

In the context of a nuclear fusion reactor it is clear that dynamic stabilization will have both economic and technical difficulties. The application of superconducting coils is hardly feasible in the frequency range where one is operating at present. The rate of dissipation of energy in the external circuits is dependent on $\omega_s$, the frequency of the dynamic stabilization field. It is, therefore, desirable to keep $\omega_s$ as low as possible. However, $\omega_s$ must be about one order of magnitude larger than the characteristic frequency of the instabilities which are to be stabilized. But apart from this, the present results seem to indicate that dynamic stabilization is most effectively enhanced by increasing the amplitude ($\epsilon$) of the oscillatory fields rather than $\omega_s$.

On the other hand, an increase in the amplitude of the oscillatory fields may introduce new problems. These are the problems associated with parametric resonances.[7,8] Depending on the effective damping of the waves being excited, this will probably first occur as dissipation of the energy fed into the dynamic stabilization coils, and by increasing the amplitude further one could drive the resonance modes unstable.

However, in spite of all these problems, the state of our present knowledge suggests that one should make further studies in order to get a better understanding of dynamic stabilization and its possible applications.

## ACKNOWLEDGMENT

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## APPENDIX A. THE DYADIC $\nabla\boldsymbol{\xi}$ IN CYLINDRICAL COORDINATES

With $\partial/\partial\theta = im/r$ we obtain

$$\begin{aligned}\nabla\boldsymbol{\xi} = {} & \mathbf{e}_r\mathbf{e}_r\frac{\partial\xi_r}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r\frac{im\xi_r - \xi_\theta}{r} + \mathbf{e}_z\mathbf{e}_r\frac{\partial\xi_r}{\partial z} \\ & + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial\xi_\theta}{\partial r} + \mathbf{e}_\theta\mathbf{e}_\theta\frac{\xi_r + im\xi_\theta}{r} + \mathbf{e}_z\mathbf{e}_\theta\frac{\partial\xi_\theta}{\partial z} \\ & + \mathbf{e}_r\mathbf{e}_z\frac{\partial\xi_z}{\partial r} + \mathbf{e}_\theta\mathbf{e}_z\frac{im\xi_z}{r} + \mathbf{e}_z\mathbf{e}_z\frac{\partial\xi_z}{\partial z}. \qquad \text{(A1)}\end{aligned}$$

Let $\mathbf{a}$ be given as

$$\mathbf{a} = a_r\mathbf{e}_r + a_z\mathbf{e}_z; \qquad \text{(A2)}$$

then it follows that

$$\begin{aligned}\mathbf{a}\cdot\nabla\boldsymbol{\xi} = {} & \left(a_r\frac{\partial\xi_r}{\partial r} + a_z\frac{\partial\xi_r}{\partial z}\right)\mathbf{e}_r + \left(a_r\frac{\partial\xi_\theta}{\partial r} + a_z\frac{\partial\xi_\theta}{\partial z}\right)\mathbf{e}_\theta \\ & + \left(a_r\frac{\partial\xi_z}{\partial r} + a_z\frac{\partial\xi_z}{\partial z}\right)\mathbf{e}_z, \qquad \text{(A3)}\end{aligned}$$

and

$$\begin{aligned}\nabla\boldsymbol{\xi}\cdot\mathbf{a} = {} & \left(a_r\frac{\partial\xi_r}{\partial r} + a_z\frac{\partial\xi_r}{\partial z}\right)\mathbf{e}_r + im\left[\left(1 - \frac{1}{m^2}\right)\frac{a_r\xi_r}{r}\right. \\ & \left. - \frac{a_r}{m^2}\frac{\partial\xi_r}{\partial r} + \frac{a_z\xi_z}{r} - \frac{a_r}{m^2}\frac{\partial\xi_z}{\partial z}\right]\mathbf{e}_\theta \\ & + \left(a_r\frac{\partial\xi_r}{\partial z} + a_z\frac{\partial\xi_z}{\partial z}\right)\mathbf{e}_z \qquad (\nabla\cdot\boldsymbol{\xi} = 0). \qquad \text{(A4)}\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}\boldsymbol{\xi}^*\cdot\nabla\boldsymbol{\xi} = {} & \mathbf{e}_r\left(\xi_r^*\frac{\partial\xi_r}{\partial r} + \xi_\theta^*\frac{im\xi_r - \xi_\theta}{r} + \xi_z^*\frac{\partial\xi_r}{\partial z}\right) \\ & + \mathbf{e}_\theta\left(\xi_r^*\frac{\partial\xi_\theta}{\partial r} + \xi_\theta^*\frac{\xi_r + im\xi_\theta}{r} + \xi_z^*\frac{\partial\xi_\theta}{\partial z}\right) \\ & + \mathbf{e}_z\left(\xi_r^*\frac{\partial\xi_z}{\partial r} + \xi_\theta^*\frac{im\xi_z}{r} + \xi_z^*\frac{\partial\xi_z}{\partial z}\right). \qquad \text{(A5)}\end{aligned}$$

Now we have from Eq. (31) that

$$i\xi_\theta = -\frac{1}{m}\frac{\partial}{\partial r}(r\xi_r) - \frac{r}{m}\frac{\partial\xi_z}{\partial z} + \frac{r}{m}\eta, \qquad \text{(A6)}$$

$$i\xi_\theta^* = \frac{1}{m}\frac{\partial}{\partial r}(r\xi_r^*) + \frac{r}{m}\frac{\partial\xi_z^*}{\partial z} - \frac{r}{m}\eta^*. \qquad \text{(A7)}$$

Using Eqs. (A6) and (A7) we obtain

$$\begin{aligned}& \xi_r^*\frac{\partial\xi_r}{\partial r} + i\xi_\theta^*\frac{m\xi_r + i\xi_\theta}{r} + \xi_z^*\frac{\partial\xi_r}{\partial z} \\ & = \left[\left(1 - \frac{1}{m^2}\right)\frac{1}{r}\frac{\partial}{\partial r}(r\,|\xi_r|^2) - \frac{r}{m^2}\left|\frac{\partial\xi_r}{\partial r}\right|^2\right], \qquad \text{(A8)}\end{aligned}$$

where we have omitted terms of order $\eta$ and $\partial\boldsymbol{\xi}/\partial z$, since they are small of order $\epsilon$. To first significant order we have

$$\xi_r^*\frac{\partial\xi_z}{\partial r} + \xi_\theta^*\frac{im\xi_z}{r} = \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r^*\xi_z). \qquad \text{(A9)}$$

From Eqs. (A5), (A8), and (A9) we readily obtain Eq. (33).

## APPENDIX B. THE EQUATION OF MOTION IN THE OSCILLATORY STATE

The forced oscillations are governed by the following equations to second order:

$$\bar{\rho}\frac{\partial \tilde{\mathbf{v}}_1'}{\partial t} = -\nabla(\tilde{p}_1 + \tilde{\mathbf{B}}_1\cdot\bar{\mathbf{B}}_0) + \bar{\mathbf{B}}_0\cdot\nabla\tilde{\mathbf{B}}_1 + \tilde{\mathbf{B}}_1\cdot\nabla\bar{\mathbf{B}}_0, \tag{B1}$$

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{\mathbf{v}}_2' + \tilde{\rho}_1\tilde{\mathbf{v}}_1') + \nabla\cdot(\bar{\rho}\tilde{\mathbf{v}}_1'\tilde{\mathbf{v}}_1') = -\nabla\left(\tilde{p} + \frac{\tilde{\mathbf{B}}^2}{2}\right)_{(2)} + \tilde{\mathbf{B}}_1\cdot\nabla\tilde{\mathbf{B}}_1 + \bar{\mathbf{B}}_0\cdot\nabla\tilde{\mathbf{B}}_2 + \tilde{\mathbf{B}}_2\cdot\nabla\bar{\mathbf{B}}_0. \tag{B2}$$

By averaging over the fast oscillations and keeping terms to significant order we obtain

$$\overline{\nabla\left(p + \frac{B^2}{2}\right)_{(2)}} = -\nabla\cdot(\bar{\rho}\tilde{\mathbf{v}}_1'\tilde{\mathbf{v}}_1') + \tilde{\mathbf{B}}_1\cdot\nabla\tilde{\mathbf{B}}_1 + \bar{B}_{0z}\frac{\partial}{\partial z}\tilde{\mathbf{B}}_2. \tag{B3}$$

Now, we have

$$\begin{aligned} -\nabla\cdot(\bar{\rho}\tilde{\mathbf{v}}_1'\tilde{\mathbf{v}}_1') + \tilde{\mathbf{B}}_1\cdot\nabla\tilde{\mathbf{B}}_1 &= \left[-\frac{\bar{\rho}}{r}\frac{\partial}{\partial r}(r\tilde{v}_{r1}'^2) - \frac{\partial}{\partial r}(\bar{\rho}\tilde{v}_{r1}'\tilde{v}_{z1}') + \frac{\partial}{\partial r}\frac{\tilde{B}_{r1}^2}{2} + \tilde{B}_{z1}\frac{\partial \tilde{B}_{r1}}{\partial z} - \frac{\partial}{\partial z}\bar{\rho}\tilde{v}_{r1}'^2\right]\mathbf{e}_r \\ &\cdot\left[-\bar{\rho}\left(\frac{\tilde{v}_{r1}'\tilde{v}_{z1}'}{r} + \frac{\partial}{\partial z}(\tilde{v}_{r1}'\tilde{\mathbf{v}}_{z1}')\right) + \tilde{B}_{r1}\frac{\partial \tilde{B}_{z1}}{\partial r} + \frac{\partial}{\partial z}\frac{\tilde{B}_{z1}^2}{2}\right]\mathbf{e}_z + \bar{B}_{0z}\frac{\partial}{\partial z}\tilde{\mathbf{B}}_2. \end{aligned} \tag{B4}$$

From Eqs. (15)–(20) we can see that $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{B}}$ have a functional form which by use of Eq. (B4) permits us to write Eq. (B3) in the form of Eq. (34).

## APPENDIX C. THE SURFACE INTEGRAL

The surface integral of Eq. (27) and the surface integral of Eq. (32) adds up to

$$\begin{aligned} S &= \overline{\int}\left[|\mathbf{n}\cdot\xi_0|^2\,\mathbf{n}\cdot\left\langle\nabla\left(p + \frac{B^2}{2}\right)\right\rangle + \mathbf{n}\cdot\xi_0^*\xi_0\cdot\nabla\left(\tilde{p} + \frac{\tilde{B}^2}{2}\right)_{(2)}\right]d\sigma \\ &= \overline{\int}|\xi_r|^2\left(\mathbf{n}\cdot\nabla\frac{\mathbf{B}_v^2}{2} - \bar{\mathbf{n}}\cdot\nabla\frac{\bar{\mathbf{B}}_i^2}{2}\right)d\sigma + \int\frac{\partial}{\partial z}g\,dz. \end{aligned} \tag{C1}$$

We have only kept leading order terms. Now, we have $\nabla\times\bar{\mathbf{B}}_i = \nabla\times\mathbf{B}_v = 0$. It, therefore, follows that

$$\left(\mathbf{n}\cdot\nabla\frac{B_v^2}{2}\right)_{(2)} = \left(B_v^2\frac{\partial^2 R}{\partial z^2}\right)_{(2)} = \bar{B}_v^2\frac{d^2\bar{R}}{dz^2} + 2\epsilon\bar{\mathbf{B}}_v\cdot\tilde{\mathbf{B}}_{v1}\frac{\partial^2\tilde{R}}{\partial z^2}, \tag{C2}$$

and

$$\bar{\mathbf{n}}\cdot\nabla\frac{\bar{B}_i^2}{2} = \bar{B}_i^2\frac{d^2\bar{R}}{dz^2}. \tag{C3}$$

Notice that

$$\mathbf{B}\cdot\nabla\mathbf{B} = \frac{\mathbf{n}}{R_c}B^2 + \mathbf{e}\frac{d}{ds}\frac{B^2}{2}, \tag{C4}$$

where $\mathbf{B} \stackrel{\text{def}}{=} \mathbf{e}B$, $(d/ds) \stackrel{\text{def}}{=} \mathbf{e}\cdot\nabla$ and $R_c$ is the radius of curvature of the magnetic field lines in the surface. The formula

$$\nabla\frac{B^2}{2} = \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{B})$$

has been used in obtaining Eqs. (C2) and (C3).

## APPENDIX D. PERTURBATION IN THE VACUUM MAGNETIC FIELD

Let the perturbation in the vacuum magnetic field be given by $\mathbf{B}_v'$. We can then write

$$\mathbf{B}_v' = \nabla\psi, \qquad \nabla^2\psi = 0 \tag{D1}$$

or

$$\mathbf{B}_v' = \nabla\times\mathbf{A}, \qquad \nabla\times(\nabla\times\mathbf{A}) = 0. \tag{D2}$$

In the first case, Eq. (D1), the magnetic field is expressed in terms of a scalar potential. This description introduces some difficulties in connection with multivaluedness of the potential $\psi$, see Lüst and Martensen.[17] In the second case, Eq. (D2), the magnetic field is expressed in terms of a vector potential. Since in our symmetric problem multivaluedness of $\psi$ does not present any difficulties, we shall study the perturbed magnetic field in terms of Eq. (D1). This means that we shall have to solve Laplace's equation. We can Fourier analyse the $\theta$ dependence, i.e., $\psi \propto \exp(im\theta)$ and the resulting equation is separable in the $r$ and $z$ dependence so one can easily write the general solution in terms of trigonometric functions of $z$ and Bessel functions of $r$. However, this solution is not very useful since it has to satisfy boundary conditions along a surface which is not a coordinate surface in cylindrical coordinates. The boundary conditions to be satisfied along the plasma–vacuum interface can be written as

$$\mathbf{n}\cdot\nabla\psi = \mathbf{n}\cdot\nabla\times(\xi\times\mathbf{B}_v). \tag{D3}$$

In view of these difficulties, we shall use the same method as Haas and Wesson[2] in obtaining an approximate solution. We shall neglect the effect of finite boundaries and mathematically treat these to be at infinity. For obvious reasons boundaries will always have a stabilizing effect unless new physical features, like a resistive layer at the boundary surface, are introduced.

By assuming the $z$ dependence in the potential $\psi$ to be weak, we solve the ordered Laplaces equation rather than the complete equation, and the proper solution is given by

$$\psi = C(z, t)\left(\frac{r}{R}\right)^{-m} \exp(im\theta). \qquad \text{(D4)}$$

We can now write

$$\overline{\int_{V_v}} |\nabla \times \mathbf{A}| \, d\mathbf{r} = \overline{\int_{V_v}} |\nabla \psi|^2 \, d\mathbf{r}$$
$$= -\overline{\int_S} \psi \mathbf{n} \cdot \nabla \psi^* \, ds. \qquad \text{(D5)}$$

Moreover, we have

$$\mathbf{n} \cdot \nabla \psi|_{r=R} = -\frac{m}{R} \psi(R) - \frac{\partial R}{\partial z} \frac{\partial \psi}{\partial z}\bigg|_{r=R} + O(\epsilon^2 \psi). \qquad \text{(D6)}$$

From the boundary condition (D3) we obtain

$$\mathbf{n} \cdot \nabla \psi = \frac{1}{\bar{R}} (\bar{G} + \tilde{G}) + O(\epsilon^2), \qquad \text{(D7)}$$

where

$$\bar{G} = \bar{B}_{z0} \bar{R}\left(\frac{d\bar{R}}{dz} \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_r}{\partial z}\right) + \xi_r\left(\bar{R} \frac{d\bar{B}_{z0}}{dz} + \bar{B}_{z0} \frac{d\bar{R}}{dz}\right) \qquad \text{(D8)}$$

and

$$\tilde{G} = \bar{B}_{z0} \frac{\partial \tilde{R}}{\partial z}\left(\bar{R} \frac{\partial \xi_r}{\partial r} + \xi_r\right)$$
$$+ \epsilon\left(\xi_r \bar{R} \frac{\partial \tilde{B}_{z1}}{\partial z} - \xi_z \bar{R} \frac{\partial \tilde{B}_{r1}}{\partial z}\right). \qquad \text{(D9)}$$

From Eq. (D6) it follows that

$$\psi(R) = -\frac{R}{m} \mathbf{n} \cdot \nabla \psi + O(\epsilon \psi). \qquad \text{(D10)}$$

From Eqs. (D5)–(D10) we then obtain to leading order

$$\overline{\int} |\nabla \times \mathbf{A}|^2 \, d\mathbf{r} = \frac{1}{m} \overline{\int_S} R \, |\mathbf{n} \cdot \nabla \psi|^2 \, ds$$
$$= \frac{2\pi}{m} \int \{|\bar{G}|^2 + \overline{|\tilde{G}|^2}\} \, dz. \qquad \text{(D11)}$$

Notice that the cross term $\bar{G}\tilde{G}$ averages to zero in leading order and that $d\ell = dz[1 + O(\epsilon^2)]$, where $d\ell$ is the arc length along a magnetic field line in the surface.

From the conservation of magnetic flux in the plasma column and the pressure jump condition one obtains

$$\frac{d\bar{B}_{z0}}{dz} \equiv -\frac{2}{\bar{R}} \frac{d\bar{R}}{dz} \bar{B}_{z0}(1 - \beta_0). \qquad \text{(D12)}$$

We can then write the expression for $\bar{G}^2$, Eq. (D8), as

$$\bar{G}^2 = \left|\bar{B}_{z0}\bar{R}\left(\frac{\partial \xi_r}{\partial z} + \frac{\partial \bar{R}}{\partial z} \frac{\partial \xi_r}{\partial r}\right) - \xi_r \bar{B}_{z0} \frac{d\bar{R}}{dz}(1 - 2\beta_0)\right|^2. \qquad \text{(D13)}$$

Moreover for $\overline{\tilde{G}^2}$, we can write (overbar means time average),

$$\overline{|\tilde{G}|^2} = \overline{\left|\bar{B}_{z0} \frac{\partial \tilde{R}}{\partial z}\left(\bar{R} \frac{\partial \xi_r}{\partial r} + \xi_r\right) + \epsilon \xi_r \bar{R} \frac{\partial \tilde{B}_{z1}}{\partial z}\right|^2}$$
$$+ \epsilon^2 \bar{R}^2 |\xi_z|^2 \overline{\left(\frac{\partial \tilde{B}_{r1}}{\partial z}\right)^2} + \frac{d}{dz} f(z) + O(\epsilon^4). \qquad \text{(D14)}$$

## APPENDIX E. A USEFUL INTEGRAL FORMULAS

The following formulas can easily be derived:

$$\iint_\Omega dr\, dz \frac{\partial}{\partial z} f(r, z) = \int dz \frac{d}{dz} \int_0^{R(z)} dr\, f(r, z)$$
$$- \int dz \frac{\partial R(z)}{\partial z} f(R(z), z)$$
$$= \int_0^{R(z_1)} dr\, f(r, z_1) - \int_0^{R(z_0)} dr\, f(r, z_0)$$
$$- \int_{z_0}^{z_1} dz \frac{\partial R(z)}{\partial z} f(R(z), z),$$

where the region of integration $\Omega$ is $r \in [0, R(z)]$ and $z \in [z_0, z_1]$.

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* Permanent address: Department of Applied Mathematics, University of Bergen, Bergen, Norway.

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# Structure of Evolving Ion-Acoustic Fronts in Collisionless Plasmas

Rodney J. Mason
*Bell Telephone Laboratories, Incorporated, Whippany, New Jersey*
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Exact analytic solutions are obtained to the linearized, two-species Maxwell–Vlasov equations for the evolution of an initial step-function density discontinuity in an initially neutral, collisionless plasma. These solutions have been evaluated numerically for hydrogen plasmas with starting temperature ratios $0 \leq \theta = T_e/T_i \leq 30$, for times $0 \leq \tau = \omega_{p_i} t \leq 100$, and over distances into the original low-density region $0 \leq \eta = x/A_i t \leq 12$, where $A_i$ is the mean thermal speed of ions. For $\tau \ll 1$ both species simply free stream. By late times ($\tau \gtrsim 30$) quasineutrality of the perturbation has been established, and it is found that: (1) the ion profiles for $\theta \lesssim 1$ exhibit the appearance of free streaming at an enhanced ion temperature, suggesting that the strong Landau damping usually attributed to the low-$\theta$ limit might better be termed "ambipolar free streaming," (2) for $\theta \simeq 5$ they are self-similar and monotonic with a steep leading edge near $\eta = \theta^{1/2}$ followed by a broad, arched trailing edge accounting for half the density rise, and (3) for $\theta \lesssim 30$ they have a steep but spreading leading edge near $\theta^{1/2}$ and general resemblance to the integrated Airy function profile obtainable through a moment equation treatment retaining only the last damped collective mode.

## I. INTRODUCTION

The existence of collisionless ion-acoustic shocks at high electron-to-ion temperature ratios is now generally accepted.[1–4] Therefore, it becomes of interest to determine how these shocks might form from steep initial conditions, the temperature dependence of their profiles, and their ultimate, long-time configuration. The last of these questions can only be answered by nonlinear analysis, but for weak, laminar shocks, by analogy with neutral gas theory,[5–8] the early-time formation process should be amenable to a linearized treatment. Thus, we examine the development of ion-acoustic fronts from an initial step-function density discontinuity.

The propagation of weak ion-acoustic waves, in accordance with the linearized Vlasov–Maxwell equations, is well understood.[9–11] Our task is simply to determine how these waves are excited by the initial conditions, and to follow the subsequent evolution of their envelope. This is readily accomplished with the aid of Fourier and Laplace transforms. The only complication lies in the double inversion of these transforms, and this is surmounted by employing the procedures of Ref. 7 with insight gained from the Vlasov–Green's function study of Weitzner.[12] Thus, in Sec. II we derive exact, analytic solutions in the form of quadratures for the linearized ion and electron profiles. Comparison is made with solutions obtained from two important approximate kinetic equations describing the motion. Also, the relationship between our linearized solutions and the corresponding nonlinear Vlasov solutions is demonstrated. Finally, we discuss the results of a numerical evaluation of the exact quadratures in Sec. III.

## II. ANALYTIC SOLUTIONS

### A. Exact Linearized Equations

We first assume that the plasma dynamics are governed by linearized Vlasov equations for both electrons and singly charged ions coupled through Poisson's equation:

$$\frac{\partial f_{1_{i,e}}}{\partial t} + u\,\frac{\partial f_{1_{i,e}}}{\partial x} \pm \frac{eE_1 n_0}{m_{i,e}}\,\frac{\partial f_{0_{i,e}}}{\partial u} = 0,$$
$$\frac{\partial E_1}{\partial x} = 4\pi e(n_{1_i} - n_{1_e}) = 4\pi e \int_{-\infty}^{+\infty} (f_{1_i} - f_{1_e})\, du. \quad (1)$$