Guiding center plasma stability of the bumpy theta pinch with weak pressure anisotropy

1976 Plasma Phys. 18 499

(http://iopscience.iop.org/0032-1028/18/7/001)

Plasma Physics, Vol. 18, pp. 499 to 510. Pergamon Press, 1976. Printed in Northern Ireland

GUIDING CENTER PLASMA STABILITY OF THE BUMPY θ PINCH WITH WEAK PRESSURE ANISOTROPY

M. J. SCHMIDT

and

G. VAHALA

Department of Physics, College of William and Mary, Williamsburg, VA 23185, U.S.A.

(*Received* 28 *July* 1975)

Abstract—The unstable modes and growth rates for the bumpy θ pinch are considered for the guiding center plasma assuming a pressure anisotropy of $\mathcal{O}(\varepsilon)$ (ε represents the slow periodic z-variation of the axisymmetric equilibrium and the stability relative to their r-variations). It is shown that any unstable mode in a guiding center plasma has the same form and growth rate as in ideal magnetohydrodynamics and *vice versa*.

1. INTRODUCTION

RECENTLY, WEITZNER (1973) has considered the one-fluid, ideal magnetohydrodynamic (MHD) stability of the bumpy θ pinch by performing expansions in two small parameters: first in ε (a measure of the slow z-variation relative to the r-variation of the equilibrium and perturbed qualities), then in δ (the bumpiness of the flux surfaces). Thus the equilibrium mangetic field have the form—in cylindrical coordinates—

$$\mathbf{B}^{(0)}(r, z) = (B_r^{(0)}, 0, B_z^{(0)}),$$

where

$$B_r^{(0)} = \varepsilon[\delta b(r) \sin \varepsilon \tilde{z} + \mathcal{O}(\delta^2 \sin 2\varepsilon z)] + \theta(\varepsilon^3)$$
$$B_z^{(0)} = [a(r) + \delta c(r) \cos \varepsilon z + \mathcal{O}(\delta^2 \cos 2\varepsilon z)] + \theta(\varepsilon^2), \tag{1}$$

with the flux function $\psi(r, z)$

$$\psi(r, z) = \psi_0(r) + \delta\psi_1(r) \cos \varepsilon z + \mathcal{O}(\delta^2 \cos 2\varepsilon z). \tag{2}$$

The equilibrium pressure balance $[p^{(0)} = p^{(0)}(\psi)]$ is

$$p^{00}(r) + \tfrac{1}{2}a^2(r) = \tfrac{1}{2}, \tag{3}$$

where $p^{(0)}(\psi) = p^{00}(r) + \delta p^{01}(r) \cos \varepsilon z + \mathcal{O}(\delta^2)$. Finally, the equilibrium relations are

$$c(r) = b'(r) + \frac{b(r)}{r} = \frac{1}{a(r)} + \frac{a'(r)b(r)}{a(r)}, \tag{4}$$

where primes denote derivatives with respect to the argument.

From the linear stability analysis, Weitzner showed that [assuming all perturbed quantities $\sim\exp(im\theta)$ in the ignorable coordinate, θ] the $m \geq 2$ transverse modes are more MHD unstable than the $m = 1$ mode. Yet experimentally, only the $m = 1$ mode has been observed [ELLIS *et al.* (1974), CANTRELL *et al.* (1974)]. It should be noted that in MHD one describes the plasma as a single fluid in local thermodynamic equilibrium obeying macroscopic moment equations with an isotropic pressure (and an assumed equation of state). Thus MHD is a *collisional* model.

Non-MHD effects, specifically finite ion larmor radius effects, have been considered by FREIDBERG (1972) and can lead to the suppression of the $m \geqslant 2$ growth rates. However, for a high β fusion plasma one can no longer rely on the finite larmor radius stabilization of the unstable $m \geqslant 2$ modes. Hence it is of considerable interest to consider other non-MHD effects. This led WEITZNER (1974, 1975) to consider the guiding-center plasma (GCP) equations of GRAD (1961, 1967).

GCP is a *collisionless* model, derivable from a small larmor radius expansion of the Vlasov equation, in which the motion of the particles are constrained to move on magnetic field lines. The field lines themselves move according to a set of macroscopic moment equations incorporating an *anisotropic* pressure tensor that is consistent with the one dimensional (collisionless) kinetic equations describing the particle motion along the field lines.

Assuming very weak pressure anisotropy of $\mathcal{O}(\varepsilon^2)$ for helically symmetric equilibria, WEITZNER (1974, 1975) concluded that the kinetic effects of the GCP model are not sufficient to eliminate any of the MHD instabilities or affect their growth rates.

In this paper we consider collisionless effects on the bumpy θ pinch but assuming a stronger pressure anisotropy, that of $\mathcal{O}(\varepsilon)$. Nevertheless we also find that the kinetic effects of the GCP model are still not sufficient to eliminate or alter the MHD growth rates. In Section 2 we briefly outline the GCP equations while in Section 3 we derive the equilibrium with assumed pressure anisotropy of $\mathcal{O}(\varepsilon)$. In the stability analysis we find exactly the same local stability criteria for $\omega^2 = \mathcal{O}(1)$ modes as presented by WEITZNER (1975)—for the non-existence of the firehose instability. Hence we omit all details of this calculation. In Section 4 we consider modes with $\omega^2 = \mathcal{O}(\varepsilon^2\delta^2)$, $m \neq 0$. We show that these modes have exactly the same growth rates as the MHD transverse modes (modes approximately perpendicular to $\mathbf{B}^{(0)}$) found by WEITZNER (1973). The case $m = 0$ is treated in the appendix. We summarize our results in Section 5.

2. GUIDING CENTER PLASMA MODEL

We consider a two species (ions and electrons) GCP plasma with the 1D kinetic equations

$$\frac{\partial f_s}{\partial t} + (\mathbf{u} + v\hat{\boldsymbol{\beta}}) \cdot \frac{\partial f_s}{\partial \mathbf{x}} + (\boldsymbol{\kappa} \cdot \mathbf{u}) v \frac{\partial f_s}{\partial v} + [\alpha_s T + (\hat{\boldsymbol{\beta}} \cdot \nabla)(\tfrac{1}{2}u^2 - \mu B)] \frac{\partial f_s}{\partial v} = 0, \tag{5}$$

for the distributions functions $f_s(\mathbf{x}, v, \mu, t)$ where v is the particle velocity along a B line and μ is its magnetic moment. Subscript s stands for the ions (i) or electrons (e), with charge e_s and mass m_s.

$$B = |\mathbf{B}(\mathbf{x}, t)|, \qquad \hat{\boldsymbol{\beta}} = \frac{\mathbf{B}}{B}$$

$$\boldsymbol{\kappa} = (\hat{\boldsymbol{\beta}} \cdot \nabla)\hat{\boldsymbol{\beta}}, \qquad \alpha_s = \frac{e_s/m_s}{(e/m)_i - (e/m)_e}, \tag{6}$$

$\mathbf{u}(\mathbf{x}, t) =$ perpendicular velocity of the B line

$T(\mathbf{x}, t) =$ effective parallel electric field

f_s is normalized so that the mass density, $\rho_s = m_s n_s$, and the mean fluid velocity along a B line, w_s, are given by

$$\begin{aligned} \rho_s(\mathbf{x}, t) &= B\int \mathrm{d}v\, \mathrm{d}\mu\, f_s(\mathbf{x}, v, \mu, t) \\ \rho_s w_s(\mathbf{x}, t) &= B\int \mathrm{d}v\, \mathrm{d}\mu\, v f_s(\mathbf{x}, v, \mu, t). \end{aligned} \tag{7}$$

Thus the charge density $q_s = e_s\rho_s/m_s$, and the current along a B line $j_s = e_s\rho_s w_s/m_s$. The total moments for the plasma are then

$$\rho = \sum_s \rho_s, \qquad \rho w = \sum \rho_s w_s.$$

The macroscopic momentum balance which governs the motion of the B line is

$$\rho\left[\frac{\partial}{\partial t} + (\mathbf{u} + \hat{\boldsymbol{\beta}} w)\cdot\nabla\right](\mathbf{u} + \hat{\boldsymbol{\beta}} w) + \mathbf{P} = (\nabla\times\mathbf{B})\times\mathbf{B}, \tag{8}$$

where $\mathbf{P}$ is the pressure tensor

$$\mathbf{P} = p_\perp \mathbf{1} + \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}(p_\parallel - p_\perp), \tag{9}$$

$\mathbf{1}$ is the unit tensor and the pressure perpendicular and parallel to the B line is given by

$$\begin{aligned} p_\perp &= \sum_s p_{\perp s} = B\sum_s \int \mathrm{d}v\, \mathrm{d}\mu\, \mu B f_s \\ p_\parallel &= \sum_s p_{\perp s} = B\sum_s \int \mathrm{d}v\, \mathrm{d}\mu (v - w)^2 f_s. \end{aligned} \tag{10}$$

The magnetic field is determined by the flux equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{11}$$

and

$$\nabla\cdot\mathbf{B} = 0. \tag{12}$$

It should be noted that, from (6)

$$\mathbf{u}(\mathbf{x}, t)\cdot\mathbf{B}(\mathbf{x}, t) = 0 \tag{13}$$

and that the component of (8) along $\hat{\boldsymbol{\beta}}$ is a consequence of the kinetic equations (5).

For consistency, and to complete this system of equations (5), (8), (11) and (12), we require (for singly charged ions)

$$\begin{aligned} q(\mathbf{x}, t) &= \sum_s \frac{e_s\rho_s}{m_s} = 0 \\ j(\mathbf{x}, t) &= \sum_s \frac{e_s\rho_s w_s}{m_s} = 0 \end{aligned} \tag{14}$$

and that the integral equation constraint for the effective electrostatic potential Φ, $(n_i = n_e = n_0)$ be satisfied

$$C\frac{\partial\Phi}{\partial B} = -\frac{1}{B}\frac{\partial}{\partial B}[p_{\parallel i} - p_{\parallel e}] + \frac{1}{B^2}\sum_s \operatorname{sgn}(e_s)[p_{\parallel s} - p_{\perp s}], \tag{15}$$

where the parallel electric field is given by

$$\left(\frac{e}{m}\right)_s E_\parallel = \alpha_s T = -\alpha_s(\hat{\beta}\,.\,\nabla)\Phi, \tag{16}$$

and $C = 2n_0 m_i m_e/(m_i + m_e)$.

3. GCP EQUILIBRIUM WITH $p_\parallel - p_\perp = 0(\varepsilon)$

We consider equilibria with no flow $(\mathbf{u} + \hat{\boldsymbol{\beta}} w = 0)$ and perform perturbation expansions in the two small parameters ε and δ introduced in Section 1. From (5) and axisymmetry, the equilibrium distribution function

$$f_s^{(0)}(\mathbf{x}, v, \mu) = f_s(\mathscr{E}_s, \mu, \psi), \tag{17}$$

where

$$\mathscr{E}_s = \tfrac{1}{2}v^2 + \mu B + \alpha_s\Phi \equiv \mathscr{E} + \alpha_s\Phi \tag{18}$$

and the flux function ψ, representing the coordinates of a B line ($\mathbf{B}^{(0)}\,\nabla\psi = 0$, with $rB_r^{(0)} = -\varepsilon\,\partial\psi/\partial\tilde{z}$, $rB_z^{(0)} = \partial\psi/\partial r$), satisfied the elliptic equation (GRAD, 1967)

$$\frac{\varepsilon^2}{r^2}\frac{\partial^2\psi}{\partial\tilde{z}^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sigma}\nabla\psi\,.\,\nabla\sigma = -\frac{1}{\sigma}\frac{\partial p_\parallel^{(0)}(\psi, B)}{\partial\psi}, \tag{19}$$

with

$$z = \varepsilon\tilde{z} \tag{20}$$

$$\sigma = 1 + \frac{p_\perp^{(0)} - p_\parallel^{(0)}}{B^2} = 1 - \frac{1}{B}\frac{\partial p_\parallel^{(0)}(\psi, B)}{\partial B}. \tag{21}$$

Superscript (0) will be omitted on equilibrium quantities when no confusion should arise.

Since we assume

$$p_\parallel^{(0)}(\psi, B) - p_\perp^{(0)}(\psi, B) = \mathcal{O}(\varepsilon), \tag{22}$$

then from (17)

$$f_s^{(0)} = f_s^{(0)}(\mathscr{E}, \psi) + \mathcal{O}(\varepsilon), \tag{23}$$

on assuming $\Phi = \mathcal{O}(\varepsilon)$. From the definitions of $p_{\parallel s}^{(0)}$ and $p_{\perp s}^{(0)}$, (10)

$$p_{\parallel s}^{(0)} = p_{\perp s}^{(0)} + \mathcal{O}(\varepsilon) = \frac{4\sqrt{2}}{3}\int_0^\infty \mathrm{d}\mathscr{E}\,\mathscr{E}^{3/2} f_s^{(0)}(\mathscr{E}, \psi) + \mathcal{O}(\varepsilon), \tag{24}$$

and of $\rho_s^{(0)}$, (7)

$$\rho_s^{(0)} = 2\sqrt{2}\int_0^\infty \mathrm{d}\mathscr{E}\,\mathscr{E}^{1/2} f_s^{(0)}(\mathscr{E}, \psi) + \mathcal{O}(\varepsilon). \tag{25}$$

It should be noted from (23) and (18)

$$\frac{\partial f_s}{\partial\mathscr{E}} = \frac{1}{v}\frac{\partial f_s^{(0)}}{\partial v} + \mathcal{O}(\varepsilon) = \frac{1}{B}\frac{\partial f_s^{(0)}}{\partial\mu} + \mathcal{O}(\varepsilon). \tag{26}$$

Since $B=\mathcal{O}(1)$ in ε, and from (21) and (22), we consider distribution functions $f_s^{(0)}$ such that

$$\begin{aligned} p_\parallel^{(0)}(\psi, B) &= \hat{p}_\parallel^{(0)}(\psi)+\varepsilon\tilde{p}_\parallel^{(0)}(\psi, B)\\ p_\perp^{(0)}(\psi, B) &= \hat{p}_\perp^{(0)}(\psi)+\varepsilon\tilde{p}_\perp^{(0)}(\psi, B), \end{aligned} \tag{27}$$

with $\hat{p}_\parallel^{(0)}(\psi)=\hat{p}_\perp^{(0)}(\psi)$.

Performing a δ expansion on the distribution functions

$$f_s^{(0)}(r, z, v, \mu)=\varphi_s(r, v, \mu)+\delta\tilde{\varphi}_s(r, v, \mu)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z}), \tag{28}$$

and on the flux function

$$\psi(r, z)=\psi_0(r)+\delta\psi_1(r)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z}), \tag{29}$$

we immediately see that

$$\begin{aligned} \hat{p}_\parallel^{(0)}(\psi) &= p_\parallel^{00}(\psi_0)+\delta\psi_1(r)\frac{\mathrm{d}p_\parallel^{00}}{\mathrm{d}\psi_0}\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})\\ &\equiv p_\parallel^{00}(r)+\delta p_\parallel^{01}(r)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z}), \end{aligned} \tag{30}$$

so that

$$p_\perp^{00}(r)=p_\parallel^{00}(r), \qquad p_\perp^{01}(r)=p_\parallel^{01}(r). \tag{31}$$

Thus, from (24) and (25)

$$\begin{aligned} p_{\perp s}^{00}(r) &= p_{\parallel s}^{00}(r)=\frac{4\sqrt{2}}{3}\int_0^\infty \mathrm{d}\mathscr{E}\,\mathscr{E}^{3/2}\varphi_s\\ \rho_s^{00}(r) &= 2\sqrt{2}\int_0^\infty \mathrm{d}\mathscr{E}\,\mathscr{E}^{1/2}\varphi_s \end{aligned} \tag{32}$$

so that the charge neutrality condition, (14), implies

$$\mathcal{O}(\delta^0):\quad \sum_s \alpha_s\int_0^\infty \mathrm{d}\mathscr{E}\,\mathscr{E}^{1/2}\varphi_s=0. \tag{33}$$

We now derive the explicit GCP equilibrium. From (21) and (22), $\sigma=1+\theta(\varepsilon)$. Thus (19) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)=-\frac{\mathrm{d}\hat{p}_\parallel^{(0)}}{\mathrm{d}\psi}+\mathcal{O}(\varepsilon), \tag{34}$$

on using (27). If we now make the identification $p^{(0)}(\psi)=\hat{p}_\parallel^{(0)}(\psi)=\hat{p}_\perp^{(0)}(\psi)$, provided the distribution functions φ_s are choosen such that (32) and (33) hold and that $f_s(\mathscr{E}, \psi)$ is a monotonically decreasing function of $\mathscr{E}$ [a sufficient condition (KULSRUD, 1962) for the linearized GCP equations to be formally self adjoint], equation (34) is the same as the elliptic equation of ψ in the MHD case, except for an error term of $\mathcal{O}(\varepsilon)$ instead of $\mathcal{O}(\varepsilon^2)$. Hence for the GCP equilibrium with $p_\parallel^{(0)}-p_\perp^{(0)}=\theta(\varepsilon)$

$$\begin{aligned} B_r &= \varepsilon[\delta b(r)\sin\tilde{z}+\theta(\delta^2\sin 2\tilde{z})]+\theta(\varepsilon^2)\\ B_z &= [a(r)+\delta c(r)\cos\tilde{z}+\theta(\delta^2\cos 2\tilde{z})]+\theta(\varepsilon)\\ B &= B_z+\theta(\varepsilon^2\delta^2), \end{aligned} \tag{35}$$

with

$$p_{\parallel}^{00} = p_{\perp}^{00} = \tfrac{1}{2}(1-a^2)$$
$$c = b' + \frac{b}{r} = \frac{1}{a} + \frac{a'b}{a}, \tag{36}$$

from (30) and the definition of ψ, we immediately see that

$$p_{\parallel}^{01} = p_{\perp}^{01} = -a'b. \tag{37}$$

Before we proceed to the stability analysis we note here certain integrals which will arise in that analysis. If we define

$$J_{mn}^{s} \equiv \int_0^\infty \mathrm{d}v \int_0^\infty \mathrm{d}\mu\, v^m \mu^n \frac{1}{v}\frac{\partial \varphi_s}{\partial v},$$

then because of charge neutrality (33),

$$\sum_s \alpha_s J_{01}^{s} = 0 = \sum_s \alpha_s J_{20}^{s}, \tag{38}$$

and by definition (10),

$$\sum_s J_{02}^{s} = -\frac{2p_{\perp}^{00}}{a^2} = \frac{2}{a}\sum_s J_{21}^{s}. \tag{39}$$

4. TRANSVERSE-LIKE MODES WITH $\bar{\omega} = 0(\varepsilon\delta)$

We now turn to the linear stability analysis of GCP normal modes with time dependence $\sim\exp(i\bar{\omega}t)$. For the case $\bar{\omega} = \mathcal{O}(1)$ in ε it is easily seen that for local stability the distribution functions must satisfy the "firehose" condition—exactly the same condition as that found by WEITZNER (1975). Hence we omit all details of the calculation.

We now consider modes with $\bar{\omega} = \mathcal{O}(\varepsilon\delta)$

$$\bar{\omega} = \varepsilon\delta\omega, \tag{40}$$

so that all perturbed quantities (denoted by superscript (1), where necessary)

$$\begin{aligned}&\sim\exp(i\bar{\omega}t + iA)\\ &\sim\exp(i\varepsilon\delta\omega t)\exp(im\theta)\exp(i\varepsilon\delta l z),\end{aligned} \tag{41}$$

where $A \equiv m\theta + i\delta l\tilde{z}$ with $m \neq 0$.

The linearized kinetic equation, from (5), is

$$\left(\frac{\partial}{\partial t} + v\hat{\boldsymbol{\beta}}^{(0)} \cdot \nabla\right) f_s^{(1)} + [\alpha_s T_{(0)} - \mu(\hat{\boldsymbol{\beta}}^{(0)} \cdot \nabla) B^{(0)}] \frac{\partial f_s^{(1)}}{\partial v}$$
$$+ (\mathbf{u} + v\hat{\boldsymbol{\beta}}^{(1)}) \cdot \nabla f_s^{(0)} + [\alpha_s T_{(1)} + \boldsymbol{\kappa}^{(0)} \cdot \mathbf{u} v - \mu\{(\hat{\boldsymbol{\beta}} \cdot \nabla) B\}^{(1)}] \frac{\partial f_s^{(0)}}{\partial v} = 0. \tag{42}$$

with the flux equation, (11),

$$\frac{\partial \mathbf{B}^{(1)}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}^{(0)}). \tag{43}$$

Hence, for non-trivial modes we require $\mathbf{u}=\theta(\bar{\omega})$ and are led to consider the following scalings (to leading order in ε)

$$u_r = i\varepsilon\delta \exp(iA)[u_r^0(r)+\delta u_r'(r)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})]\exp(i\bar{\omega}t)$$

$$u_\theta = \varepsilon\delta \exp(iA)[u_\theta^0(r)+\delta u_\theta'\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})]\exp(i\bar{\omega}t), \tag{44}$$

and

$$u_z = \mathcal{O}(\varepsilon^2\delta^2),$$

since $\mathbf{u}\cdot\mathbf{B}^{(0)}=0$, by (13). Since (42) is of $\mathcal{O}(\varepsilon\delta)$,

$$f_s^{(1)} = \exp(iA)[g_s^0(r,v,\mu)+\delta g_s^1(r,v,\mu)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})]\exp(i\bar{\omega}t), \tag{45}$$

$$T_{(1)} = \varepsilon\delta\exp(iA)[iT^0(r)+T_r'\sin\tilde{z}+\mathcal{O}(\delta)]\exp(i\bar{\omega}t). \tag{46}$$

From (6), (35) and (36), $\boldsymbol{\kappa}^{(0)}=\mathcal{O}(\varepsilon^2)$ and

$$(\hat{\boldsymbol{\beta}}^{(0)}\cdot\nabla)B^{(0)} = -\varepsilon\delta\frac{\sin\tilde{z}}{a}+\mathcal{O}(\varepsilon\delta^2\sin 2\tilde{z}), \tag{47}$$

with the operator $(\hat{\boldsymbol{\beta}}^{(0)}\cdot\nabla)=\mathcal{O}(\varepsilon\delta)$. The equilibrium effective electric field

$$T_{(0)} = -(\hat{\boldsymbol{\beta}}^{(0)}\cdot\nabla)\Phi = \mathcal{O}(\varepsilon^2\delta). \tag{48}$$

Since $\Phi=\mathcal{O}(\varepsilon)$. From (43) $B_r^{(1)}=\mathcal{O}(\varepsilon\delta)$, $B_\theta^{(1)}=\mathcal{O}(\varepsilon\delta)$ and $B_z^{(1)}=\mathcal{O}(1)$ so that

$$\{(\hat{\boldsymbol{\beta}}\cdot\nabla)B\}^{(1)} = -\frac{\varepsilon\delta}{\omega}\exp(iA)\left[ilaD^0+\sin\tilde{z}\left(b\frac{\mathrm{d}D^0}{\mathrm{d}r}-aD'-\frac{D^0}{a}+\frac{a'}{a^2}u_r^0\right)\right]+\mathcal{O}(\varepsilon\delta^2), \tag{49}$$

where

$$D^0(r) = \frac{1}{r}[ru_r^0(r)]'+\frac{m}{r}u_\theta^0(r)$$
$$D^1(r) = \frac{1}{r}[ru_r^1(r)]'+\frac{m}{r}u_\theta^1(r). \tag{50}$$

On substituting (28), (35), (43)–(50) into the linearized kinetic equation (42) we find that

$$\mathcal{O}(\varepsilon\delta):\quad g_s^0+\frac{u_r^0}{\omega}\frac{\partial\varphi_s}{\partial r}+\left(\alpha_sT^0+\frac{la}{\omega}\mu D^0\right)\frac{1}{\omega+vl}\frac{\partial\varphi_s}{\partial v}=0, \tag{51}$$

$$\mathcal{O}(\varepsilon\delta\sin\tilde{z}):\quad -g_s^1+\frac{b}{a}\frac{\partial g_s^0}{\partial r}+\frac{1}{a}\frac{\mu}{v}\frac{\partial g_s^0}{\partial v}+\frac{1}{\omega}\left(-\frac{mb}{ra}u_\theta^0-u_r^1-\frac{u_r^0}{a^2}+\frac{b}{a}D^0\right)\frac{\partial\varphi_s}{\partial r}$$
$$+\left[\alpha_sT^1+\frac{\mu}{\omega}\left(b\frac{\partial D^0}{\partial r}-aD^1-\frac{D^0}{a}+\frac{a'}{a^2}u_r^0\right)\right]\frac{1}{v}\frac{\partial\varphi_s}{\partial v}=0, \tag{52}$$

T^0 and T^1 are determined from the charge neutrality constraint. For the equilibrium distribution function this constraint implies [see (14)]

$$\sum_s\alpha_s\int\mathrm{d}v\,\mathrm{d}\mu\,\varphi_s=0, \tag{53}$$

while for the perturbed distribution functions

$$\sum_s\alpha_s\int\mathrm{d}v\,\mathrm{d}\mu\,g_s^0=0=\sum_s\alpha_s\int\mathrm{d}v\,\mathrm{d}\mu\,g_s^1. \tag{54}$$

Hence, operating on (51) and (52) with $\sum_s \alpha_s \int \mathrm{d}v\, \mathrm{d}\mu$, yields

$$T^0 = -D^0 \frac{la}{\omega} \frac{\sum_s \alpha_s I_{01}{}^s}{\sum_s \alpha_s^2 I_{00}{}^s} \tag{55}$$

where

$$I_{mn}{}^s = \int \mathrm{d}v\, \mathrm{d}\mu \frac{v^m \mu^n}{v+\omega/l} \frac{\partial \varphi_s}{\partial v}, \tag{56}$$

and

$$T_1' \sum_s \alpha_s^2 J_{00}{}^s + \left(-aD' + \frac{a'}{a^2} u_r^0\right) \frac{1}{\omega} \sum_s \alpha_s J_{01}{}^s + \frac{1}{a} \sum_s \alpha_s \int \mathrm{d}v\, \mathrm{d}\mu\, \mu \frac{1}{v} \frac{\partial g_s^0}{\partial v} = 0, \tag{57}$$

where $J_{mn}{}^s$ is defined by (38) $\left[J_{mn}{}^s = I_{mn}{}^s\left(\frac{\omega}{l} = 0\right)\right]$. Note that $\sum_s \alpha_s J_{01}{}^s = 0$, by (38).

4.1 $D^0 = 0 = D^1$ FOR $M \neq 0$

To proceed further we now turn to the momentum balance equation, (8), and show that $D^0 = 0 = D^1$. On linearizing (8) and (9), and expanding $\nabla \cdot \mathbf{P}^{(1)}$ using $p_\parallel^{(0)} - p_\perp^{(0)} = \mathcal{O}(\varepsilon)$, we find

$$\begin{aligned} &i\varepsilon\delta\omega\rho_0(\mathbf{u} + \hat{\boldsymbol{\beta}}^{(0)} w) + \nabla(p_\perp^{(1)} + \mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)}) \\ &\quad + \hat{\boldsymbol{\beta}}^{(0)}(\mathbf{B}^{(0)} \cdot \nabla)\left(\frac{p_\parallel - p_\perp}{B}\right)^{(1)} + \hat{\boldsymbol{\beta}}^{(0)}(\mathbf{B}^{(1)} \cdot \nabla)\left[\frac{p_\parallel - p_\perp}{B}\right]^{(0)} + \boldsymbol{\kappa}^{(0)}(p_\parallel^{(1)} - p_\perp^{(1)}) \\ &\quad = (\mathbf{B}^{(0)} \cdot \nabla)\mathbf{B}^{(1)} + (\mathbf{B}^{(1)} \cdot \nabla)\mathbf{B}^{(0)} + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{58}$$

The third, fourth and fifth terms on the left hand side of (58) do not occur in MHD. From the r, θ components of (58), using (44),

$$\vec{\nabla}(p_\perp^{(1)} + \mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)}) = 0 + \mathcal{O}(\varepsilon^2), \tag{59}$$

in ε, where superscript $\rightarrow$ denotes vectors restricted to the r, θ plane. Since we are considering modes with $m \neq 0$ (see the appendix for the case $m = 0$) then

$$p_\perp^{(1)} + \mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)} = 0 + \mathcal{O}(\varepsilon^2). \tag{60}$$

From the definition of $p_\perp^{(1)}$, (10), (60) becomes

$$\frac{2\mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)}}{(B^{(0)})^2} [p_\perp{}^0 + \tfrac{1}{2}(B^{(0)})^2] + \sum_s (B^{(0)})^2 \int \mathrm{d}v\, \mathrm{d}\mu\, \mu f_s^{(1)} = 0 + \mathcal{O}(\varepsilon^2). \tag{61}$$

Performing δ expansion on the $\mathcal{O}(\varepsilon^0)$ part of (61) yields after some manipulations with equilibrium relations

$$\mathcal{O}(\varepsilon^0\delta^0): \quad \frac{D^0}{\omega} \exp(iA)(1 + a^3\Gamma) = 0, \tag{62}$$

where

$$\Gamma = \sum_s I_{02}{}^s - \frac{\left(\sum_s \alpha_s I_{01}{}^s\right)^2}{\sum_s \alpha_s^2 I_{00}{}^s}. \tag{63}$$

However, from the local stability criterion for the firehose instability (derived for $\bar{\omega}=\mathcal{O}(1)$ modes, see WEITZNER (1975) for details), $1+a^3\Gamma>0$. Therefore (62) implies

$$D^0=0. \tag{64}$$

Equation (55) then yields

$$T^0=0, \tag{65}$$

and from (51)

$$g_s^{\,0}=-\frac{u_r^{\,0}}{\omega}\frac{\partial\varphi_s}{\partial r}. \tag{66}$$

To show $D^1=0$ we consider the $\mathcal{O}(\varepsilon^0\delta)$ equation of (61). After some algebra we find

$$D^1\left[1+a^3\Gamma\left(\frac{\omega}{l}=0\right)\right]$$
$$=u_r^{\,0}\left[\frac{4a'}{a^3}-\frac{2a'}{a}-a^2\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\sum_s J_{02}{}^s}{a}\right)-\left(\frac{\sum_s \alpha_s J_{01}{}^s}{\sum_s \alpha_s^{\,2}J_{00}{}^s}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\sum_s \alpha_s J_{01}{}^s}{a}\right)\right\}\right]=0$$

on using (36)–(39). Hence $\left(\text{since for local firehouse stability, } 1+a^3\Gamma\left(\frac{\omega}{l}=0\right)>0\right)$

$$D^1=0, \tag{67}$$

so that from (57) and (66)

$$T^1=\frac{u_r^{\,0}}{a\omega}\frac{\sum_s \alpha_s J_{01}{}^s}{\sum_s \alpha_s^{\,2}J_{00}{}^s}=0 \tag{68}$$

by (38). Thus, to leading order, the parallel component of the perturbed electric field is zero.

4.2 TRANSVERSE-LIKE MODES

We now proceed to eliminate w from the momentum balance equation (58). By applying the operator $\mathbf{1}-\hat{\boldsymbol{\beta}}^{(0)}\hat{\boldsymbol{\beta}}^{(0)}$ to (58) and remembering that $\mathbf{u}\cdot\hat{\boldsymbol{\beta}}^{(0)}=0=\hat{\boldsymbol{\beta}}^{(0)}\cdot\boldsymbol{\kappa}^{(0)}$, we find

$$i\varepsilon\delta\omega\rho_0(\bar{\mathbf{u}}+u_z\hat{\mathbf{z}})+\varepsilon^2\nabla\Omega+\hat{\boldsymbol{\beta}}^{(0)}G+\boldsymbol{\kappa}^{(0)}(p_\parallel^{(1)}-p_\perp^{(1)})$$
$$=(\mathbf{B}^{(0)}\cdot\nabla)\mathbf{B}^{(1)}+(\mathbf{B}^{(1)}\cdot\nabla)\mathbf{B}^{(0)}+\theta(\varepsilon^3), \tag{69}$$

where, from (60)

$$p_\perp^{(1)}+\mathbf{B}^{(0)}\cdot\mathbf{B}^{(1)}=\varepsilon^2\Omega, \tag{70}$$

and we define

$$G\equiv(\mathbf{B}^{(0)}\cdot\nabla)B_z^{(1)}+(\mathbf{B}^{(1)}\cdot\nabla)B_z^{(0)}. \tag{71}$$

The second, third and fourth terms of the left hand side of (69) contain contributions from the GCP model and are thus non-MHD like terms. If these third and fourth terms can be shown to be higher order in δ, namely $\mathcal{O}(\varepsilon^2\delta^3)$, then

(69) is exactly that equation one would obtain if one were looking at the $\omega^2 = \mathcal{O}(\varepsilon^2\delta^2)$ transverse modes for the MHD stability of the bumpy θ pinch (WEITZNER 1973) for perturbed flow $\mathbf{u}$ with

$$\frac{\partial \mathbf{B}^{(1)}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}^{(0)}). \tag{72}$$

Since: (i) $\mathcal{O}(u_z/\tilde{\mathbf{u}}) = \mathcal{O}(\varepsilon\delta)$, i.e., the flow $\tilde{\mathbf{u}}$ is mainly perpendicular to $\mathbf{B}$ and so a transverse mode.

(ii) $\nabla \cdot \tilde{\mathbf{u}} = 0 + \mathcal{O}(\delta^2)$, since $D^0 = 0 = D^1$, i.e., $\tilde{\mathbf{u}}$ is incompressible to at least second order in δ (in MHD, $\tilde{\mathbf{u}}$ is incompressible to all orders in δ however the growth rate of these MHD modes are unaffected provided $\tilde{\mathbf{u}}$ is incompressible to *second order* in δ) and we can introduce a stream function χ.

(iii) Because of the incompressibility, the second term in (69) plays no role for these transverse modes, i.e., as its counterpart in MHD, one eliminates $\nabla\Omega$ by taking the curl of (69). The eigenvalue equation for the stream function χ is then the z-component of the curl of (69). This would then be exactly that set of equations found by WEITZNER (1973) for the MHD bumpy θ pinch.

4.3 TO SHOW THAT $G = 0(\varepsilon\delta^2)$, $P_\parallel^{(1)} - P_\perp^{(1)} = 0(\delta^2)$

From the definition of G, (71), we readily find that

$$G = \varepsilon\delta \frac{\exp(iA)}{\omega} a^2 D^1 \sin \tilde{z} + \mathcal{O}(\varepsilon\delta^2)$$
$$= \mathcal{O}(\varepsilon\delta^2). \tag{73}$$

Since, (67), $D^1 = 0$. From (60),

$$p_\perp^{(1)} = -\mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)} + \mathcal{O}(\varepsilon^2)$$
$$= \frac{\exp(iA)}{\omega} [aa' u_r^0 + \delta \cos \tilde{z}(a^2 D^1 + aa' u_r^1 + \{ac\}' u_r^0)] + \mathcal{O}(\delta^2) + \mathcal{O}(\varepsilon^2) \tag{74}$$

Finally, from (10)

$$p_\parallel^{(1)} = \frac{\mathbf{B}^{(0)} \cdot \mathbf{B}^{(1)}}{(B^{(0)})^2} p_\parallel^{(0)} + \sum_s B^{(0)} \int \mathrm{d}\mu \, \mathrm{d}v \, v^2 f_s^{(1)}$$

Substitution the δ expansion of $f_s^{(1)}$, (45), (51) and (52) and performing the integration, we find

$$p_\parallel^{(1)} = \frac{\exp(iA)}{\omega}\Big[aa' u_r^0 - \delta \cos \tilde{z}\Big(\frac{a'}{a} p_\parallel^{01} u_r^0 + p_\parallel^{00}\Big\{D^1 - \frac{a'c}{a^2} u_r^0 + \frac{a'}{a} u_r^1 + \frac{c'}{a} u_r^0\Big\}$$
$$+ u_r^0 \frac{\mathrm{d}}{\mathrm{d}r} \sum_s J_{21}{}^s + b\Big\{u_r^0\Big(\frac{p_\parallel^{00}}{a}\Big)'\Big\}' + \Big\{\frac{mb}{r} u_\theta^0 + au_r^1 + cu_r^0 - \frac{u_r^0}{a}\Big\}\Big\{\frac{p_\parallel^{00}}{a}\Big\}'$$
$$- \omega T^1 a \sum_s \alpha_s J_{20}{}^s + \Big\{a^2 D^1 - \frac{a'}{a} u_r^0\Big\} \sum_s J_{21}{}^s\Big)\Big] + \mathcal{O}(\delta^2). \tag{75}$$

Since $p_\parallel{}^{01} = -a'b$, $p_\parallel{}^{00} = \frac{1}{2}(1 - a^2)$, and $D^1 = T^1 = 0$ with (38) and (39), we see that

$$p_\parallel^{(1)} - p_\perp^{(1)} = \mathcal{O}(\delta^2). \tag{76}$$

The eigenvalue equation for ω^2 and for the stream function $\chi_0\chi/\varepsilon\delta = \exp(iA)[\chi_0(r)+\delta\chi_1(r)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})]\exp(i\bar{\omega}t)$—with $ru_r = m\chi$, $u_\theta = -\dfrac{\partial\chi}{\partial r}$ [since $D^0 = 0 = D^1$; (50), (64) and (67)]—is obtained from the z-component of the curl of (69). It is of $\mathcal{O}(\varepsilon^2\delta^2)$. The third term of (69) now will have order, from (73),

$$\mathcal{O}(\beta_r G) = \mathcal{O}(\varepsilon^2\delta^3), \tag{77}$$

since to leading order $\beta_r = \dfrac{\varepsilon\delta b\sin z}{a}$, $\beta_\theta = 0$. The fourth term will now be

$$\mathcal{O}(\kappa_r[p_\parallel^{(1)} - p_\perp^{(1)}]) = \mathcal{O}(\varepsilon^2\delta^2), \tag{78}$$

by (76) since $\kappa_r = (\varepsilon^2\delta b\cos\tilde{z})/a + \mathcal{O}(\varepsilon^2\delta^2)$, $\kappa_\theta = 0$.

We thus conclude that the collisionless effects of the GCP model are not sufficient either to stabilize or even alter the growth of the unstable MHD transverse modes, under a weak pressure anisotropy of $\mathcal{O}(\varepsilon)$.

5. CONCLUSIONS

In this paper we have considered the guiding center plasma stability of the bumpy θ pinch assuming a weak pressure anisotropy of $\theta(\varepsilon)$, It was seen in Section 3 that a fairly wide class of distribution functions could be chosen such that the GCP system was formally self-adjoint and that (together with charge and current neutrality satisfied) the GCP equilibrium was, to leading order in ε, the same as in MHD, on identifying

$$p_\parallel^{(0)} = p_\perp^{(0)} = p_{\mathrm{MHD}}^{(0)}$$

The stability analysis showed that any unstable mode in MHD has exactly the same growth in a GCP and *vice versa.* It should be noted that we *do* have effects of the parallel magnetic field force in the kinetic equations, while for helical symmetry this force is a higher order effect; i.e., for axisymmetry

$$(\mathbf{B}^{(0)}\cdot\nabla)B^{(0)} = \mathcal{O}(\varepsilon\delta)$$

while for helically symmetry (and pressure anisotropy of $\mathcal{O}(\varepsilon^2)$ or $\mathcal{O}(\varepsilon)$)

$$(\mathbf{B}^{(0)}\cdot\nabla)B^{(0)} = \mathcal{O}(\varepsilon^2)\cdot$$

(The kinetic equation is of $\mathcal{O}(\varepsilon\delta)$, for the unstable modes $\omega = \mathcal{O}(\varepsilon\delta)$).

Acknowledgements—The authors are grateful to Harold Weitzner for valuable discussions and comments.

REFERENCES

FRIEDBERG J. P. (1972) *Physics Fluids* **15,** 1102.
GRAD H. (1961) in "*Proc. Sym. on Electromagnetics and Fluid Dynamics of Gaseous Plasma*". p. 37. Polytechnic Press, Brooklyn New York.
GRAD H. (1967) in "*Proc. Sym. in Applied Mathematics*", (H. Grad, Ed.) Vol. 18, p. 162. Am. Math. Soc. Providence, Rhode Island.
KULSRUD R. M. (1962) *Physics Fluids* **5,** 192.
WEITZNER H. (1973) *Physics Fluids* **10,** 237.
WEITZNER H. (1974) Sherwood Theoretical Meeting, April, Berkeley, California.
WEITZNER H. (1975) *Physics Fluids* (Submitted).

CANTRELL E. L., ELLIS W. R., HARRIS H. W., JAHODA F. C., KRISTAL R., MACHALEK M. D., MCCONNELL J. R., QUINN W. E., RIBE F. L., SAWYER G. A., SEIBEL F. T. and SIEMON R. E. In *Proc. 5 IAEA Conf. on Plasma Physics and Controlled Fusion Res.* Tokyo.
ELLIS W. R., JAHODA F. C., KRISTAL R., QUINN W. E., RIBE F. L., SAWYER G. A. and SIEMON R. E. (1974) *Nucl. Fusion* **14,** 860.

APPENDIX. MODES WITH $m=0$

We now turn our attention to modes with $m=0$. From (59)

$$p_\perp^{(1)}+\mathbf{B}^{(0)}\cdot\mathbf{B}^{(1)}=f(\tilde{z})+\mathcal{O}(\varepsilon^2). \tag{A1}$$

Applying $\hat{\boldsymbol{\beta}}^{(0)}$ to (58) gives

$$\mathcal{O}(\varepsilon^3)+i\varepsilon\delta\omega\rho_0 w+\varepsilon f'(\tilde{z})+(\mathbf{B}^{(0)}\cdot\nabla)\left[\frac{p_\parallel-p_\perp}{B}\right]^{(1)}+(\mathbf{B}^{(1)}\cdot\nabla)\left[\frac{p_\parallel-p_\perp}{B}\right]^{(0)}=G, \tag{A2}$$

where G is given by (71). We shall assume that the plasma extends to the wall so that the boundary conditions at the wall $r=R_w$ is

$$\hat{\mathbf{n}}\cdot(\mathbf{u}+\hat{\boldsymbol{\beta}}^{(0)}w)\big|_{r=R_w}=0$$

Since $\hat{\mathbf{n}}=\nabla\psi/|\nabla\psi|$, $\hat{\boldsymbol{\beta}}^{(0)}\cdot\hat{\mathbf{n}}=0$. Hence to leading order in ε,

$$[u_r]\big|_{r=R_w}=0 \tag{A3}$$

From (A2) we conclude that $w=\mathcal{O}(1)[\mathbf{u}$ is $\theta(\varepsilon\delta)]$ and so we assume this longitudinal mode (significant flow along B line)

$$\begin{aligned} w&=\exp(il\,\delta\tilde{z})[w_0(r)+\delta w_1(r)\cos\tilde{z}+\mathcal{O}(\delta^2\cos 2\tilde{z})]\exp(i\bar{\omega}t)\\ f(\tilde{z})&=-\frac{\varepsilon\delta}{\bar{\omega}}\exp(il\,\delta\tilde{z})[f_0+f_1\sin\tilde{z}+\mathcal{O}(\delta)] \end{aligned} \tag{A4}$$

where f_0 and f_1 are constants. u_r is now expanded in the form

$$u_r=i\varepsilon\delta\exp(il\,\delta\tilde{z})[u_0(r)+u_1(r)\sin\tilde{z}+\mathcal{O}(\delta)]\exp(i\bar{\omega}t) \tag{A5}$$

As in the MHD calculation we need consider only that part of the leading order term in (A1) and (A2) which has no trigonometric dependence on $\tilde{z}$. Hence from (A2), and orthogonality,

$$i\omega^2\rho_0(r)w_0(r)=ilf_0. \tag{A6}$$

Hence

$$w_0(r)=\frac{1}{\rho_0(r)}, \tag{A7}$$

$$\omega^2=lf_0. \tag{A8}$$

From (A1) and (A4) we have, from orthogonality (with $m=0$),

$$(1+a^3\Gamma)\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru_0)=f_0 \tag{A9}$$

so that, using (A8)

$$ru_0(r)=\frac{\omega^2}{l}\int_0^r\frac{r'\,\mathrm{d}r'}{1+a^3(r')\Gamma}. \tag{A10}$$

Hence, using the boundary condition (A3), we see that $\omega^2=0$. That is, these longitudinal modes with large flows along a B line do not exist.